Birefringent breakup of Dirac fermions on a square optical lattice

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(Received 5 November 2010; revised manuscript received 16 March 2011; published 31 May 2011)

We introduce a lattice model for fermions in a spatially periodic magnetic field that also has spatially periodic hopping amplitudes. We discuss how this model might be realized with cold atoms in an artificial magnetic field on a square optical lattice. When there is an average flux of half a flux quantum per plaquette, the spectrum of low-energy excitations can be described by massless Dirac fermions in which the usually doubly degenerate Dirac cones split into cones with different "speeds of light." These gapless birefringent Dirac fermions arise because of broken chiral symmetry in the kinetic energy term of the effective low-energy Hamiltonian. We characterize the effects of various perturbations to the low-energy spectrum, including staggered potentials, interactions, and domain-wall topological defects.

DOI: 10.1103/PhysRevA.83.053636 PACS number(s): 67.85.−d, 05.30.Fk, 37.10.Jk, 71.10.Pm

I. INTRODUCTION

The discovery of graphene [1] and topological insulators [2] has led to much recent interest in systems whose low-energy excitations can be described with Dirac fermions. In parallel there has been exploration of the possibility of generating artificial magnetic fields for cold atoms confined in an optical lattice. Neutral bosonic cold atoms cannot couple to a magnetic field directly, so there have been numerous proposals [3,4] of approaches to couple atoms to an artificial gauge field, several of which have been implemented experimentally [5].

Quantum particles in a uniform magnetic field on a lattice have the well-known Hofstadter spectrum [6]. We study a Hamiltonian with a tunable Hofstadter-like spectrum that arises from the combination of hopping and an artificial magnetic field with a nonzero mean that are both periodically modulated in the x *and* the y directions. The presence of spatial periodicity in the *amplitude* as well as the phase of the hopping is the key difference between the model we consider here and previous work on the spectrum of particles in the presence of magnetic fields that are periodic in both the x and y directions [7]. This difference facilitates the unusual Dirac-like spectrum that we discuss here.

Our main result is that in the lattice model we introduce [Eq. (3)], when there is an average of half a flux quantum per plaquette, and at half filling, the low-energy degrees of freedom can be described by a Dirac Hamiltonian with the unusual property that chiral symmetry is broken in the kinetic energy rather than via mass terms. This has the consequence that the doubly degenerate Dirac cone for massless fermions splits into two cones with tunable distinct slopes, analogous to a situation in which there are two speeds of light for fermionic excitations, similar to birefringence of light in crystals such as calcite. We discuss the meaning of broken chiral symmetry in our effective model and explore the effects of various perturbations, such as staggered potentials, domain walls, and interactions between fermions.

The paper is structured as follows: in Sec. II we discuss a possible scheme to realize the Hamiltonian Eq. (3), illustrate its Hofstadter-like spectrum, and demonstrate that there are Dirac points in the spectrum when there is an average of half a flux quantum per plaquette. In Sec. III we discuss the low-energy theory in the vicinity of the Dirac points, and we give brief conclusions and discussion in Sec. IV.

II. EFFECTIVE HAMILTONIAN

We use a generalization of the approach introduced by Sørensen *et al.* [4] to obtain an artificial magnetic field. However, our results regarding the spectrum of the model are independent of any particular experimental scheme used to realize the model. The scheme in Ref. [4] was presented for bosons but applies equally well to a starting point of spinless fermions (corresponding to only one available hyperfine state for cold atoms) with Hamiltonian

$$H = -\mathcal{J}\sum_{\langle i,j\rangle}(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i), \tag{1}$$

where $\hat{c}_i^\dagger$ and $\hat{c}_i$ are fermionic creation and annihilation operators respectively at site i and the notation $\langle i,j\rangle$ indicates that we restrict the sum in the hopping term to nearest neighbors only. There can be no Hubbard-like interaction for spinless fermions, and since nearest-neighbor interactions in an optical lattice system are weak, we initially ignore interactions.

In Ref. [4], two steps are required to generate an artificial magnetic field. First, a time-varying quadrupolar potential $V(t) = V_{qp}\sin(\omega t)\hat{x}\hat{y}$ is applied to the system, and second, the hopping is modulated as a function of time. During the course of one oscillation of the quadrupolar potential, hopping in the x direction is turned on for a very short period of time $\tau \ll t_0 = \frac{2\pi}{\omega}$ at times $t = nt_0$, where n is an integer, and hopping in the y direction is turned on for time τ around $t = (n+\frac{1}{2})t_0$. Due to the periodic oscillation in the Hamiltonian, the time evolution operator after m periods may be written as $U(t = mt_0) = U(t = t_0)^m$.

Our modification to the proposal in Ref. [4] is that when hopping is turned on in the x direction at time $t = nt_0$, hopping is also turned on in the y direction with an amplitude $0 \leqslant \beta \leqslant 1$ relative to the hopping in the x direction. At time $t = (n+\frac{1}{2})t_0$, hopping is turned on in the y direction, and hopping in the x direction is turned on with amplitude β relative to the hopping in the y direction as illustrated in Fig. 1. The operator

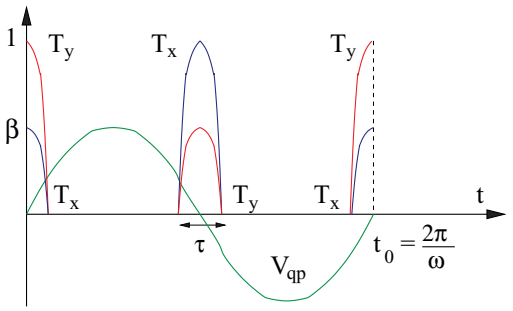

FIG. 1. (Color online) Time dependence of the hopping and the quadrupolar potential during the course of one period of the quadrupolar potential.

for hopping in the x direction is $\hat{T}_x = -\mathcal{J}\sum_{x,y}(\hat{c}^\dagger_{x+1,y}\hat{c}_{x,y} +$ H.c.) with a similar expression for $\hat{T}_y$. We may write the time evolution operator as

$$U(t = mt_0) = [e^{-(i\tau/2\hbar)(\beta\hat{T}_x+\hat{T}_y)}e^{2\pi i\alpha\hat{x}\hat{y}}e^{-(i\tau/\hbar)(\hat{T}_x+\beta\hat{T}_y)} \times e^{-2\pi i\alpha\hat{x}\hat{y}}e^{-(i\tau/2\hbar)(\beta\hat{T}_x+\hat{T}_y)}]^m, \quad (2)$$

where $\alpha = V_{qp}/\pi\hbar\omega$ and we have set the lattice constant to unity. To lowest order in $\mathcal{J}\tau/\hbar$ we can write this in the form

$$U = e^{-iH_{\rm eff}t/\hbar},$$

$$H_{\rm eff} = -J_0\sum_{x,y}\{[(1+\beta e^{2\pi i\alpha x})\hat{c}^\dagger_{x,y+1}\hat{c}_{x,y} + \text{H.c.}] + [(\beta + e^{2\pi i\alpha y})\hat{c}^\dagger_{x+1,y}\hat{c}_{x,y} + \text{H.c.}]\}, \quad (3)$$

with $J_0 = \tau\mathcal{J}/t_0$.

A. Hofstadter-like spectrum

A more conventional way to write this Hamiltonian is in the form

$$H_{\rm eff} = -\sum_{ij}\left[t_{ij}e^{(ie/\hbar)\int_j^i \mathbf{A}\cdot d\mathbf{l}}\hat{c}^\dagger_i\hat{c}_j + \text{H.c.}\right], \quad (4)$$

from which we may identify the amplitude of the hopping

$$t_{x+1,y} = J_0\sqrt{1+\beta^2+2\beta\cos(2\pi\alpha y)}, \quad (5)$$

$$t_{x,y+1} = J_0\sqrt{1+\beta^2+2\beta\cos(2\pi\alpha x)}, \quad (6)$$

and the artificial magnetic field

$$B_z = \frac{2\pi\alpha\hbar}{e}\left\{\frac{\beta^2+\beta\cos(2\pi\alpha x)}{1+\beta^2+2\beta\cos(2\pi\alpha x)} - \frac{1+\beta\cos(2\pi\alpha y)}{1+\beta^2+2\beta\cos(2\pi\alpha y)}\right\}. \quad (7)$$

This field is the sum of a spatially uniform piece with magnitude $\frac{2\pi\hbar\alpha}{e}$ and a piece that is spatially periodic in both the x and y directions. If $\beta = 0$, the hopping amplitude is J_0 and the field is uniform with strength $\frac{2\pi\hbar\alpha}{e}$, corresponding to a flux of $\alpha\phi_0$ per plaquette (where ϕ_0 is the flux quantum) as found in Ref. [4], and there is a Hofstadter spectrum. If $\beta = 1$, then $B_z = 0$, but the hopping parameters are still spatially periodic. At β intermediate between 0 and 1, both the hopping and the magnetic field are spatially periodic in x and y. This illustrates the essential difference between the model we consider and previous work on quantum particles in a periodic magnetic field on a lattice—there is spatial periodicity of $1/\alpha$ in the amplitude of the hopping as well as in the magnetic field. For finite β the spectrum (illustrated for $\beta = 1$ in Fig. 2) as a function of α is reminiscent of the Hofstadter spectrum.

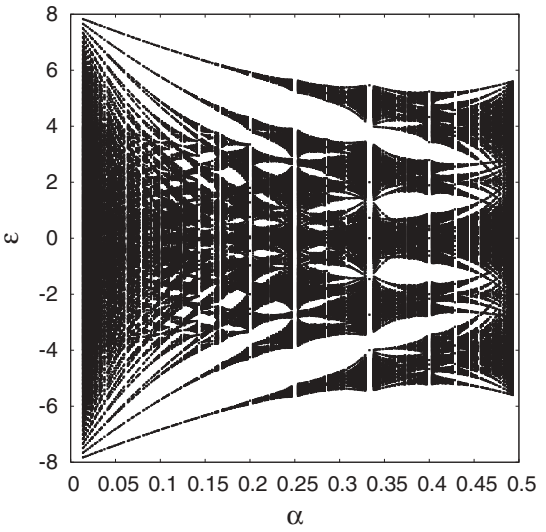

FIG. 2. Energy spectrum (in units of J_0) as a function of α when $\beta = 1$: there is no artificial magnetic field, yet due to the periodic hopping, the spectrum has some similarities with the Hofstadter spectrum.

B. Half a flux quantum per plaquette

When $\alpha = 1/2$ there is an average of half a flux quantum per plaquette and $H_{\rm eff}$ simplifies to a tight-binding model with four sites in the unit cell as shown in Fig. 3(a).

Labeling the four sites in the unit cell as A, B, C, and D, and Fourier transforming in space, we may rewrite the effective Hamiltonian in the following form:

$$H = \sum_{\mathbf{k}}\psi^\dagger_k[E_k - \mathcal{H}_k]\psi_k, \quad (8)$$

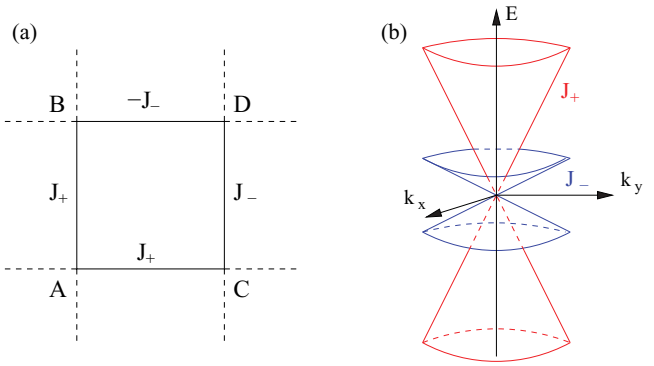

FIG. 3. (Color online) (a) Unit cell of tight-binding model with hopping parameters indicated. (b) Dirac cones corresponding to J_+ and J_- bands.

with

$$\mathcal{H}_k = 2\begin{pmatrix} 0 & J_+\cos k_y & J_+\cos k_x & 0 \\ J_+\cos k_y & 0 & 0 & -J_-\cos k_x \\ J_+\cos k_x & 0 & 0 & J_-\cos k_y \\ 0 & -J_-\cos k_x & J_-\cos k_y & 0 \end{pmatrix},$$

where $J_\pm = J_0(1 \pm \beta)$ and $\psi_k^T = (c_{Ak}, c_{Bk}, c_{Ck}, c_{Dk})$. We find that the dispersion is

$$E_k = \pm J_\pm \sqrt{\cos^2 k_x + \cos^2 k_y},$$

and we note that when $\beta = 1$, $J_- = 0$, so there will be a flatband at $E_k = 0$ and a dispersing band associated with J_+. We can also see that in the vicinity of the points $\mathbf{K}_{\pm,\pm} = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, the spectrum is linear,

$$E_q = \pm J_\pm \sqrt{q_x^2 + q_y^2}, \tag{9}$$

where $\mathbf{q} = \mathbf{k} - (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, and there are cones with two different slopes, corresponding to $J_\pm$ respectively, as illustrated in Fig. 3(b). When $\beta = 0$, the two slopes are identical, whereas as $\beta \to 1$, the J_- band becomes flat, and the J_+ band remains conical. Several authors recently considered lattice models closely related to the $\beta = 1$ limit of our model. In these models there are three bands, one flat, and one Dirac-like [8]. The $\beta = 1/2$ spectrum matches that of Weyl fermions considered in Refs. [9,10]. When $\beta \neq 1$, the underlying Dirac structure of the problem is exposed, which allows us to use a symmetry approach to understand this unusual dispersion.

III. LOW-ENERGY THEORY

We expand around the Dirac points and represent the low-energy theory (with $\mathbf{k}$ measured with respect to $\mathbf{K}$) as

$$H_k = 2J_0[(\gamma^0\gamma^1 + i\beta\gamma^3)k_x + (\gamma^0\gamma^2 + i\beta\gamma^5)k_y], \tag{10}$$

where we use a nonstandard representation of the γ matrices in which $\gamma^0 = \sigma_3 \otimes \sigma_3$, $\gamma^1 = i\sigma_2 \otimes I_2$, $\gamma^2 = i\sigma_3 \otimes \sigma_2$, $\gamma^3 = -i\sigma_1 \otimes I_2$, and $\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = -i\sigma_3 \otimes \sigma_1$. The matrices γ^0, γ^1, γ^2, and γ^3 satisfy the Clifford algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ with Minkowski metric $g^{\mu\nu}$.

The dimension of the minimal representation of the Clifford algebra in $2+1$ dimensions is 2, allowing for the 2×2 Pauli matrices as a choice for the γ's. A nonminimal 4×4 representation as we have used above leads to a freedom in the choice of the γ^0 matrix, i.e., a matrix with $(\gamma^0)^2 = I_4$ that anticommutes with γ^1 and γ^2. Candidates for γ^0 are then $\{\gamma^0, \gamma^0\gamma^3, \gamma^0\gamma^5, \gamma^1\gamma^2\}$. The matrices $\{\gamma^0, \gamma^0\gamma^3, \gamma^0\gamma^5\}$ form a triplet and $\gamma^1\gamma^2$ forms a singlet with respect to the SU(2) "chiral"-symmetry group with generators $\{\frac{i}{2}\gamma^3, \frac{i}{2}\gamma^5, \frac{i}{2}\gamma^{35}\}$ (where $\gamma^{35} \equiv \gamma^3\gamma^5$). Each different choice of γ_0 corresponds to a different labeling of the four sites in the unit cell. The elements of the chiral group generate transformations between each labeling. For example, the generator γ^5 translates the plaquette indices to the labeling of the neighboring lattice cell along the y direction, while γ^3 translates the plaquette indices to the neighboring cell in the x direction:

$$e^{(\pi/2)\gamma^5}\begin{pmatrix} c_A \\ c_B \\ c_C \\ c_D \end{pmatrix} = i\begin{pmatrix} c_B \\ c_A \\ -c_D \\ -c_C \end{pmatrix}, \quad e^{(\pi/2)\gamma^3}\begin{pmatrix} c_A \\ c_B \\ c_C \\ c_D \end{pmatrix} = i\begin{pmatrix} c_C \\ c_D \\ c_A \\ c_B \end{pmatrix}. \tag{11}$$

Similarly, γ^{35} translates the plaquette one lattice cell along the x and one lattice cell along the y direction. When $\beta = 0$, the elements of the chiral group are symmetries of H_k. When $\beta \neq 0$, the γ^3 and γ^5 terms in H_k break the chiral symmetry and shifts along either the x or y direction do not leave H_k invariant. We emphasize that this manifest chiral symmetry breaking is inherently different from the conventional notion of spontaneous chiral symmetry breaking in field theoretical models, which is the signature of mass generation [11].

An additional discrete symmetry of H_k (which arises from the hopping structure in $\mathcal{H}_k$) that holds even when $\beta \neq 0$ is

$$\Gamma = \frac{i}{2}(\gamma^1\gamma^3 + \gamma^2\gamma^5) - \frac{i}{2}(\gamma^2\gamma^3 - \gamma^1\gamma^5),$$

which corresponds to a reflection about the diagonal AD in the unit cell, with $c_A \to c_A$, $c_B \to c_C$, $c_C \to c_B$, and $c_D \to -c_D$. The action of Γ on H_k is to exchange k_x and k_y.

A. Fermion birefringence

As illustrated in Fig. 3(b) the dispersion equation (9) admits massless fermions with two different "speeds of light" controlled by β. The eigenvectors (written as row vectors) for the positive- and negative-energy J_+ bands are $\Psi_1 = \frac{1}{\sqrt{2}}(1, -\sin\theta, -\cos\theta, 0)$ and $\Psi_2 = \frac{1}{\sqrt{2}}(1, \sin\theta, \cos\theta, 0)$; while the eigenvectors for the J_- bands are $\Psi_3 = \frac{1}{\sqrt{2}}(0, \cos\theta, -\sin\theta, 1)$ and $\Psi_4 = \frac{1}{\sqrt{2}}(0, -\cos\theta, \sin\theta, 1)$, where we write $k_x = k\cos\theta$ and $k_y = k\sin\theta$. The linear combinations $\Psi_1 + \Psi_2$ and $\Psi_3 + \Psi_4$ have nonzero amplitude only on the A and D sites, respectively. Any other state will break up into fast (J_+) and slow (J_-) fermionic excitations, analogous to fast and slow modes in an optically birefringent medium.

B. Staggered potentials

Staggered on-site potentials are a natural perturbation to H_k in the context of cold atoms on an optical lattice. The most general form of such a potential is

$$\Delta = \sum_k \psi_k^\dagger[\Delta_0 I_4 + \Delta_1\gamma^0 + \Delta_2(i\gamma^1\gamma^3 + i\gamma^2\gamma^5) + \Delta_3(i\gamma^1\gamma^3 - i\gamma^2\gamma^5)]\psi_k, \tag{12}$$

where we may set $\Delta_0 = 0$ since this just corresponds to a uniform shift of the chemical potential. The Δ_1 term violates chiral symmetry in the usual way but is Lorentz invariant and hence introduces a gap in the dispersion of the fermions,

$$E_k = \pm\sqrt{\Delta_1^2 + 4J_\pm^2 k^2}. \tag{13}$$

When $\beta = 1$ there are flatbands at $E = \pm\Delta_1$ that intersect the J_+ bands only at $(k_x, k_y) = (0,0)$. The birefringence property discussed above is unaffected by the Δ_1 term. We combine $i\gamma^1\gamma^3$ and $i\gamma^2\gamma^5$ into a Lorentz-invariant term (Δ_2) and a

Lorentz-violating term (Δ_3). There are two cases in which we have obtained simple analytic solutions for the spectrum: case I, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, $\Delta_3 = 0$, for which

$$E_k = \begin{cases} \Delta_2 \pm \sqrt{(\Delta_1+\Delta_2)^2 + 4J_+^2 k^2}, \\ -\Delta_2 \pm \sqrt{(\Delta_1-\Delta_2)^2 + 4J_-^2 k^2}; \end{cases}$$

and case II, $\Delta_1 \neq 0$, $\Delta_2 = 0$, $\Delta_3 \neq 0$, for which

$$E_k = \begin{cases} \Delta_3 \pm \sqrt{(\Delta_1-\Delta_3)^2 + 4J_+^2 k_y^2 + 4J_-^2 k_x^2}, \\ -\Delta_3 \pm \sqrt{(\Delta_1+\Delta_3)^2 + 4J_+^2 k_x^2 + 4J_-^2 k_y^2}. \end{cases}$$

In case I the dispersion is isotropic in momentum space and there are flatbands when $\beta = 1$, whereas in case II, the dispersion is anisotropic, with the anisotropy governed by β through $J_\pm$. In both cases, there is a shift in the spectrum and at least one set of massive modes (however, in both cases there can be a set of massless modes whose dispersion is given by the upper half of a cone if $\Delta_1 = \pm\Delta_{2,3}$ and $\Delta_0 = \mp\Delta_{2,3}$).

C. Interactions

As we consider spinless fermions, there will be no on-site Hubbard interaction, so we consider nearest-neighbor interactions of the extended Hubbard type (for cold atoms in an optical lattice these will generally be weak):

$$H_{\text{int}} = \sum_{\langle ij \rangle} V_{ij} n_i n_j. \tag{14}$$

Setting all of the $V_{ij} = V_0$, we can write the interaction Hamiltonian in terms of spinors as

$$H_{\text{int}} = \frac{V_0}{16} \sum_k [(\bar{\psi}_k \gamma^0 \psi_k)^2 - (\bar{\psi}_k \psi_k)^2], \tag{15}$$

with $\bar{\psi}_k = \psi_k^\dagger \gamma^0$. The identity and γ^0 that appear in the kernels of the quartic interaction terms are the only elements of the Clifford algebra that either commute or anticommute with all of the elements of the Lorentz group and the chiral group, ensuring that the interactions remain invariant under any rotation of the lattice by the Lorentz group or relabeling of the plaquette indices by the chiral group. At the mean-field level, the $(\bar{\psi}\psi)^2$ term breaks chiral symmetry by introducing an effective-mass term $m_0\gamma^0$, and the $(\bar{\psi}\gamma^0\psi)^2$ term renormalizes the chemical potential as δI_4. For weak interactions, the mean-field interaction Hamiltonian is

$$H_{\text{int}}^{\text{MF}} = \sum_k \psi_k^\dagger [(\delta I_4 + m_0\gamma^0) + (m_1\gamma^0\gamma^1 + m_2\gamma^0\gamma^2 + m_3 i\gamma^3 + m_5 i\gamma^5)]\psi_k, \tag{16}$$

where $\delta = \langle n_A \rangle + \langle n_B \rangle + \langle n_C \rangle + \langle n_D \rangle$, and the order parameter for staggered charge-density-wave order $m_0 = \langle n_A \rangle - \langle n_B \rangle - \langle n_C \rangle + \langle n_D \rangle$ arises from the Hartree term. The remaining masses m_1, m_2, m_3, and m_5 arise from the Fock term—if these are dropped and $\beta = 0$, we recover the mean-field approximation of the Gross-Neveu model [12]. Similarly to a $\Delta_1\gamma^0$ staggered potential, the Hartree term leads to massive excitations, but does not destroy fermion birefringence. The detailed study of interactions when $\beta \neq 0$ is a topic for future investigation.

For small values of β, when H_{int} is added to Eq. (3) there is a mapping between the weak-interaction-strength regime considered above to the strong-interaction-strength limit that preserves the property of birefringence:

$$E_k(\beta, V_0) = \beta E_k(\beta^{-1}, \beta^{-1} V_0). \tag{17}$$

This arises from the appearance of the chiral symmetry generators γ^3 and γ^5 in the kinetic energy and their duality with Lorenz group generators $\gamma^0\gamma^1, \gamma^0\gamma^2$. Upon choosing a different representation of Clifford algebra elements, one can transform $\gamma^0\gamma^1 \leftrightarrow \gamma^3, \gamma^0\gamma^2 \leftrightarrow \gamma^5$.

D. Topological defects

Broken chiral symmetry at $\beta \neq 0$ implies that there cannot be straightforward vortices, but domain walls of the form $\Delta_1(x)\gamma^0$ where $\lim_{x\to\infty} \Delta_1(x) = \Delta$ and $\lim_{x\to-\infty} \Delta_1(x) = -\Delta$ can occur. If $\beta = 0$ the solutions are well known. When $\beta \neq 0$ we can find zero-energy bound states with different spatial extents for the $\pm$ solutions:

$$\psi_+(x) = e^{-\kappa_+ \int_0^x ds \Delta_1(s)} u_+, \quad \psi_-(x) = e^{-\kappa_- \int_0^x ds \Delta_1(s)} u_-,$$

where $u_+ = (1,0,i,0)$, $u_- = (0,-i,0,1)$, and $\kappa_\pm = 1/2J_\pm$.

IV. DISCUSSION

The model we introduce here has the unusual feature of low-energy excitations that are birefringent massless fermions arising from broken chiral symmetry. We suggested a particular procedure based on Ref. [4] as a way to realize this model with cold atoms in an optical lattice. This might not be the only route to realize the model: recent work by Lan *et al.* [10] on cold atoms suggested a procedure that gives a model with the same spectrum as the $\alpha = 1/2$, $\beta = 1/2$ version of our model. Additionally, as noted in a similar context [13], it might be possible to engineer an appropriate semiconductor heterostructure.

An important feature of the birefringent fermion dispersion that we find here is that the slopes of the J_+ and J_- bands can be controlled by the parameter β. Possible applications of birefringent properties might include a filter for Dirac fermions in cold atom systems (via spatially varying β) or birefringent Klein tunneling [10]. Flatbands such as Landau levels enhance interactions, suggesting that the flatbands we observe here when $\beta = 1$, which are robust to the addition of a staggered potential and weak interactions may allow for interesting correlated phases when interactions beyond the mean field are taken into account [14]. Future avenues for research on this model could include the study of such correlated phases when $\beta \neq 0$. Generalization of the model to fermions with spin would allow for on-site Hubbard interactions, that could require an approach similar to those used to study QED_3 in high-temperature superconductors [15].

ACKNOWLEDGMENTS

The authors acknowledge very helpful discussions with Igor Herbut and Bitan Roy. We acknowledge support from NSERC and the use of Westgrid computing facilities.

[1] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, Science **306**, 666 (2004); K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature (London) **438**, 197 (2005) ; G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).

[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature (London) **452**, 970 (2008).

[3] N. R. Cooper and N. K. Wilkin, Phys. Rev. B **60**, 16279 (1999); D. Jaksch and P. Zoller, New. J. Phys. **5**, 56 (2003); N. R. Cooper, Adv. Phys. **57**, 539 (2008); A. Klein and D. Jaksch, Europhys. Lett. **85**, 13001 (2009); M. Rosenkranz, A. Klein, and D. Jaksch, Phys. Rev. A **81**, 013607 (2010); L.-K. Lim, A. Hemmerich, and C. M. Smith, *ibid.* **81**, 023404 (2010); F. Gerbier and J. Dalibard, New. J. Phys. **12**, 033007 (2010); A. Bermudez, L. Mazza, M. Rizzi, N. Goldman, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. **105**, 190404 (2010); T.-L. Ho and S. Zhang, e-print arXiv:1007.0650; D. Makogon, I. B. Spielman, and C. Morais Smith, e-print arXiv:1007.0782; E. Kapit and E. J. Mueller, Phys. Rev. A **83**, 033625 (2011); N. R. Cooper, Phys. Rev. Lett. **106**, 175301 (2011).

[4] A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **94**, 086803 (2005).

[5] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, and E. A. Cornell, Phys. Rev. Lett. **92**, 040404 (2004); Y.-J. Lin, R. L. Compton, K. Jiménez-García, J. V. Porto, and I. B. Spielman, Nature (London) **462**, 628 (2009).

[6] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[7] A. Barelli, J. Bellissard, and R. Rammal, J. Phys. (France) **51**, 2167 (1990); M. C. Chang and Q. Niu, Phys. Rev. B **50**, 10843 (1994); G.-Y. Oh, *ibid.* **60**, 1939 (1999); M. Ando, S. Ito, S. Katsumoto, and Y. Iye, J. Phys. Soc. Jpn. **68**, 3462 (1999); M.-C. Chang and M.-F. Yang, Phys. Rev. B **69**, 115108 (2004); Y. Iye, E. Kuramochi, M. Hara, A. Endo, and S. Katsumoto, *ibid.* **70**, 144524 (2004); Y.-F. Wang and C.-D. Gong, *ibid.* **74**, 193301 (2006).

[8] R. Shen, L. B. Shao, B. Wang, and D. Y. Xing, Phys. Rev. B **81**, 041410(R) (2010); D. Bercioux, D. F. Urban, H. Grabert, and W. Häusler, Phys. Rev. A **80**, 063603 (2009); V. Apaja, M. Hyrkäs, and M. Manninen, *ibid.* **82**, 041402(R) (2010).

[9] H. Watanabe, Y. Hatsugai, and H. Aoki, e-print arXiv:1009.1959.

[10] Z. Lan, N. Goldman, A. Bermudez, W. Lu, and P. Ohberg, e-print arXiv:1102.5283.

[11] V. A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).

[12] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).

[13] B. Seradjeh, C. Weeks, and M. Franz, Phys. Rev. B **77**, 033104 (2008).

[14] D. Green, L. Santos, and C. Chamon, Phys. Rev. B **82**, 075104 (2010).

[15] I. F. Herbut, Phys. Rev. Lett. **88**, 047006 (2002); Z. Tesanovic, O. Vafek, and M. Franz, Phys. Rev. B **65**, 180511(R) (2002); M. Franz, Z. Tesanovic, and O. Vafek, *ibid.* **66**, 054535 (2002); I. F. Herbut, *ibid.* **66**, 094504 (2002).