More stringent confinement limit of the Dirac particle in one dimension

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Unanyan, Otterbach, and Fleischhauer [Phys. Rev. A **79**, 044101 (2009)] found that the confinement limit of a one-dimensional Dirac particle in a symmetric potential is half its corresponding Compton length and can be derived from the Dirac equation. Then, Toyama and Nogami [Phys. Rev. A **81**, 044106 (2010)] conjectured that a more stringent limit holds for any symmetric potential. I would like to show, in this Brief Report, that the confinement limit conjectured by Toyama and Nogami can be derived from the Dirac equation.

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Recent interest in the coordinate uncertainty of Dirac particles has been stimulated in the paper by Unanyan, Otterbach, and Fleischhauer [1], where it was shown that for any Dirac particle with mass m in a one-dimensional symmetric scalar potential, the lower confinement limit is strictly larger than half its corresponding Compton length $\lambda_C = \hbar/(mc)$. The conclusion that Unanyan, Otterbach, and Fleischhauer draw from the Dirac equation does not depend on the Heisenberg uncertainty relation and the explicit form of the scalar potential.

Let us briefly review the derivation by Unanyan, Otterbach, and Fleischhauer in order to follow the discussion. The time-independent one-dimensional Dirac equation for a particle in a scalar potential U is

$$(c\alpha\hat{p} + \beta mc^2 + U)\psi(z) = E\psi(z), \tag{1}$$

where two 2×2 Dirac matrices may be chosen as $\alpha = \sigma_y$ and $\beta = \sigma_z$, then Eq. (1) becomes real and has the following form:

$$-c\hbar\frac{d\psi_2}{dz} + mc^2\psi_1 = (E - U)\psi_1, \tag{2}$$

$$c\hbar\frac{d\psi_1}{dz} - mc^2\psi_2 = (E - U)\psi_2, \tag{3}$$

where $(\psi_1,\psi_2)^T = \psi$. Two components ψ_1 and ψ_2 can be chosen as real functions. Equations (2) and (3) lead to the identity

$$4\psi_1\psi_2 = \lambda_C\frac{d\rho}{dz}, \tag{4}$$

where $\rho = \psi_1^2 + \psi_2^2$ is the normalized density distribution. Using integration by parts, Eq. (4) gives

$$2\int_{-\infty}^{+\infty} z\psi_1\psi_2\, dz = \left[\frac{\lambda_C z\rho}{2}\right]_{-\infty}^{+\infty} - \frac{\lambda_C}{2}. \tag{5}$$

When the Dirac particle is in an appropriate bound state, the boundary term on the right-hand side vanishes. From (5), and using the arithmetic mean inequality $\psi_1^2 + \psi_2^2 \geqslant 2|\psi_1\psi_2|$ and the Cauchy-Schwarz inequality $\langle z^2\rangle = \langle |z|^2\rangle \geqslant \langle |z|\rangle^2$, it follows that $\sqrt{\langle z^2\rangle} \geqslant \lambda_C/2$. This gives the lower limit of coordinate uncertainty:

$$\Delta z = \sqrt{\langle z^2\rangle - \langle z\rangle^2} \geqslant \frac{\lambda_C}{2}. \tag{6}$$

Here, $\langle z\rangle = 0$ since the scalar potential U is symmetric; the density ρ is also symmetric.

The limiting case $\Delta z = \lambda_C/2$ has been studied by Toyama and Nogami [2]. They pointed out that this equality holds for $\psi_1 = (z/|z|)\psi_2$ or $\psi_1 = -(z/|z|)\psi_2$. Furthermore, based on the analysis of point interaction and finite-ranged potentials, Toyama and Nogami conjectured that a more stringent confinement limit $\Delta z \geqslant \lambda_C/\sqrt{2}$ holds for any symmetric potential. In this Brief Report, we find that the confinement limit $\Delta z \geqslant \lambda_C/\sqrt{2}$ can also be derived from the Dirac equation.

By using the inequality

$$\rho \geqslant -2\psi_1\psi_2 \tag{7}$$

and Eq. (4), it is easy to see that

$$\int_0^{+\infty} z^2\rho\, dz \geqslant -2\int_0^{+\infty} z^2\psi_1\psi_2\, dz = -\left[\frac{\lambda_C z^2\rho}{2}\right]_0^{+\infty} + \lambda_C\int_0^{+\infty} z\rho\, dz. \tag{8}$$

By discarding the boundary term and using Eqs. (7) and (5), we arrive at the confinement limit

$$\Delta z \geqslant \frac{\lambda_C}{\sqrt{2}}. \tag{9}$$

Here we have used the identity

$$\int_{-\infty}^{0} z^2\rho\, dz = \int_0^{+\infty} z^2\rho\, dz \tag{10}$$

since the density ρ is symmetric.

It is easy to check that the limiting case $\Delta z = \lambda_C/\sqrt{2}$ holds for

$$\psi_1 = -(z/|z|)\psi_2. \tag{11}$$

A direct calculation indicates that it would be impossible to satisfy the limiting case if $\psi_1 = (z/|z|)\psi_2$. To see this, assuming that $\Delta z = \lambda_C/\sqrt{2}$ holds for $\psi_1 = (z/|z|)\psi_2$, we obtain, after using Eq. (4) and integrating by parts,

$$\int_0^{+\infty} z^2\rho\, dz = -\lambda_C\int_0^{+\infty} z\rho\, dz, \tag{12}$$

which is incorrect.

In conclusion, we investigated a more stringent confinement limit of a one-dimensional Dirac particle in a symmetric potential. We have shown that the confinement limit conjectured by Toyama and Nogami is correct and can be derived from the Dirac equation. A calculation shows explicitly that the limiting case holds only when the identity Eq. (11) is satisfied.

[1] R. G. Unanyan, J. Otterbach, and M. Fleischhauer, Phys. Rev. A **79**, 044101 (2009).

[2] F. M. Toyama and Y. Nogami, Phys. Rev. A **81**, 044106 (2010).