Berry phase and Hannay's angle in a quantum-classical hybrid system

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The Berry phase, which was discovered more than two decades ago, provides very deep insight into the geometric structure of quantum mechanics. Its classical counterpart, Hannay's angle, is defined if closed curves of action variables return to the same curves in phase space after a time evolution. In this paper we study the Berry phase and Hannay's angle in a quantum-classical hybrid system under the Born-Oppenheimer approximation. By the term quantum-classical hybrid system, we mean a composite system consists of a quantum subsystem and a classical subsystem. The effects of subsystem-subsystem couplings on the Berry phase and Hannay's angle are explored. The results show that the Berry phase has been changed sharply by the couplings, whereas the couplings have a small effect on the Hannay's angle.

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I. INTRODUCTION

The concept of Berry phase has paved a new way for understanding quantum physics. It has attracted much attention since Berry's discovery [1–3] and found potential applications in fields ranging from chemistry to condensed matter physics. Shortly after Berry's discovery, Simon gave a mathematical interpretation to this phase, that it can be regarded as the holonomy in a Hermitian line bundle since the adiabatic theorem naturally defines a connection in such a bundle [4]. After these remarkable discoveries, Hannay found that this geometrical phase not only exists in quantum systems but also in the classical world [5]. Analogous with the Berry phase, the angle variable of classical integrable systems [6] acquires an additional angle shift as the system slowly cycles in phase space. This angle shift is called Hannay's angle. It was later proved by Berry that the geometric phase and Hannay's angle possess a natural relation under semiclassical approximation [7]. As a matter of course, this quantum-classical correspondence gives rise to many impressive explorations [8–10].

It is known that a quantum Hilbert space carries the same structure of a classical phase space [1,6]. By this virtue, a quantum system can be treated like a classical system without loss of physics [11,12]. Particularly in Ref. [12], the authors introduced a general framework for testing nonlinear quantum mechanics. In this generalized theory, the elements of quantum mechanics, such as wave functions, observable symmetries, and time evolution, all can be treated classically. Based on this instructive work, some interesting efforts have been devoted to nonlinear quantum systems and quantum-classical hybrid systems [13–19]. Some questions have arisen: Since quantum mechanics can be put into the framework of classic mechanics, can the Berry phase be presented into the form of Hannay's angle? What is the Berry phase or Hannay's angle of the quantum-classical hybrid system? How do the subsystem-subsystem couplings affect Hannay's angle and the Berry phase? We shed light on these questions in this paper.

The paper is organized as follows. In Sec. II we first present a quantum system in the framework of classical theory by Weinberg's method [12] and then calculate Hannay's angle of the quantum system. Next we compare this Hannay's angle with the Berry phase of the original quantum system and find that Hannay's angle and the Berry phase differ only by a sign [14]. An example is given to show this result. In Sec. III we present a quantum-classical hybrid system in classical mechanics based on the Born-Oppenheimer approximation, a unified one-form which can deduce both the Berry phase and Hannay's angle. By this one-form we calculated Hannay's angles and the Berry phase and study the effect of subsystem-subsystem couplings on Hannay's angle. Finally, we conclude our results in Sec. IV.

II. BERRY PHASE AND HANNAY'S ANGLE

In quantum theory, observables are represented by Hermitian matrices F or real bilinear functions $\langle\psi|\hat{F}|\psi\rangle$. Weinberg has generalized the representation into real nonbilinear functions $f(\boldsymbol{\psi},\boldsymbol{\psi}^*)$ to include nonlinearity. Let $\{|\varphi_n\rangle\}$ be an orthonormal basis of Hilbert space. The Schrödinger equation can then be rewritten as

$$i\hbar\frac{d\psi_n}{dt}=\frac{\partial h}{\partial\psi_n^*},\tag{1}$$

with $|\psi\rangle=\sum\psi_n|\varphi_n\rangle$, $\boldsymbol{\psi}=(\psi_1,\ldots,\psi_n,\ldots,\psi_N)^T$, and $h(\boldsymbol{\psi},\boldsymbol{\psi}^*)$ the total energy of the system. If we decompose ψ_n into real and imaginary parts $\psi_n=(q_n+ip_n)/\sqrt{2\hbar}$, the Schrödinger equation and its complex conjugate can be written as Hamiltonian canonical equations [11,12],

$$\dot{q}_n=\frac{\partial h}{\partial p_n},\quad \dot{p}_n=-\frac{\partial h}{\partial q_n}.\tag{2}$$

The Hamiltonian function $h(\psi,\psi^*)$ can be transformed into $h(\boldsymbol{q},\boldsymbol{p})$. It is amazing to note that the quantum Hermitian structure becomes the symplectic structure of classical mechanics, $dp_n\wedge dq_n=i\hbar d\psi_n^*\wedge d\psi_n$.

Now, let us turn to the first question. Consider a quantum system with N levels, whose Hamiltonian function $h(\psi,\psi^*,\boldsymbol{X})$ depends on a set of slowly-varying parameters $\boldsymbol{X}=(X_1,Y_1,\ldots)$ and its quantum state $|\psi\rangle$. By the procedure mentioned above, i.e., by decomposing $|\psi\rangle$ with basis $\{|n\rangle\}$, $|\psi\rangle=\sum_n\psi_n(t)|n\rangle$, and setting $\psi_n(t)=[q_n(t)+ip_n(t)]/\sqrt{2\hbar}$, we can write the Hamiltonian function in terms of "position variable $\boldsymbol{q}$" and "momentum variable $\boldsymbol{p}$" as

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$h(\boldsymbol{p}(t),\boldsymbol{q}(t))$. Of course, the state vector $|\psi\rangle$ can also be expanded by taking the Hamiltonian eigenstates $|E_k(\boldsymbol{X})\rangle$ as a basis, $|\psi\rangle = \sum_k \psi_k'(t)|E_k(\boldsymbol{X})\rangle$. By the adiabatic theorem, the occupation probability of each eigenstate $|\psi_k'(t)|^2$ remains unchanged in the adiabatic limit. One then can introduce a new pair of variables $(\boldsymbol{\theta},\boldsymbol{I})$ by

$$\psi_k'(t) = \sqrt{\frac{I_k}{\hbar}} e^{-i\theta_k}, \tag{3}$$

and write the Hamiltonian function as [12,14]

$$\bar{h} = \bar{h}(\boldsymbol{I},\boldsymbol{X}) = \sum_k E_k(\boldsymbol{X}) I_k/\hbar, \tag{4}$$

where $E_k(\boldsymbol{X})$ is the eigenvalue of the Hamiltonian $\hat{H}$ with corresponding eigenstate $|E_k(\boldsymbol{X})\rangle$. It has been proved that the two new variables $\boldsymbol{\theta}$ and $\boldsymbol{I}$ satisfy the same canonical equations as the angle-action variables in classical mechanics [12],

$$\dot{\theta}_k = \frac{\partial \bar{h}}{\partial I_k}, \dot{I}_k = 0. \tag{5}$$

So far, the quantum Hamiltonian $\hat{H}$ was transformed into classical Hamiltonian function $\bar{h}(\boldsymbol{I},\boldsymbol{X})$, and the quantum unitary transformation $|E_k(\boldsymbol{X})\rangle = \sum_n C_{kn}(\boldsymbol{X})|n\rangle$ turns out to be a classical canonical transformation $(\boldsymbol{q},\boldsymbol{p}) \rightarrow (\boldsymbol{\theta},\boldsymbol{I})$.

$$\begin{aligned} p_n &= \sum_k \sqrt{2I_k}\,\{\cos\theta_k \mathrm{Im}[C_{kn}(\boldsymbol{X})] - \sin\theta_k \mathrm{Re}[C_{kn}(\boldsymbol{X})]\}, \\ q_n &= \sum_k \sqrt{2I_k}\,\{\cos\theta_k \mathrm{Re}[C_{kn}(\boldsymbol{X})] + \sin\theta_k \mathrm{Im}[C_{kn}(\boldsymbol{X})]\}. \end{aligned} \tag{6}$$

According to Berry's theory [7], the Hannay's angle of the system can be written as

$$\Delta\theta_k(\boldsymbol{I};\boldsymbol{X}) = -\frac{\partial}{\partial I_k} \oint A_H(\boldsymbol{I};\boldsymbol{X}), \tag{7}$$

where

$$\begin{aligned} A_H(\boldsymbol{I};\boldsymbol{X}) &= \langle \boldsymbol{p}(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{X}) d_{\boldsymbol{X}} \boldsymbol{q}(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{X})\rangle_\theta \\ &= \frac{1}{(2\pi)^N} \oint d\theta \sum_n p_n(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{X}) d_{\boldsymbol{X}} q_n(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{X}) \end{aligned} \tag{8}$$

is the angle one-form for Hannay's angle [17]. The angular brackets $\langle\cdots\rangle_\theta$ denote an averaging over all angles $\boldsymbol{\theta}$, and $d_{\boldsymbol{X}}$ is defined as $d_{\boldsymbol{X}} F(\boldsymbol{X}) = \frac{\partial F(X)}{\partial X} \cdot d\boldsymbol{X}$. Substituting Eq. (6) into Eq. (8), we obtain

$$\begin{aligned} A_H &= \sum_k I_k \sum_n i[C_{kn}^*(\boldsymbol{X}) d_{\boldsymbol{X}} C_{kn}(\boldsymbol{X})] \\ &= \sum_k i I_k \langle E_k(\boldsymbol{X})| d_{\boldsymbol{X}} E_k(\boldsymbol{X})\rangle = \sum_k i I_k A_B(k;\boldsymbol{X}). \end{aligned} \tag{9}$$

We note that $A_B(k;\boldsymbol{X})$ is nothing but the one-form for Berry phase [17]. This means that the Hannay's angles exactly equal the minus Berry phases of the original quantum system [14],

$$\Delta\theta_k(\boldsymbol{I};\boldsymbol{X}) = -\frac{\partial}{\partial I_k} \oint A_H = -i \oint A_B(k;\boldsymbol{X}) = -\gamma_k(C). \tag{10}$$

The appearance of the minus is because the angle variables of the effective classical system correspond to the opposite numbers of the phases in Eq. (3). Therefore we can choose $A(\boldsymbol{I};\boldsymbol{X}) = A_H$ to be a general one-form,

$$\Delta\theta_k = -\gamma_k = -\frac{\partial}{\partial I_k} \oint A(\boldsymbol{I};\boldsymbol{X}). \tag{11}$$

To shed more light on this result, we consider the adiabatic evolution of a spin-half particle with magnetic moment μ in an external magnetic field $\boldsymbol{B}$. The Hamiltonian reads

$$\hat{H} = -\mu\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{B}, \tag{12}$$

where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)$ are Pauli matrices. As mentioned previously, if we choose the two spin eigenstates $|\pm\rangle$ as the basis, the Hamiltonian in Eq. (12) can be transformed into a Hamiltonian function

$$\begin{aligned} h(\boldsymbol{p},\boldsymbol{q};\boldsymbol{B}) = \frac{-\mu}{\hbar}\big[&(q_1q_2 + p_1p_2)B_1 + (p_1q_2 - p_2q_1)B_2 \\ &+ \frac{1}{2}\big(p_2^2 + q_2^2 - p_1^2 - q_1^2\big)B_3\big], \end{aligned} \tag{13}$$

with $|\psi\rangle = \psi_1|-\rangle + \psi_2|+\rangle$ and $\psi_j = (q_j + ip_j)/\sqrt{2\hbar}$, $(j = 1,2)$. The canonical variables $(\boldsymbol{q},\boldsymbol{p})$ satisfy the normalization condition [11],

$$\sum_{j=1}^{2} \big(p_j^2 + q_j^2\big) = 2\hbar. \tag{14}$$

It is interesting to note that by defining a vector, $\boldsymbol{S} = (S_1,S_2,S_3)$, the Hamiltonian function can be written as

$$h(\boldsymbol{S};\boldsymbol{B}) = -\mu\boldsymbol{S} \cdot \mathbf{B}, \tag{15}$$

where

$$\begin{cases} S_1 = (q_1q_2 + p_1p_2)/\hbar, \\ S_2 = (p_1q_2 - p_2q_1)/\hbar, \\ S_3 = \big(p_2^2 + q_2^2 - p_1^2 - q_1^2\big)/(2\hbar). \end{cases} \tag{16}$$

The normalization condition in terms of $\boldsymbol{S}$ is $S^2 \equiv S_1^2 + S_2^2 + S_3^2 = 1$, and their Poisson bracket has a relation with the quantum commutator as [11]

$$\{S_i,S_j\} = 2\epsilon_{ijk}S_k/\hbar = \frac{1}{i\hbar}\langle\psi|[\hat{\sigma}_i,\hat{\sigma}_j]|\psi\rangle. \tag{17}$$

Moreover, if we choose $|\pm\rangle$ as the basis, $\boldsymbol{S}$ is nothing but the Stokes parameters which span the Poincare sphere,

$$\begin{cases} I = |\psi_2|^2 + |\psi_1|^2 = S^2 = 1, \\ U = 2\mathrm{Re}(\psi_2\psi_1^*) = S_1 S, \\ V = 2\mathrm{Im}(\psi_2\psi_1^*) = S_2 S, \\ Q = |\psi_2|^2 - |\psi_1|^2 = S_3 S. \end{cases} \tag{18}$$

We now move to calculate the Hannay's angle. Since the Hamiltonian in Eq. (12) has two eigenstates,

$$\begin{aligned} |E_1\rangle &= \sqrt{\frac{B + B_3}{2B}}|+\rangle + \frac{B_1 + iB_2}{\sqrt{2B(B + B_3)}}|-\rangle, \\ |E_2\rangle &= -\sqrt{\frac{B - B_3}{2B}}|+\rangle + \frac{B_1 + iB_2}{\sqrt{2B(B - B_3)}}|-\rangle, \end{aligned} \tag{19}$$

with eigenenergies $-\mu B$ and μB, respectively, where $B = \sqrt{B_1^2 + B_2^2 + B_3^2}$. The canonical transformation $(\boldsymbol{q},\boldsymbol{p}) \rightarrow$

$(\boldsymbol{\theta},\boldsymbol{I})$ and the transformed Hamiltonian function can be written as

$$\begin{aligned} q_1 &= \sqrt{2I_1}\left[\frac{B_1\cos\theta_1}{\sqrt{2B(B+B_3)}}+\frac{B_2\sin\theta_1}{\sqrt{2B(B+B_3)}}\right] \\ &\quad+\sqrt{2I_2}\left[\frac{B_1\cos\theta_2}{\sqrt{2B(B-B_3)}}+\frac{B_2\sin\theta_2}{\sqrt{2B(B-B_3)}}\right], \\ q_2 &= \sqrt{\frac{2I_1(B+B_3)}{2B}}\cos\theta_1-\sqrt{\frac{2I_2(B-B_3)}{2B}}\cos\theta_2, \end{aligned} \tag{20}$$

$$\begin{aligned} p_1 &= \sqrt{2I_1}\left[\frac{B_2\cos\theta_1}{\sqrt{2B(B+B_3)}}-\frac{B_1\sin\theta_1}{\sqrt{2B(B+B_3)}}\right] \\ &\quad+\sqrt{2I_2}\left[\frac{B_2\cos\theta_2}{\sqrt{2B(B-B_3)}}-\frac{B_1\sin\theta_2}{\sqrt{2B(B-B_3)}}\right], \\ p_2 &= \sqrt{\frac{2I_2(B-B_3)}{2B}}\sin\theta_2-\sqrt{\frac{2I_1(B+B_3)}{2B}}\sin\theta_1, \\ \bar{h}(\boldsymbol{I};\boldsymbol{B}) &= \mu B(I_2-I_1), \end{aligned}$$

with $\boldsymbol{q}=(q_1,q_2)$, $\boldsymbol{p}=(p_1,p_2)$, $\boldsymbol{\theta}=(\theta_1,\theta_2)$, and $\boldsymbol{I}=(I_1,I_2)$. Therefore we obtain the angle one-form by Eq. (8),

$$A=\frac{B_2dB_1-B_1dB_2}{2B(B+B_3)}I_1+\frac{B_2dB_1-B_1dB_2}{2B(B-B_3)}I_2. \tag{21}$$

The Hannay's angles can thus be obtained by Eq. (11),

$$\begin{aligned} \Delta\theta_1 &= -\oint\frac{B_2dB_1-B_1dB_2}{2B(B+B_3)}, \\ \Delta\theta_2 &= -\oint\frac{B_2dB_1-B_1dB_2}{2B(B-B_3)}, \end{aligned} \tag{22}$$

which differ from the Berry phases for the original quantum Hamiltonian [1] by only a sign.

III. BERRY PHASE AND HANNAY'S ANGLE IN HYBRID SYSTEM

Based on formalism in the last section, we now turn to the second question raised in the Introduction. Consider a hybrid system consisting of a classical and quantum subsystem. The Hamilton function of this quantum-classical hybrid system under Born-Oppenheimer approximation can be written as [15,19]

$$H_{hybrid}=\langle\psi|\hat{H}_1(\boldsymbol{Q},\boldsymbol{X}_1)|\psi\rangle+H_2(\boldsymbol{P},\boldsymbol{Q};\boldsymbol{X}_2), \tag{23}$$

where $|\psi\rangle$ is the state of the fast quantum subsystem. The Hamiltonian function H_2 describes a slow classical subsystem with momentum $\boldsymbol{P}$, and coordinate $\boldsymbol{Q}$, $\boldsymbol{X}_1=(X_1,Y_1,\ldots)$ and $\boldsymbol{X}_2=(X_2,Y_2\ldots)$ are slowly-varying parameters of the quantum and classical subsystems, respectively. The subsystem-subsystem coupling is included in $\hat{H}_1(\boldsymbol{Q},\boldsymbol{X}_1)$. Following the procedure given in the last section, we first choose a basis and then expand the state $|\psi\rangle$ in this basis. We next decompose the expansion coefficients into real parts $\boldsymbol{q}$ and imaginary parts $\boldsymbol{p}$, and finally represent the hybrid system by a classical Hamiltonian function,

$$H=H_1(\boldsymbol{p},\boldsymbol{q};\boldsymbol{Q},\boldsymbol{X}_1)+H_2(\boldsymbol{P},\boldsymbol{Q};\boldsymbol{X}_2). \tag{24}$$

As is known, the Hamiltonian of quantum subsystem $H_1(\boldsymbol{p},\boldsymbol{q};\boldsymbol{Q},\boldsymbol{X}_1)$ can be transformed into $\bar{H}_1(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)$ by a canonical transformation $(\boldsymbol{q},\boldsymbol{p})\to(\boldsymbol{\theta},\boldsymbol{I})$, but the new Hamiltonian $\bar{H}_1$ differs from the old one, taking [6,7] the form

$$\begin{aligned} \bar{H}_1(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1) &= \mathscr{H}_1(\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)+\frac{\partial S}{\partial t} \\ &= \mathscr{H}_1(\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)+\dot{\boldsymbol{Q}}\cdot\left(\frac{\partial\mathscr{S}}{\partial\boldsymbol{Q}}-\boldsymbol{p}\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{Q}}\right) \\ &\quad+\dot{\boldsymbol{X}}_1\cdot\left(\frac{\partial\mathscr{S}}{\partial\boldsymbol{X}_1}-\boldsymbol{p}\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{X}_1}\right), \end{aligned} \tag{25}$$

where

$$\mathscr{H}_1(\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)\equiv H_1[\boldsymbol{p}(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1),\boldsymbol{q}(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1);\boldsymbol{Q},\boldsymbol{X}_1], \tag{26}$$

and $S(\boldsymbol{q},\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)$ is the generating function of the transformation and the single-valued function $\mathscr{S}\equiv S(\boldsymbol{q}(\boldsymbol{\theta},\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1),\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)$ is introduced to give an explicit form for $\bar{H}_1$ [7]. Since the variables $\boldsymbol{Q}$ can be treated as slowly-varying parameters like $\boldsymbol{X}_1$, an average over θ may be taken to approximate to Eq. (25) [6],

$$\langle\bar{H}_1\rangle_\theta=\mathscr{H}_1(\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)-\dot{\boldsymbol{Q}}\cdot\left\langle\boldsymbol{p}\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{Q}}\right\rangle_\theta-\dot{\boldsymbol{X}}_1\cdot\left\langle\boldsymbol{p}\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{X}_1}\right\rangle_\theta, \tag{27}$$

where the angular brackets $\langle\cdots\rangle_\theta$ denote an averaging over all angles $\boldsymbol{\theta}$, and the terms $\dot{\boldsymbol{Q}}\cdot\langle\frac{\partial\mathscr{S}}{\partial\boldsymbol{Q}}\rangle_\theta$ and $\dot{\boldsymbol{X}}_1\cdot\langle\frac{\partial\mathscr{S}}{\partial\boldsymbol{X}_1}\rangle_\theta$ are dropped for zero contribution to the equations of motion. Thus the Hamiltonian of total system can be written as

$$\begin{aligned} H_{av} &= \mathscr{H}_1(\boldsymbol{I};\boldsymbol{Q},\boldsymbol{X}_1)+H_2(\boldsymbol{P},\boldsymbol{Q};\boldsymbol{X}_2) \\ &\quad-\dot{\boldsymbol{Q}}\cdot\left\langle\sum_i p_i\frac{\partial q_i}{\partial\boldsymbol{Q}}\right\rangle-\dot{\boldsymbol{X}}_1\cdot\left\langle\sum_i p_i\frac{\partial q_i}{\partial\boldsymbol{X}_1}\right\rangle, \end{aligned} \tag{28}$$

where the first two terms are the effective Hamiltonian under Born-Oppenheimer approximation and the latter two terms are related to the Berry phase of the quantum subsystem, because the quantum one-form is defined as $A_1(\boldsymbol{I};\boldsymbol{X}_1,Q)=-\left(\dot{\boldsymbol{Q}}\cdot\left\langle\sum_i p_i\frac{\partial q_i}{\partial\boldsymbol{Q}}\right\rangle+\dot{\boldsymbol{X}}_1\cdot\left\langle\sum_i p_i\frac{\partial q_i}{\partial\boldsymbol{X}_1}\right\rangle\right)dt$, like Eq. (11).

Since $\boldsymbol{I}$ can be treated as a constant in the adiabatic limit, the Hamiltonian in Eq. (28) contains only the variables of the classical subsystem. Therefore we can transform this Hamiltonian into a function of angle-action variables via the canonical transformation $(\boldsymbol{Q},\boldsymbol{P})\to(\boldsymbol{\phi},\boldsymbol{J})$. By averaging over $\boldsymbol{\phi}$, we finally obtain an averaged Hamiltonian for the hybrid system,

$$\langle H_{av}\rangle=\mathscr{H}(\boldsymbol{I},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2)+\frac{A(\boldsymbol{I},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2)}{dt}, \tag{29}$$

with $\mathscr{H}(\boldsymbol{I},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2)\equiv\mathscr{H}_1[\boldsymbol{I};\boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2),\boldsymbol{X}_1]+H_2[\boldsymbol{Q}(\boldsymbol{\phi},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2),\boldsymbol{P}(\boldsymbol{\phi},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2);\boldsymbol{X}_2]$. According to classical mechanics [6], the time-dependent canonical transformation will bring out a geometric term in the classical Hamiltonian. This, together with the quantum one-form A_1, defines a general one-form for the whole system,

$$A(\boldsymbol{I},\boldsymbol{J};\boldsymbol{X}_1,\boldsymbol{X}_2)\equiv\langle A_1\rangle_\phi-\dot{\boldsymbol{X}}_2\cdot\left\langle\sum_i P_i\frac{\partial Q_i}{\partial\boldsymbol{X}_2}\right\rangle_\phi dt. \tag{30}$$

By this one-form we can obtain the Berry phases γ_k and the Hannay's angles $\Delta\phi_l$ uniformly for the hybrid system by Eq. (10),

$$\gamma_k = \frac{\partial}{\partial I_k}\oint A, \Delta\phi_l = -\frac{\partial}{\partial J_l}\oint A. \tag{31}$$

To illustrate the result in Eq. (31), we now consider a spin-half particle in an external magnetic field $\boldsymbol{B}$ coupled with a classical harmonic oscillator. This model is widely used since the beginning of quantum mechanics (e.g., see [20]). The Hamiltonian for such a system reads

$$H = \langle\psi|\hat{H}_1|\psi\rangle + (XQ^2 + 2YPQ + ZP^2), \tag{32}$$

where $\hat{H}_1 = -\mu\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{B} - \mu\lambda\hat{\sigma}_3 Q$ is the Hamiltonian of the quantum particle coupled to the magnetic field $\boldsymbol{B} \equiv B(\cos\varphi, \sin\varphi, 0)$ with magnetic moment μ and coupling constant λ, and $\boldsymbol{X} = (X,Y,Z)$ are the time-dependent parameters of the classical subsystems. The quantum state $|\psi\rangle$ is defined by the spin eigenstates $|\pm\rangle$, i.e., $|\psi\rangle = [(q_- + ip_-)|-\rangle + (q_+ + ip_+)|+\rangle]/\sqrt{2\hbar}$. It is easy to obtain the eigenfunctions $|E_\pm\rangle$ and the corresponding eigenvalues for $\hat{H}_1$,

$$\begin{aligned} E_\pm &= \pm\mu B_{tot}, \\ |E_+\rangle &= \cos\tfrac{\Theta}{2}|+\rangle + e^{i\varphi}\sin\tfrac{\Theta}{2}|-\rangle, \\ |E_-\rangle &= -\sin\tfrac{\Theta}{2}|+\rangle + e^{i\varphi}\cos\tfrac{\Theta}{2}|-\rangle, \end{aligned} \tag{33}$$

with $B_{tot} \equiv \sqrt{B^2 + \lambda^2 Q^2}$ and $\cos\Theta \equiv \lambda Q/B_{tot}$. Following the proposed procedures, we transform the quantum Hamiltonian $\hat{H}_1$ into a classical form with a canonical transformation $(q_\pm, p_\pm) \to (\theta_\pm, I_\pm)$, where $|\psi\rangle = \sqrt{I_+/\hbar}e^{-i\theta_+}|E_-\rangle + \sqrt{I_-/\hbar}e^{-i\theta_-}|E_+\rangle$. After averaging over the angles $\theta_\pm$, we obtain

$$\begin{aligned} H_{av} &= \mu B_{tot}(I_+ - I_-) + \frac{A_1(\boldsymbol{I};Q,\boldsymbol{B})}{dt} \\ &\quad + \frac{1}{2}(XQ^2 + 2YPQ + ZP^2), \end{aligned} \tag{34}$$

where

$$A_1(\boldsymbol{I};Q,\boldsymbol{B}) = -\tfrac{1}{2}\left[I_+(1-\cos\Theta) + I_-(1+\cos\Theta)\right]d\varphi \tag{35}$$

is the phase one-form for the quantum subsystem. So far, the Hamiltonian is fully classical and contains only variables of a classical subsystem. It is difficult to derive the action variable for this Hamiltonian, because the dependence of B_{tot} on Q is complicated. However, in the weak coupling limit $\lambda \ll |\frac{B}{Q}|$, the problem becomes easy. By expanding B_{tot} to the second order in λ,

$$B_{tot} \approx B + \frac{\lambda^2 Q^2}{2B}, \tag{36}$$

we obtain the Hamiltonian in the weak coupling limit,

$$\begin{aligned} H_{av} &\approx \mu B(I_+ - I_-) + \frac{\mu\lambda^2 Q^2(I_+ - I_-)}{2B} + \frac{A_1(\boldsymbol{I};Q,\boldsymbol{B})}{dt} \\ &\quad + \frac{1}{2}(XQ^2 + 2YPQ + ZP^2). \end{aligned} \tag{37}$$

According to Berry's theory [7], we introduce the canonical transformation $(Q,P) \to (\phi, J)$

$$\begin{aligned} Q &= \left(\frac{2ZJ}{\Omega}\right)^{1/2}\cos\phi, \\ P &= -\left(\frac{2ZJ}{\Omega}\right)^{1/2}\left(\frac{Y}{Z}\cos\phi + \frac{\Omega}{Z}\sin\phi\right), \end{aligned} \tag{38}$$

with $\Omega \equiv \{[X + \mu\lambda^2(I_+ - I_-)/B]Z - Y^2\}^{1/2}$. By substituting Eq. (38) into Eq. (34) and averaging it over ϕ, we obtain the Hamiltonian for the hybrid system,

$$\langle H_{av}\rangle = \mu B(I_+ - I_-) + J\Omega + \frac{A(\boldsymbol{I},J;\boldsymbol{B},\boldsymbol{X})}{dt}, \tag{39}$$

where the general one-form for the hybrid system is given by $(\langle\cos\Theta\rangle_\phi = 0)$,

$$A(\boldsymbol{I},J;\boldsymbol{B},\boldsymbol{X}) = -\frac{(I_+ + I_-)}{2}d\varphi - \frac{YJ}{2Z}d\left(\frac{Z}{\Omega}\right). \tag{40}$$

Thus the Berry phases and Hannay's angle can be given uniformly by a straightforward calculation

$$\begin{aligned} \gamma_\pm &= \oint\left[-\frac{1}{2}d\varphi \mp \frac{\mu\lambda^2 Z^2 J}{4\Omega^3 B}d\left(\frac{Y}{Z}\right)\right], \\ \Delta\phi &= \oint\frac{Y}{2Z}d\left(\frac{Z}{\Omega}\right). \end{aligned} \tag{41}$$

The range of integration is determined by the common period of $\boldsymbol{X}$ and $\boldsymbol{B}$. It is easy to find that when $\lambda = 0$, the Berry phases are the solid angle π on a Bloch sphere. The subsystem-subsystem couplings not only add a correction to the Berry phases but also change the range of integration, and the interaction gives the Hannay's angle a modification $\omega \equiv (XZ - Y^2)^{1/2} \to \Omega$ through Ω.

The second example is a quantum harmonic oscillator coupled with a classical one. The mean-field Hamiltonian [19] for this hybrid system is

$$\hat{H} = \langle\psi|\hat{H}_1|\psi\rangle + \tfrac{1}{2}(X_2Q^2 + 2Y_2PQ + Z_2P^2), \tag{42}$$

where $\hat{H}_1 = \frac{1}{2}[X_1\hat{q}^2 + Y_1(\hat{p}\hat{q} + \hat{q}\hat{p}) + Z_1\hat{p}^2] + K\hat{q}Q$ includes the free Hamiltonian of the quantum subsystem and the subsystem-subsystem coupling, K is the coupling constant, and $\boldsymbol{X}_1 = (X_1,Y_1,Z_1)$, $\boldsymbol{X}_2 = (X_2,Y_2,Z_2)$ are the time-dependent parameters of the two subsystems. Similar to Ref. [7], the eigenfunctions and eigenvalues of $\hat{H}_1$ are

$$\begin{aligned} E_n &= \left(n + \frac{1}{2}\right)\hbar\omega - \frac{Z_1K^2Q^2}{2\omega^2}, \\ \psi_n &= \sqrt{\alpha}\chi_n\left[\alpha\left(q + \frac{KZ_1Q}{\omega^2}\right)\right]\exp\left(\frac{-iY_1q^2}{2Z_1\hbar}\right), \end{aligned} \tag{43}$$

with $\omega = \sqrt{X_1Z_1 - Y_1^2}$, $\alpha = \sqrt{\frac{\omega}{Z_1\hbar}}$, and the normalized Hermite functions $\chi_n(\xi)$. By the transformation $(\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{\theta},\boldsymbol{I})$ we obtain the averaged Hamiltonian by averaging over the angles $\boldsymbol{\theta}$,

$$H_{av} = \sum_n (n+1/2) I_n \omega - \frac{Z_1 K^2 Q^2}{2\omega^2} + \frac{A_1(\boldsymbol{I};\boldsymbol{X_1},Q)}{dt} + \frac{1}{2}(X_2 Q^2 + 2Y_2 PQ + Z_2 P^2), \quad (44)$$

where

$$A_1(\boldsymbol{I};\boldsymbol{X}_1,Q) = \sum_n I_n \left[\frac{(2n+1)Z_1}{4\omega} + \frac{K^2 Z_1^2 Q^2}{2\hbar\omega^4}\right] d\left(\frac{Y_1}{Z_1}\right) \quad (45)$$

is the phase one-form for the quantum subsystem. Note that the action variables $\boldsymbol{I}$ are invariant and we can treat them as constants. After the canonical transformation $(Q,P) \to (\phi, J)$ (elliptic case),

$$Q = \left(\frac{2Z_2 J}{\Omega}\right)^{1/2} \cos\phi, \quad P = -\left(\frac{2Z_2 J}{\Omega}\right)^{1/2} \left(\frac{Y_2}{Z_2}\cos\phi + \frac{\Omega}{Z_2}\sin\phi\right), \quad (46)$$

and substituting Eq. (46) into Eq. (44) as well as averaging over $\boldsymbol{\phi}$, the Hamiltonian of the hybrid system becomes

$$\langle H_{av}\rangle = \sum_n \left(n+\frac{1}{2}\right) I_n\omega + J\Omega + \frac{A(\boldsymbol{I},J;\boldsymbol{X_1},\boldsymbol{X_2})}{dt} \quad (47)$$

with $\Omega = [\frac{(\omega^2 X_2 - Z_1 K^2)Z_2}{\omega^2} - Y_2^2]^{1/2}$. The general one-form for the hybrid system is then

$$A(\boldsymbol{I},J;\boldsymbol{X_1},\boldsymbol{X_2}) = \sum_n I_n \left[\frac{(2n+1)Z_1}{4\omega} + \frac{K^2 Z_1^2 Z_2 J}{2\hbar\omega^4\Omega}\right] \times d\left(\frac{Y_1}{Z_1}\right) - \frac{Y_2 J}{2Z_2} d\left(\frac{Z_2}{\Omega}\right). \quad (48)$$

Therefore the Berry phases and Hannay's angle can be given by

$$\gamma_n = \oint \left[\frac{(2n+1)Z_1}{4\omega} + \frac{K^2 Z_1^2 Z_2 J}{2\hbar\omega^4\Omega}\right] d\left(\frac{Y_1}{Z_1}\right), \quad \Delta\phi = \oint \left[\frac{Y_2}{2Z_2} d\left(\frac{Z_2}{\Omega}\right) - \frac{K^2 Z_1^2 Z_2}{2\omega^4\Omega} d\left(\frac{Y_1}{Z_1}\right)\right]. \quad (49)$$

The limits of the integrals are decided by the common period of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

Now we take a specific choice of the periodic parameters to see an exact result of our theory. Set

$$\begin{cases} X_1 = A_1\mu_1[1+\epsilon\cos(\omega_1 t)] \\ Y_1 = -A_1\epsilon\sin(\omega_1 t) \\ Z_1 = \frac{A_1}{\mu_1}[1-\epsilon\cos(\omega_1 t)] \end{cases}, \quad \begin{cases} X_2 = A_2\mu_2[1+\epsilon\cos(\omega_2 t)] \\ Y_2 = -A_2\epsilon\sin(\omega_2 t) \\ Z_2 = \frac{A_2}{\mu_2}[1-\epsilon\cos(\omega_2 t)] \end{cases}, \quad (50)$$

where ω_1, ω_2 are the frequencies of the parameters, ϵ is a dimensionless constant, and the units of A and μ are s^{-1} and kg/s, respectively. If the frequency ratio ω_1/ω_2 is rational,

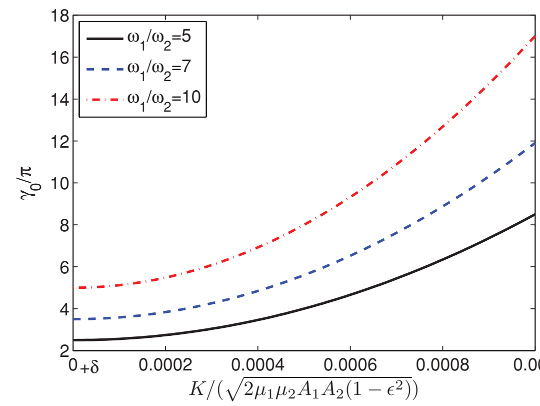

FIG. 1. (Color online) The Berry phase of the ground state γ_0 as a function of coupling constant K. The parameters are $\epsilon = \sqrt{3}/2$, $A_1/A_2 = 10^8$, and $J/\hbar = 10^{13}$, and ω_1/ω_2 takes different values for different lines.

$\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ have a common period T. After a straightforward calculation, we obtain the phases and angle in Eq. (49) as

$$\gamma_n = \gamma_{n0} + \gamma_I, \quad \Delta\phi = \Delta\phi_0 + \Delta\phi_I. \quad (51)$$

The noninteracting phases γ_{n0} and angle $\Delta\phi_0$ are

$$\gamma_{n0} = \frac{(2n+1)(1-\sqrt{1-\epsilon^2})T\omega_1}{4\sqrt{1-\epsilon^2}}, \quad \Delta\phi_0 = -\int_0^T \left[\frac{\omega_2\epsilon^2\sin^2(\omega_2 t)dt}{2\Omega[1-\epsilon\cos(\omega_2 t)]} + \frac{\epsilon A_2 \sin(\omega_2 t) d\Omega}{2\Omega^2}\right], \quad (52)$$

and the coupling terms follow

$$\gamma_I = \int_0^T \frac{-\epsilon\omega_1 A_2 D^2 J[1-\epsilon\cos(\omega_2 t)][\epsilon - \cos(\omega_1 t)]dt}{\hbar A_1(1-\epsilon^2)\Omega} = -\frac{J}{\hbar}\Delta\phi_I, \quad (53)$$

with the definition $D \equiv K/[\sqrt{2\mu_1\mu_2 A_1 A_2(1-\epsilon^2)}]$ and $\Omega = A_2\sqrt{1-\epsilon^2 - 2D^2[1-\epsilon\cos(\omega_1 t)][1-\epsilon\cos(\omega_2 t)]}$. If we take the limit $D \ll \sqrt{1-\epsilon^2}$, $\Omega \approx A_2\sqrt{1-\epsilon^2}$ can be treated as a constant, and γ_{n0} and $\Delta\phi_0$ then satisfy the relation

$$\gamma_{n0} \approx -\left(n+\frac{1}{2}\right)\frac{\omega_1\Delta\phi_0}{\omega_2}. \quad (54)$$

The coupling Berry phase and Hannay's angle can then be approximated as

$$\gamma_I \approx \frac{\epsilon^2 A_2 J D^2 T\omega_1}{\hbar A_1(1-\epsilon^2)\sqrt{1-\epsilon^2}}, \quad \Delta\phi_I \approx -\frac{\epsilon^2 A_2 D^2 T\omega_1}{A_1(1-\epsilon^2)\sqrt{1-\epsilon^2}}. \quad (55)$$

Considering the elliptic condition $[1-\epsilon^2 - 2D^2(1+\epsilon)^2 > 0]$, we perform numerical calculation for Eqs. (52) and (53) with $\epsilon = \sqrt{3}/2$, $A_1/A_2 = 10^8$, and $J/\hbar = 10^{13}$. The results are presented in Figs. 1 and 2.

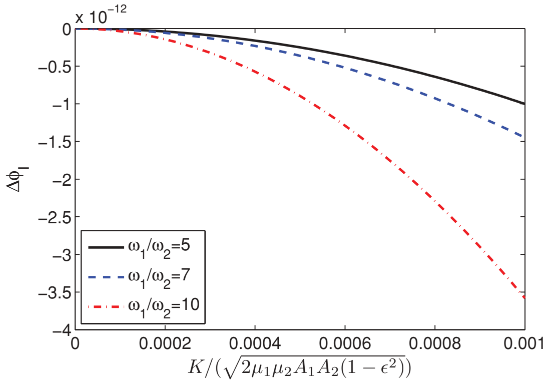

FIG. 2. (Color online) The coupling Hannay's angle $\Delta\phi_I$ as a function of coupling constant K, with the same parameters as Fig. 1.

Figure 1 shows how the Berry phase of the ground state changes with K. The original $0+\delta$ on the horizontal axis denotes an infinitely small positive number. Because when $K=0$ there is no interaction between the two subsystems, the range of integration of the phases and the angle is determined by the period of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively, rather than their common period. The relation between Berry phases and Hannay's angle $\gamma_{n0}=\frac{(2n+1)T\omega_1}{4}\approx-(n+\frac{1}{2})\frac{\omega_1\Delta\phi_0}{\omega_2}$ agrees with the prediction in Ref. [7], $\gamma_n=(n+\frac{1}{2})\pi=-(n+\frac{1}{2})\Delta\phi$. From the figure we can see that the coupling constant K has a remarkable influence on the Berry phase, and as the coupling increases, the influence becomes larger. This means that the classical subsystem plays the role of driving field for the quantum subsystem. It can also be seen from Fig. 1 that the frequency ratio of the parameters ω_1/ω_2 can change both Berry phase γ_I and γ_{n0}, since it determines the common period T of the parameters $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. In contrast, the Hannay angle $\Delta\phi_I$ is much smaller than $\Delta\phi_0\approx-\pi T\omega_2$, as Fig. 2 shows. Therefore we conclude that not only can a classical system drive a quantum system to obtain an extra Berry phase, but also the quantum system can react back on the classic system and induce a correction to Hannay's angle. This is because the quantum system possesses the same structure as the classical system, and the quantum parallel transport can be treated as a parallel transport in the effective classical system. The quantum one-form may also affect the classical subsystem due to the interaction.

It is worth addressing that this quantum-classical hybrid model is different from the full quantum coupled quantum generalized harmonic oscillator model, since they possess different structures. For a full quantum model, the Hamiltonian can be written as

$$\hat{H}=\tfrac{1}{2}[X_1\hat{q}^2+Y_1(\hat{p}\hat{q}+\hat{q}\hat{p})+Z_1\hat{p}^2]+K\hat{q}\hat{Q} +\tfrac{1}{2}[X_2\hat{Q}^2+Y_2(\hat{P}\hat{Q}+\hat{Q}\hat{P})+Z_2\hat{P}^2]. \quad (56)$$

After a canonical transformation for Q and q by

$$\begin{pmatrix} R_1\\ R_2\end{pmatrix}=\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix} q/\sqrt{Z_1}\\ Q/\sqrt{Z_2}\end{pmatrix}, \quad (57)$$

with

$$\sin\beta\equiv\left[\frac{\omega_2^2-\omega_1^2+\sqrt{(\omega_1^2-\omega_2^2)^2+4K^2Z_1Z_2}}{2\sqrt{(\omega_1^2-\omega_2^2)^2+4K^2Z_1Z_2}}\right]^{1/2}, \quad \cos\beta=\left[\frac{\omega_1^2-\omega_2^2+\sqrt{(\omega_1^2-\omega_2^2)^2+4K^2Z_1Z_2}}{2\sqrt{(\omega_1^2-\omega_2^2)^2+4K^2Z_1Z_2}}\right]^{1/2}, \quad (58)$$

the frequencies of the two oscillators $\omega_1\equiv(X_1Z_1-Y_1^2)^{1/2}$ and $\omega_2\equiv(X_2Y_2-Y_2^2)^{1/2}$ change into

$$\Omega_1\equiv\sqrt{\frac{\omega_1^2+\omega_2^2+\sqrt{(\omega_1^2-\omega_2^2)^2+4K^2Z_1Z_2}}{2}}, \quad \Omega_2\equiv\sqrt{\frac{\omega_1^2+\omega_2^2-\sqrt{(\omega_1^2-\omega_2^2)^2+4K^2Z_1Z_2}}{2}}. \quad (59)$$

The eigenfunctions of $\hat{H}$ can be written as the product of the eigenfunctions of two effective harmonic oscillators,

$$\Psi_{mn}(R_1,R_2;\boldsymbol{X}_1,\boldsymbol{X}_2)=\varphi_{1m}(R_1;\boldsymbol{X}_1,\boldsymbol{X}_2)\varphi_{2n}(R_2;\boldsymbol{X}_1,\boldsymbol{X}_2), \quad (60)$$

where

$$\varphi_{kn}(R_k;\boldsymbol{X}_1,\boldsymbol{X}_2)=\left(\frac{\Omega_k}{\hbar}\right)^{1/4}\chi_n\left(R_k\sqrt{\frac{\Omega_k}{\hbar}}\right)\exp\left(-\frac{iY_1q^2}{2Z_1\hbar}-\frac{iY_2Q^2}{2Z_2\hbar}\right), \quad (61)$$

and χ_n are the normalized Hermite functions. Inserting Eq. (60) into Eqs. (9) and (10), we obtain the one-form and the phase

$$A_{mn}^B=\frac{-iZ_1}{4}\left[\frac{(2m+1)\cos^2\beta}{\Omega_1}+\frac{(2n+1)\sin^2\beta}{\Omega_2}\right]d\left(\frac{Y_1}{Z_1}\right) -\frac{iZ_2}{4}\left[\frac{(2m+1)\sin^2\beta}{\Omega_1}+\frac{(2n+1)\cos^2\beta}{\Omega_2}\right]d\left(\frac{Y_2}{Z_2}\right), \quad (62)$$

$$\begin{aligned}\gamma_{mn}&=i\oint A_{mn}^B\\ &=\oint\left\{\frac{Z_1}{4}\left[\frac{(2m+1)\cos^2\beta}{\Omega_1}+\frac{(2n+1)\sin^2\beta}{\Omega_2}\right]d\left(\frac{Y_1}{Z_1}\right)\right.\\ &\quad\left.+\frac{Z_2}{4}\left[\frac{(2m+1)\sin^2\beta}{\Omega_1}+\frac{(2n+1)\cos^2\beta}{\Omega_2}\right]d\left(\frac{Y_2}{Z_2}\right)\right\}.\end{aligned} \quad (63)$$

It is easy to find that when there is no interaction between the two oscillators ($K=0$), the Berry phase in Eq. (63) returns to a sum of the Berry phases of two generalized harmonic oscillators [7],

$$\gamma_{mn}=\oint\frac{(2m+1)Z_1}{4\omega_1}d\left(\frac{Y_1}{Z_1}\right)+\oint\frac{(2n+1)Z_2}{4\omega_2}d\left(\frac{Y_2}{Z_2}\right). \quad (64)$$

If $\hat{q}$ denotes the coordinate of a light particle and $\hat{Q}$ the coordinate of a heavy one, we can take the Born-Oppenheimer approximation, treating the heavy particle coordinate $\hat{Q}$ as a parameter for the light particle. The Hamiltonian for the light particle then takes

$$\hat{H}_1 = \frac{1}{2}[X_1\hat{q}^2 + Y_1(\hat{p}\hat{q} + \hat{q}\hat{p}) + Z_1\hat{p}^2] + K\hat{q}Q. \quad (65)$$

Its eigenfunctions $\psi_n(q;Q,\boldsymbol{X}_1)$ and eigenvalues $E_n(Q,\boldsymbol{X}_1)$ are given by Eq. (43),

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{Z_1K^2Q^2}{2\omega^2}, \quad \psi_n = \sqrt{\alpha}\chi_n\left[\alpha\left(q + \frac{KZ_1Q}{\omega^2}\right)\right]\exp\left(\frac{-iY_1q^2}{2Z_1\hbar}\right), \quad (66)$$

with $\omega = \sqrt{X_1Z_1 - Y_1^2}$ and $\alpha = \sqrt{\frac{\omega}{Z_1\hbar}}$. Note that $E_n(Q)$ enters into the Hamiltonian for the heavy particle as a potential, and the Hamiltonian for the heavy particle then takes,

$$H_n^{eff} = \frac{1}{2}[X_2\hat{Q}^2 + Y_2(\hat{P}\hat{Q} + \hat{Q}\hat{P}) + Z_2\hat{P}^2] + E_n(Q,\boldsymbol{X}_1). \quad (67)$$

The eigenvalues and eigenfunctions of the effective Hamiltonian can be calculated straightforwardly,

$$E_{mn}^{eff}(\boldsymbol{X}_1,\boldsymbol{X}_2) = \left(m + \tfrac{1}{2}\right)\hbar\Omega + \left(n + \tfrac{1}{2}\right)\hbar\omega, \quad \varphi_m(Q;\boldsymbol{X}_1,\boldsymbol{X}_2) = \sqrt{\alpha'}\chi_m(\alpha' Q)\exp\left(\frac{-iY_2Q^2}{2Z_2\hbar}\right), \quad (68)$$

where the effective frequency is defined by $\Omega = [\frac{(\omega^2X_2 - Z_1K^2)Z_2}{\omega^2} - Y_2^2]^{1/2}$ and $\alpha' = \sqrt{\frac{\Omega}{Z_2\hbar}}$ $E_{mn}^{eff}(\boldsymbol{X}_1,\boldsymbol{X}_2)$, and $(m,n = 1,2,3,\ldots)$ are the eigenvalues of the total Hamiltonian. The corresponding eigenvectors are

$$\Psi_{mn}^{tot}(q,Q;\boldsymbol{X}_1,\boldsymbol{X}_2) \approx \varphi_m(Q;\boldsymbol{X}_1,\boldsymbol{X}_2)\psi_n(q;Q,\boldsymbol{X}_1). \quad (69)$$

Therefore the Berry phase of the total system can be calculated by Eqs. (9) and (10) as

$$\begin{aligned}\gamma_{mn} &= \int\int dq\,dQ \oint \psi_n^*(q;Q,\boldsymbol{X}_1)\varphi_m^*(Q;\boldsymbol{X}_1,\boldsymbol{X}_2) \\ &\quad\times d_X[\varphi_m(Q;\boldsymbol{X}_1,\boldsymbol{X}_2)\psi_n(q;Q,\boldsymbol{X}_1)] \\ &= \oint\left\{\left[\frac{(2n+1)Z_1}{4\omega} + \frac{(2m+1)K^2Z_1^2Z_2}{4\omega^4\Omega}\right]d\left(\frac{Y_1}{Z_1}\right) + \left[\frac{(2m+1)Z_2}{4\Omega}\right]d\left(\frac{Y_2}{Z_2}\right)\right\}.\end{aligned} \quad (70)$$

Interestingly, if we take the Bohr-Sommerfeld quantization rule $J = (m + 1/2)\hbar$ [7,21] into account and notice $\oint d(Y_2/\Omega) = 0$, the contribution of γ_{mn} to the Berry phase for the light particle [the first two term in Eq. (70)] exactly matches the Berry phase for our quantum-classical hybrid oscillator in Eq. (49), and the contribution to the Berry phase for the heavy particle [the second two term in Eq. (70)] satisfies $\Delta\phi = -\partial\gamma_{mn}/\partial m$ (see [7]), where Hannay's angle takes Eq. (49). This is exactly the relation between the Berry phase and Hannay's angle. The description for the quantum and classical system is different in physics. To treat them uniformly, we apply Weinberg's theory and express the quantum subsystem classically. Since the classical particle is much heavier than the quantum particle, the Born-Oppenheimer approximation turns out to be a good approximation for this problem; the predictions made in this paper are reasonable.

IV. CONCLUSION

The Berry phase and Hannay's angle in coupled quantum-classical hybrid systems have been studied in this paper. To uniformly calculate the Berry phase and Hannay's angle, we introduced a one-form connection, by which we obtain both the Berry phase and Hannay's angle for the hybrid system. In this sense, the Berry phase and Hannay's angle in the quantum and classical subsystem can be treated uniformly. To illustrate the formalism, we give two examples. The first example is a spin-half particle coupled to a classical oscillator. In the second example, we calculated the Berry phase and the Hannay's angle for two coupled oscillators, one of which is quantum while another is classical. The effects of subsystem-subsystem coupling on the phase and angle are given and discussed. The results show that the classical subsystem provides the quantum subsystem a large correction to the Berry phase, while the quantum subsystem gives the classical Hannay angle a small perturbation. These predictions depend on the features of the quantum-classical hybrid system and their mutual interactions. We also found that the frequency ratio affects the phases and the angle, since it can control the evolution periods of the quantum and classical subsystems. Finally, we have calculated the Berry phase for a fully quantum version of the two coupling systems.

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