System of fermions confined in a harmonic potential and subject to a magnetic field or a rotational motion

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Making use of the Bloch density matrix technique, we derive exact analytical expressions for the density profile in Fourier space, for the current density and the so-called integrated current for fermionic systems confined by a two-dimensional harmonic oscillator, in the presence of a magnetic field or in a rotating trap of arbitrary strength. We present numerical, illustrative examples with or without magnetic field (with or without rotation).

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I. INTRODUCTION

The theoretical studies of the properties of harmonically trapped Fermi gases have gained increasing attention since the experimental observation by DeMarco and Jin of a quantum degenerate atomic Fermi gas [1]. In these experiments, temperatures of the order of a fraction of the Fermi temperatures were reached—a regime where these ultracold atomic gases constitute dilute systems in which the interparticle interactions are weak. For the description of the physics of dilute neutral Fermi gases, the ideal Fermi gas constitutes a natural starting point. As a first approximation the interactions can, in fact, be neglected, or treated as a small perturbation, which can be done for an entirely polarized system, where interactions are strongly quenched due to the Pauli exclusion principle suppressing s-wave scattering. The research on degenerate quantum gases for both bosons and fermions is presently an active field where an impressive amount of experimental and theoretical work has been carried out [2–6]. This has led to the study of many interesting issues related to this field. Among these is the study of the response of a quantum atomic gas to a rotation. Indeed, for bosons in rotating harmonic traps, intensive investigations have been performed both theoretically [2–5] and experimentally [7–9], showing interesting phenomena such as the appearance of quantized vortices, the number of which increases with increasing angular velocity [10].

To study the interplay between rotation and the effects of quantum statistics, we have chosen to restrict our study to fermionic gases. Indeed, a number of theoretical studies have recently appeared that examine rotating Fermi gases. These studies reveal that a Landau-level-like energy spectrum appears in the fast-rotating regime and the influence of the shell structure on the density profiles is increasingly important [11–14]. A precursor to these studies is the interesting work of Ref. [15], where the energy spectrum of a noninteracting Fermi gas in a harmonic trap is examined as function of the rotational angular velocity and where the connection to the quantum Hall effect is pointed out. Very recently, anharmonic traps with an additional quartic potential were also considered [6] and a possible experiment with an ultracold Fermi gas in one of the so-called Paris traps (see, e.g., Ref. [9]) was suggested in Ref. [16]. It is worth mentioning in this context the recent work of Dalibard *et al.* [17] on the use of an atom-light interaction to generate artificial gauge potentials acting on neutral matter—a case where the Hamiltonian contains an artificial gauge field, similar to the case of particles in a rotating frame.

Owing to the well-known assimilation of the Coriolis pseudoforce as a Lorentz magnetic force, a two-dimensional (2D) spin-polarized neutral Fermi gas confined in a rotating harmonic trap can be mapped onto a spin-polarized charged gas subject to a uniform magnetic field [12]. To make the total system neutral, one supposes that the gas is immersed in a uniformly charged background medium, with charge of opposite sign so that the Coulomb interactions between charges can at first approximation be neglected.

Recently, the thermodynamical properties of a charged ideal Fermi gas in two dimensions, made up of N fermions of charge q in a perpendicular uniform magnetic field, were investigated by the authors of Refs. [18]. The authors use the inverse Laplace transform (ILT) technique of the Bloch density matrix (BDM) to obtain exact analytical expressions for the particle and kinetic energy densities that are valid for an arbitrary strength of the external magnetic field. The present work deals with this class of studies following the phase-space density study of Ref. [19]. Here we focus on two specific physical aspects: (i) the density profile in Fourier space and (ii) the induced current density.

For the sake of definiteness, we refer throughout the paper to the 2D noninteracting charged Fermi system in a uniform magnetic field, but our results are easily transposable, as stated above in the mapping correspondence, to a rotating neutral polarized Fermi gas.

The paper is organized as follows. After a brief survey of the approach in Sec. II, we present a derivation of exact and mathematically simple analytical expressions for the density in Fourier space (Sec. III) and the orbital current-density distribution (Sec. IV) of that system. An exact expression is also obtained in Sec. V for the so-called integrated current. Finally, Sec. VI draws some conclusions.

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II. A BRIEF SURVEY OF THE METHOD

Let us consider a system of N independent fermions in the xy plane, confined by an isotropic 2D harmonic potential $m^*\omega_o^2\vec{r}^{\,2}/2$ with frequency ω_o and in the presence of a uniform magnetic field $\vec{B}=\vec{\nabla}\times\vec{A}=B_o\vec{e}_z$ applied along the z axis. The one-body Hamiltonian H of this system is given by

$$H=\frac{1}{2m^*}(\vec{p}-q\vec{A})^2+\frac{1}{2}m^*\omega_o^2\vec{r}^{\,2}. \tag{1}$$

Note that we have introduced an effective electron mass m^* which is always possible through an appropriate redefinition of the frequency ω_o. Here $\vec{p}=-i\hbar\vec{\nabla}$, $\vec{r}^{\,2}=x^2+y^2$, $q=-e$ $(e>0)$ is the electron charge, and $\vec{A}$ is the vector potential defined within our choice of gauge by

$$\vec{A}=\frac{1}{2}(\vec{B}\times\vec{r})=\frac{B_o}{2}(-y\,\vec{e}_x+x\,\vec{e}_y). \tag{2}$$

The Hamiltonian, Eq. (1), can be rewritten in coordinate representation as

$$H=-\frac{\hbar^2}{2m^*}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{2}m^*\Omega^2(x^2+y^2)+\omega_L L_z, \tag{3}$$

where $\vec{L}=\vec{r}\times\vec{p}$ is the orbital angular momentum of the particle and L_z is its z component. The Larmor frequency is

$$\omega_L=\frac{eB_o}{2m^*}, \tag{4}$$

and

$$\Omega=\sqrt{\omega_o^2+\omega_L^2}. \tag{5}$$

Instead of explicitly using the eigensolutions associated with H to calculate the static properties of such a system, we rather introduce the BDM, defined at zero temperature by

$$C(\vec{r},\vec{r}\,';\beta)=\sum_i\varphi_i(\vec{r}\,)\,\varphi_i^*(\vec{r}\,')e^{-\beta\varepsilon_i}, \tag{6}$$

where $\varphi_i(\vec{r}\,)$ and ε_i are the eigenfunctions and eigenenergies of H. Note that β is a parameter that has the dimension of an inverse energy, but it should not be confused with an inverse temperature since, as pointed out above, Eq. (6) defines the BDM at zero temperature which is the solution of the Bloch equation

$$HC(\vec{r},\vec{r}\,';\beta)=-\frac{\partial C(\vec{r},\vec{r}\,';\beta)}{\partial\beta}. \tag{7}$$

The interest in the BDM lies in the fact that its knowledge gives direct access to the single-particle density matrix $\rho(\vec{r},\vec{r}\,')$ at zero and nonzero temperatures through an appropriate ILT [20,21]. Let us start with the zero-temperature case, where the density matrix for a given Fermi energy λ is given by

$$\rho(\vec{r},\vec{r}\,')=\mathcal{L}_\lambda^{-1}\left[\frac{C(\vec{r},\vec{r}\,';\beta)}{\beta}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\beta\frac{e^{\beta\lambda}}{\beta}C(\vec{r},\vec{r}\,';\beta). \tag{8}$$

The expression of the BDM associated with the Hamiltonian (1), obtained by March and Tosi [22], may be expressed (see Ref. [18]) in terms of the center of mass and relative coordinates $\vec{R}=(\vec{r}+\vec{r}\,')/2$ and $\vec{s}=\vec{r}-\vec{r}\,'$:

$$\begin{aligned}C(\vec{R},\vec{s}\,;\beta)&=\frac{m^*\Omega}{2\pi\hbar\sinh(\beta\hbar\Omega)}\exp\left\{i\,\frac{m^*\Omega}{\hbar}\frac{\sinh(\beta\hbar\omega_L)}{\sinh(\beta\hbar\Omega)}(\vec{s}\times\vec{R})\cdot\vec{e}_z\right\}\\&\times\exp\left\{-\frac{m^*\Omega}{\hbar}\left[\coth(\beta\hbar\Omega)-\frac{\cosh(\beta\hbar\omega_L)}{\sinh(\beta\hbar\Omega)}\right]\vec{R}^{\,2}\right\}\\&\times\exp\left\{-\frac{m^*\Omega}{4\hbar}\left[\coth(\beta\hbar\Omega)+\frac{\cosh(\beta\hbar\omega_L)}{\sinh(\beta\hbar\Omega)}\right]\vec{s}^{\,2}\right\}.\end{aligned} \tag{9}$$

In the following section, we use this expression to derive an exact analytical expression of the density profile in Fourier space.

III. FOURIER TRANSFORM OF THE PARTICLE DENSITY

In spatial coordinates, exact analytical expressions of the particle density were obtained in Ref. [23] for the case of harmonically confined quantum gases in arbitrary dimensions. Recently, simple analytical expressions of this density in Fourier space have been found [24]. In this section, we calculate this latter quantity in the 2D case in the presence of a uniform magnetic field. The Fourier transform $n(\vec{k})$ of the one-particle density $\rho(\vec{r}\,)$ may be rewritten with Eq. (8) in the form

$$n(\vec{k}\,)=\mathcal{L}_\lambda^{-1}\left[\frac{1}{\beta}\int d^2r\,C(\vec{r}\,;\beta)e^{-i\vec{k}\cdot\vec{r}}\right], \tag{10}$$

where $C(\vec{r}\,;\beta)$ is the diagonal part of the BDM, $C(\vec{r},\vec{s}\,;\beta)$, which may be expressed with Eq. (9) as

$$C(\vec{r};\beta)=\frac{m^*\Omega}{2\pi\hbar\sinh(\beta\hbar\Omega)}\,e^{-g(\beta)r^2}. \tag{11}$$

Above, we have introduced the function

$$g(\beta)=\frac{m^*\Omega}{\hbar}\left[\coth(\beta\hbar\Omega)-\frac{\cosh(\beta\hbar\omega_L)}{\sinh(\beta\hbar\Omega)}\right]. \tag{12}$$

Inserting Eq. (11) into Eq. (10) and using the identity

$$\int d^2u\,\exp(-i\vec{b}\cdot\vec{u}-a\vec{u}^2)=\frac{\pi}{a}\exp\left[-\frac{\vec{b}^2}{4a}\right], \tag{13}$$

one obtains

$$n(\vec{k})=\frac{m^*\Omega}{2\hbar}\,\mathcal{L}_\lambda^{-1}\left[\frac{1}{\beta g(\beta)\sinh(\beta\hbar\Omega)}\exp\left(-\frac{\vec{k}^{\,2}}{4g(\beta)}\right)\right]. \tag{14}$$

Upon inserting the explicit expression of $g(\beta)$ and making use of some relations between hyperbolic functions, such as

$$\frac{\sinh(x)}{\cosh(x)-\cosh(y)}=\frac{1}{2}\left[\coth\left(\frac{x+y}{2}\right)+\coth\left(\frac{x-y}{2}\right)\right] \tag{15}$$

with $x=\beta\hbar\Omega$ and $y=\beta\hbar\omega_L$, Eq. (14) becomes

$$n(\vec{k})=\mathcal{L}_\lambda^{-1}\left[\frac{1}{\beta}\left\{\frac{\exp\left[-\frac{\hbar}{8m^*\Omega}\coth(\frac{\beta\hbar\Omega_-}{2})\vec{k}^{\,2}\right]}{2\sinh(\frac{\beta\hbar\Omega_-}{2})}\times\frac{\exp\left[-\frac{\hbar}{8m^*\Omega}\coth(\frac{\beta\hbar\Omega_+}{2})\vec{k}^{\,2}\right]}{2\sinh(\frac{\beta\hbar\Omega_+}{2})}\right\}\right] \tag{16}$$

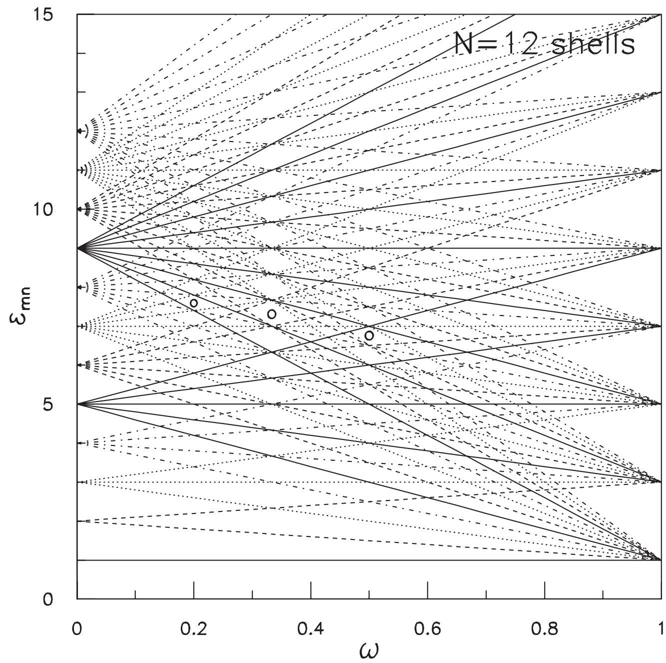

FIG. 1. Energy spectrum, in units of $\hbar\Omega$, as a function of the magnetic field strength $\omega = \omega_L/\Omega$. Circles indicate gaps at the Fermi surface for a system of $N = 60$ particles for ω values $\omega \sim 1/5, 1/3, 1/2$.

with $\Omega_\pm = \Omega \pm \omega_L$. Note that the partition function $Z = \int d^2r C(\vec{r};\beta)$ of the problem under study is the same as that of an anisotropic two-dimensional harmonic oscillator with frequencies Ω_- and Ω_+. In fact, using Eq. (11), one easily finds $Z = 1/[4\sinh(\beta\hbar\Omega_+/2)\sinh(\beta\hbar\Omega_-/2)]$. Note also that the eigenenergies of the quantum Hamiltonian in Eq. (1) are given by

$$\begin{aligned}\varepsilon_{n,m} &= \hbar\Omega_+(n+1/2) + \hbar\Omega_-(m+1/2)\\ &= \hbar\Omega(n+m+1) + \hbar\omega_L(n-m), \quad n,m = 0,1,2,\ldots .\end{aligned} \tag{17}$$

This energy spectrum $\varepsilon_{n,m}$ (in units of $\hbar\Omega$) is shown in Fig. 1 as function of the magnetic field strength characterized by the dimensionless variable $\omega = \omega_L/\Omega$. Let us now calculate the inverse Laplace transform in Eq. (16). We use the following expansion involving Laguerre polynomials [25]:

$$\frac{e^{-x\coth y}}{\sinh y} = 2e^{-x}\sum_{n=0}^{+\infty} L_n(2x)\, e^{-(2n+1)y} \tag{18}$$

with $x = \hbar/(8m^*\Omega)$ and $y = \beta\hbar\Omega_\pm/2$. This allows us to rewrite $n(\vec{k})$, as given in Eq. (16), as

$$n(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} L_n\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)L_m\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right) \times \mathcal{L}_\lambda^{-1}\left[\frac{e^{-\beta\varepsilon_{n,m}}}{\beta}\right] \tag{19}$$

and, upon using (see, e.g., Ref. [26])

$$\mathcal{L}_\lambda^{-1}\left\{\frac{e^{-\beta\varepsilon_{n,m}}}{\beta}\right\} = \Theta(\lambda - \varepsilon_{n,m}) \tag{20}$$

where Θ is the Heaviside step function, one obtains the final result,

$$n(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} L_n\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)L_m\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right) \times \Theta(\lambda - \varepsilon_{n,m}). \tag{21}$$

We thus obtain an exact expression which turns out to be rather compact and easy to use in numerical calculations. In Eq. (21), the quantum numbers n and m are restricted, so

$$\varepsilon_{n,m} = \hbar\Omega_+(n+1/2) + \hbar\Omega_-(m+1/2) \leqslant \lambda. \tag{22}$$

The highest possible value for n, $n_{\max}$, is given by

$$n_{\max} = \text{Int}\left(\frac{\lambda - \hbar\Omega}{\hbar\Omega_+}\right), \tag{23}$$

where $\text{Int}(x)$ denotes the integer part of x with x positive. For a given allowed value of n the maximum allowed value for m is then

$$m_{\max}(n) = \text{Int}\left(\frac{\lambda - \hbar\Omega - n\hbar\Omega_+}{\hbar\Omega_-}\right). \tag{24}$$

To obtain the Fermi energy λ for a given number of particles, N, one can use the trivial normalization relation $n(\vec{k} = \vec{0}) = N$, which yields, together with Eq. (21),

$$N = \sum_{n=0}^{n_{\max}}\sum_{m=0}^{m_{\max}(n)} 1. \tag{25}$$

Figure 2 shows the density profile $n(\vec{k})$ in Fourier space for a system of $N = 60$ particles and its variation with the magnetic field strength characterized by ω. The three values of ω all correspond to a shell closure of the studied system (see Fig. 1). It appears that significant changes of the magnetic field do not lead to drastic changes of the density in Fourier space, but shell oscillations play an important role. As is also obvious from Eq. (21), one has $n(\vec{k} = \vec{0}) = N$.

The simplicity of the result in Eq. (21) allows us to deduce the Fourier transform of the density $|\phi_{n,m}(\vec{r})|^2$, where $\phi_{n,m}(\vec{r})$ stands for the single-particle wave function of the Hamiltonian (1) with energy $\varepsilon_{n,m}$. In coordinate space, the single-particle density $\rho(\vec{r})$ is, of course, given in terms of the $\phi_{n,m}(\vec{r})$ as

$$\rho(\vec{r}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\phi_{n,m}(\vec{r})|^2\Theta(\lambda - \varepsilon_{n,m}) \tag{26}$$

and, taking the Fourier transform of the above equation and using the result of Eq. (21), one immediately finds

$$\int d^2r e^{-i\vec{k}\cdot\vec{r}}\, |\phi_{n,m}(\vec{r})|^2 = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)L_n\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right) \times L_m\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right). \tag{27}$$

Hence, we have obtained the Fourier transform of the density $|\phi_{n,m}(\vec{r})|^2$ without the explicit use of the analytical expression of the single-particle wave function $\phi_{n,m}(\vec{r})$. This result is used later on.

Notice that the analytical form of the density profile in Fourier space, Eq. (21), can be immediately used to write the

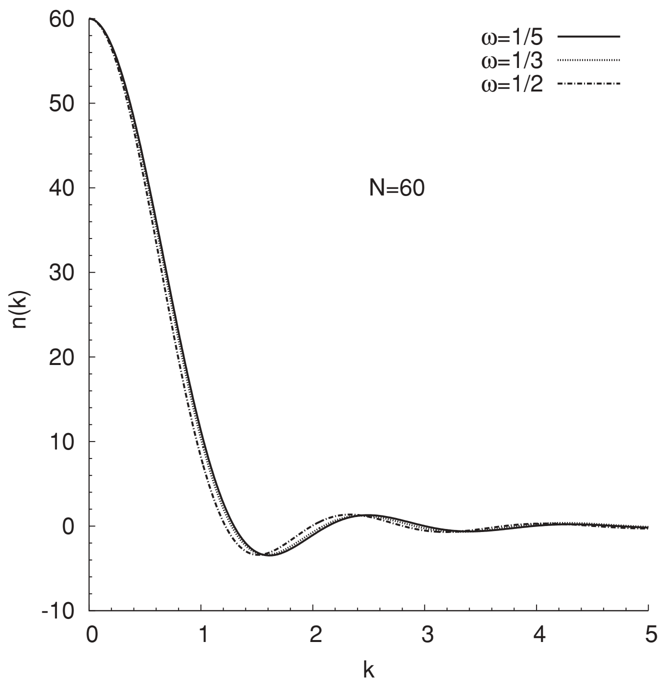

FIG. 2. Density profile in Fourier space for $N = 60$ particles at three different values of the magnetic field strength as characterized by the dimensionless parameter ω (see Fig. 1).

expression of such a density at finite temperature T, which for Fermions is then simply given by

$$n_T^F(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} L_n\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)L_m\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right) \times \frac{1}{\exp\left(\frac{\varepsilon_{n,m}-\mu}{k_BT}\right)+1} \tag{28}$$

upon considering the Heaviside step function as a Fermi distribution in the zero-temperature limit. Here k_B is the Boltzmann constant and μ is the chemical potential, which has to be chosen such that $n_T^F(\vec{k}=\vec{0}) = N$. The Fermi distribution function enters here in a trivial way, which immediately suggests an examination of the corresponding case of bosons. All that is required is, in fact, to replace the Fermi function by the corresponding Bose distribution to obtain

$$n_T^B(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} L_n\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)L_m\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right) \times \frac{1}{\exp\left(\frac{\varepsilon_{n,m}-\mu}{k_BT}\right)-1}. \tag{29}$$

Equation (29) can be used as a starting point to examine, in Fourier space, the thermodynamical properties of a system of bosons in a rotating trap. This study can be performed in a similar way as was done in Ref. [27] for noninteracting Bose gases in d-dimensional anisotropic harmonic traps but without rotation. In this latter reference, exact results were given, in particular, for the particle density for arbitrary temperatures. It is not the aim, however, of our paper to present a study of thermodynamics of Bose gases. Nevertheless, we briefly discuss the high- and low-temperature limits of the density profile of Eq. (29) in the Appendix.

In the absence of a magnetic field (where only the harmonic confinement remains), our results reduce to the correct limits [24], as shown now. Since the following is not affected by the Fermi or Bose types of distribution, we can limit ourselves to the zero-temperature case.

In the limit of vanishing magnetic field, where $\omega_L = 0$ and $\Omega = \omega_o$, one has $\Omega_+ = \Omega_- = \omega_o$ and Eq. (21) becomes

$$n^{(\vec{B}=\vec{0})}(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right)\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} L_n\left(\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right) \times L_m\left(\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right)\Theta(\lambda - \hbar\omega_o(n+m+1)). \tag{30}$$

Defining $j = n + m$ and M as the maximal possible value of j, determined by the Fermi energy $\lambda = \hbar\omega_o(M+1)$, one obtains

$$n^{(\vec{B}=\vec{0})}(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right)\sum_{j=0}^{M}\sum_{m=0}^{j} L_{j-m}\left(\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right) \times L_m\left(\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right). \tag{31}$$

The relations (see, e.g., Ref. [28])

$$\sum_{m=0}^{n} L_{n-m}(x)\, L_m(y) = L_n^1(x+y) \quad \text{and} \quad \sum_{j=0}^{M} L_j^1(x) = L_M^2(x), \tag{32}$$

where $L_n^\alpha(x)$ is the generalized Laguerre polynomial of order α, reduce Eq. (31) to

$$n^{(\vec{B}=\vec{0})}(\vec{k}) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right)\sum_{j=0}^{M} L_j^1\left(\frac{\hbar\vec{k}^2}{2m^*\omega_o}\right) = \exp\left(-\frac{\hbar\vec{k}^2}{4m^*\omega_o}\right)L_M^2\left(\frac{\hbar\vec{k}^2}{2m^*\omega_o}\right), \tag{33}$$

which is the result obtained in two dimensions in Ref. [24].

It appears that the results given in Eqs. (21) and (28) can be of practical use in the theory of light scattering. (For a presentation of the underlying theory of light scattering from a confined atomic cloud see, e.g., Ref. [29].) To explore the shell structure appearing in the density profile of a rotating gas of spin-polarized fermions, the authors of Ref. [12] discussed the effects of elastic scattering of light waves on the atoms of the confined gas. If the light frequency is far from the resonance of the internal atomic states, the elastic scattering entirely determines the angular distribution, which is characterized by the elastic structure function [12,29]

$$S_e(\vec{k}) = \left|\int d^2r\rho(\vec{r})e^{-i\vec{k}\cdot\vec{r}}\right|^2 = |n(\vec{k})|^2, \tag{34}$$

where $\vec{k} = \vec{k}_{\rm inc} - \vec{k}_{\rm sc}$ is the momentum transfer between the incoming and the scattered waves. As can be seen from Eq. (34), the analytical results obtained in Eqs. (21), (28), (29), and (33) can therefore be useful in evaluating exactly such elastic structure functions.

IV. ORBITAL CURRENT DENSITY DISTRIBUTION

Together with the particle density $\rho(\vec{r}\,)$ or its Fourier transform $n(\vec{k}\,)$, the knowledge of the current distribution $\vec{j}(\vec{r}\,)$ is necessary for a better understanding of the properties of the system. In fact, in the description of the properties of systems in the presence of a magnetic field, the matter density $\rho(\vec{r}\,)$ and the kinematical current density $\vec{j}_{\rm kin}(\vec{r}\,)$ are the two basic ingredients used in the so-called current-density-functional theory [30].

The customary definition of the current density in the presence of a magnetic field is

$$\vec{j}(\vec{r}\,) = \vec{j}_{\rm kin}(\vec{r}\,) + \frac{e}{m^*}\,\rho(\vec{r}\,)\,\vec{A}, \tag{35}$$

where $\vec{j}_{\rm kin}(\vec{r})$ is the kinematical current density

$$\vec{j}_{\rm kin}(\vec{r}\,) = \left[\frac{\hbar}{im^*}\vec{\nabla}_{\vec{s}}\,\rho(\vec{r},\vec{s})\right]_{\vec{s}=\vec{0}} \tag{36}$$

and where the density matrix ρ has been expressed in terms of the center of mass and relative coordinates $\vec{r}$ and $\vec{s}$. The presence of the term proportional to the vector potential $\vec{A}$ guarantees that the total current density given by Eq. (35) is gauge invariant and satisfies the continuity equation

$$\vec{\nabla}\cdot\vec{j}(\vec{r}\,) = 0. \tag{37}$$

The kinematical part of the current density may be written, upon using Eqs. (8) and (9), as

$$\vec{j}_{\rm kin}(\vec{r}\,) = -\frac{m^*\Omega^2}{2\pi\hbar}\,\mathcal{L}_\lambda^{-1}\Bigg[\frac{\sinh(\beta\hbar\omega_L)}{\beta\,\sinh^2(\beta\hbar\Omega)} \times e^{-\frac{m^*\Omega\vec{r}^{\,2}}{\hbar}\left[\coth(\beta\hbar\Omega)-\frac{\cosh(\beta\hbar\omega_L)}{\sinh(\beta\hbar\Omega)}\right]}\Bigg](\vec{e}_z\times\vec{r}). \tag{38}$$

Rewriting the vector potential as $\vec{A} = B_o r\vec{e}_\phi/2$, where $\vec{e}_\phi = \vec{e}_z\times\vec{e}_r$ is the usual azimuthal unit vector, we obtain

$$\vec{j}(\vec{r}\,) = j_\phi(\vec{r}\,)\vec{e}_\phi \tag{39}$$

with

$$j_\phi(\vec{r}) = \omega_L r\rho(\vec{r}\,) - \frac{m^*\Omega^2 r}{2\pi\hbar}\,\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{\beta\,\sinh^2(\beta\hbar\Omega)}e^{-g(\beta)\vec{r}^{\,2}}\right]. \tag{40}$$

Note that the current density distribution $\vec{j}(\vec{r}\,)$ possesses only an azimuthal component. The associated flow thus performs a circular motion around the z axis along the equipotential lines of the harmonic potential $m^*\omega_o^2r^2/2$.

Expanding the exponential factor in Eq. (38) in a power series in $\vec{r}^{\,2}$, one obtains

$$\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{\beta\,\sinh^2(\beta\hbar\Omega)}e^{-g(\beta)\vec{r}^{\,2}}\right] = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}\left(\frac{m^*\Omega}{\hbar}r^2\right)^{n+m}\mathcal{L}_\lambda^{-1}\left[\frac{(e^{\beta\hbar\omega_L}-e^{-\beta\hbar\omega_L})e^{(n-m)\beta\hbar\omega_L}}{\beta}\,\frac{e^{-\frac{m^*\Omega}{\hbar}\vec{r}^{\,2}\coth(\beta\hbar\Omega)}}{(e^{\beta\hbar\Omega}-e^{-\beta\hbar\Omega})^{n+m+2}}\right]. \tag{41}$$

Making use of the generating function for the associated Laguerre polynomials [28]

$$\frac{\exp\left(-x\frac{t}{1-t}\right)}{(1-t)^{\alpha+1}} = \sum_{\ell=0}^{\infty}L_\ell^\alpha(x)t^\ell \tag{42}$$

with $x = 2\frac{m^*\Omega}{\hbar}r^2$ and $t = e^{-\beta\hbar\Omega}$ leads readily to the following identity [18]:

$$\frac{e^{-\frac{m^*\Omega}{\hbar}\vec{r}^{\,2}\coth(\beta\hbar\Omega)}}{(e^{\beta\hbar\Omega}-e^{-\beta\hbar\Omega})^{\alpha+1}} = e^{-\frac{m^*\Omega}{\hbar}r^2}\sum_{\ell=0}^{\infty}L_\ell^\alpha\left(\frac{2m^*\Omega}{\hbar}r^2\right)e^{-(2\ell+\alpha+1)\beta\hbar\Omega}, \tag{43}$$

which, used in Eq. (41) for $\alpha = n+m+1$ and taking into account the identity recalled in Eq. (20), yields

$$\begin{aligned}\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{\beta\,\sinh^2(\beta\hbar\Omega)}e^{-g(\beta)\vec{r}^{\,2}}\right] &= 2e^{-\frac{m^*\Omega}{\hbar}r^2}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}\left(\frac{m^*\Omega}{\hbar}r^2\right)^{n+m}L_\ell^{n+m+1}\left(\frac{2m^*\Omega}{\hbar}r^2\right)\\ &\quad\times\mathcal{L}_\lambda^{-1}\left\{\frac{e^{\{-(2\ell+n+m+2)\hbar\Omega+(n-m+1)\hbar\omega_L\}\beta}-e^{\{-(2\ell+n+m+2)\hbar\Omega+(n-m-1)\hbar\omega_L\}\beta}}{\beta}\right\}\\ &= 2e^{-\frac{m^*\Omega}{\hbar}r^2}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}\left(\frac{m^*\Omega}{\hbar}r^2\right)^{n+m}L_\ell^{n+m+1}\left(\frac{2m^*\Omega}{\hbar}r^2\right)\\ &\quad\times\{\Theta(\lambda-(2\ell+n+m+2)\hbar\Omega+(n-m+1)\hbar\omega_L)\\ &\quad-\Theta(\lambda-(2\ell+n+m+2)\hbar\Omega+(n-m-1)\hbar\omega_L)\}.\end{aligned} \tag{44}$$

Substituting this result into Eq. (39), we find

$$\begin{aligned}j_\phi(\vec{r}\,) &= r\omega_L\rho(\vec{r}\,) - \frac{m^*\Omega^2 r}{\pi\hbar}\,e^{-\frac{m^*\Omega}{\hbar}r^2}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}\left(\frac{m^*\Omega}{\hbar}r^2\right)^{n+m}L_\ell^{n+m+1}\left(\frac{2m^*\Omega}{\hbar}r^2\right)\\ &\quad\times\{\Theta(\lambda-(2\ell+n+m+2)\hbar\Omega+(n-m+1)\hbar\omega_L)-\Theta(\lambda-(2\ell+n+m+2)\hbar\Omega+(n-m-1)\hbar\omega_L)\}.\end{aligned} \tag{45}$$

An expansion for the density $\rho(\vec{r}\,)$ in terms of the associated Laguerre polynomials had already been given in Ref. [18]. In the present work, we have chosen to obtain this result without having recourse to a scaling transformation as done there:

$$\rho(\vec{r}) = \frac{m^*\Omega}{\pi\hbar} e^{-\frac{m^*\Omega}{\hbar}r^2} \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!m!}\left(\frac{m^*\Omega r^2}{\hbar}\right)^{n+m} L_\ell^{n+m}\left(\frac{2m^*\Omega}{\hbar}r^2\right)\Theta(\lambda-(2\ell+n+m+1)\hbar\Omega\ +(n-m)\hbar\omega_L). \quad (46)$$

With this expression, it is now obvious that the total current density $j_\phi(r)$ as given by Eq. (45) depends only on the radial variable r. It is displayed in Fig. 3 for the three different values of the magnetic field strength studied previously in Figs. 1 and 2. One observes that an increase of the field strength leads to a larger spatial extension of the current density.

With some algebraic effort, one can show that the current density $\vec{j}(r)$ given by Eq. (45) indeed satisfies the continuity Eq. (37). One can also easily show that if the confining potential is switched off, so that $\omega_o = 0$, the current density vanishes everywhere, as it should since the system becomes homogeneous in that case.

V. INTEGRATED CURRENT

We observe from Eq. (45) that the current density is the sum of two currents which are both azimuthal, yet flowing in opposite directions. To investigate their net effect, we evaluate the total azimuthal current through a radial cross section. As done in Refs. [31–33], we define the so-called integrated azimuthal current by

$$I = \int_0^\infty j_\phi(r)\,dr. \quad (47)$$

It should be noted here that in the aforementioned references the current was evaluated in the high-magnetic-field limit only. It has also been predicted there that, for systems in the presence of a magnetic field, the integrated current I is quantized in integer multiples of $\omega_L/2\pi$. Here we give an exact analytical expression for this current, valid for an arbitrary magnetic field strength. To evaluate the above integral, it is better not to use for $j_\phi(r)$ the form given in Eq. (45) but rather the initial

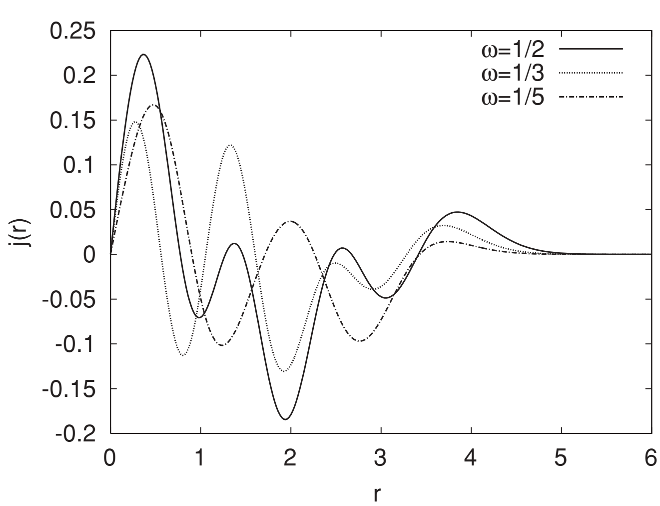

FIG. 3. Total current density for a system of $N = 60$ particles for three different values of the uniform magnetic field as given by ω.

expression of Eq. (40), which, after insertion into Eq. (47), yields

$$\begin{aligned} I &= \omega_L\int_0^\infty r\rho(r)dr - \frac{m^*\Omega^2}{2\pi\hbar}\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{\beta\sinh^2(\beta\hbar\Omega)}\int_0^\infty re^{-g(\beta)\vec{r}^{\,2}}dr\right] \\ &= \omega_L\int_0^\infty r\rho(r)\,dr - \frac{m^*\Omega^2}{2\pi\hbar}\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{2g(\beta)\beta\sinh^2(\beta\hbar\Omega)}\right]. \end{aligned} \quad (48)$$

Using the normalization of the density and the fact that $\rho(\vec{r}\,)$ depends only on the radial coordinate r

$$\int_0^\infty r\,\rho(r)\,dr = \frac{N}{2\pi}, \quad (49)$$

one obtains from Eq. (48)

$$I = \frac{N\omega_L}{2\pi} - \frac{m^*\Omega^2}{2\pi\hbar}\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{2g(\beta)\beta\sinh^2(\beta\hbar\Omega)}\right]. \quad (50)$$

To calculate the above ILT, it is better to rewrite the function $g(\beta)$ of Eq. (12) in the following form:

$$g(\beta) = \frac{2m^*\Omega}{\hbar\sinh(\beta\hbar\Omega)}\left[\sinh\frac{\beta\hbar\Omega_-}{2}\sinh\frac{\beta\hbar\Omega_+}{2}\right]. \quad (51)$$

Inserting this expression into Eq. (50), one obtains

$$\begin{aligned} I &= \frac{N\omega_L}{2\pi} - \frac{\Omega}{8\pi} \\ &\quad\times\mathcal{L}_\lambda^{-1}\left[\frac{\sinh(\beta\hbar\omega_L)}{\beta\sinh(\beta\hbar\Omega)\sinh(\beta\hbar\Omega_-/2)\sinh(\beta\hbar\Omega_+/2)}\right], \end{aligned} \quad (52)$$

and, using the power expansion of $\sinh x$, one finally gets I in the following form:

$$\begin{aligned} I &= \frac{N\omega_L}{2\pi} \\ &\quad -\frac{\Omega}{2\pi}\mathcal{L}_\lambda^{-1}\left[\frac{1}{\beta}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{e^{-(2\ell+n+m+2)\beta\hbar\Omega+(n-m+1)\beta\hbar\omega_L}\right. \\ &\quad\left. -e^{-(2\ell+n+m+2)\beta\hbar\Omega+(n-m-1)\beta\hbar\omega_L}\}\right]. \end{aligned} \quad (53)$$

The three above summations can, in fact, be reduced to two, since the first exponential term can be rewritten as

$$\begin{aligned} &\sum_{\ell=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=0}^{\infty}e^{-(2\ell+n'+m+1)\beta\hbar\Omega+(n'-m)\beta\hbar\omega_L} \\ &\quad=\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-(2\ell+n+m+1)\beta\hbar\Omega+(n-m)\beta\hbar\omega_L} \\ &\qquad-\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}e^{-(2\ell+m+1)\beta\hbar\Omega-m\beta\hbar\omega_L}, \end{aligned} \quad (54)$$

and in a similar way the second exponential term can be written as

$$\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m'=1}^{\infty} e^{-(2\ell+n+m'+1)\beta\hbar\Omega+(n-m')\beta\hbar\omega_L} = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-(2\ell+n+m+1)\beta\hbar\Omega+(n-m)\beta\hbar\omega_L} - \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} e^{-(2\ell+n+1)\beta\hbar\Omega+n\beta\hbar\omega_L}, \tag{55}$$

so that the term involving three summations in Eq. (54) cancels and one is finally left with

$$I = \frac{N\omega_L}{2\pi} - \frac{\Omega}{2\pi}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{L}_\lambda^{-1}\left[\frac{1}{\beta}(e^{-(2\ell+n+1)\beta\hbar\Omega+n\beta\hbar\omega_L} - e^{-(2\ell+n+1)\beta\hbar\Omega-n\beta\hbar\omega_L})\right] \tag{56}$$

which, with the help of Eq. (20), finally yields

$$I = \frac{N\omega_L}{2\pi} - \frac{\Omega}{2\pi}\sum_{\ell=0}^{\infty}\sum_{n=1}^{\infty}\{\Theta(\lambda-(2\ell+n+1)\hbar\Omega+n\hbar\omega_L) - \Theta(\lambda-(2\ell+n+1)\hbar\Omega-n\hbar\omega_L)\}, \tag{57}$$

where we have started the sum over n at $n=1$ since the $n=0$ contribution vanishes identically. As shown in Ref. [34], the integrated current I defined in Eq. (47) is simply the magnetization at the center of the electron system. Since the current density obeys the continuity Eq. (37), it is possible to write the current density in the form $\vec{j} = \vec{\nabla}\times\vec{M}(\vec{r})$, where $\vec{M}(\vec{r}) = M(\vec{r})\,\vec{e}_z$ is the local orbital magnetization. It is then easy to show that $I = M(\vec{r}=0)$. Our aim here is simply to show that *exact* expressions can be obtained for the integrated current for an arbitrary magnetic field strength. The fact that this quantity is, in fact, quantized has not been addressed here and is the subject of a subsequent investigation.

VI. SUMMARY AND CONCLUSIONS

Using the inverse Laplace transform of the Bloch density matrix, we have presented an analytical study of a fermionic particle system confined by a harmonic potential, in two dimensions, in the presence of an external magnetic field or within a rotating trap. We have derived rather simple analytical expressions for the density in Fourier space and for the orbital current density. An exact analytical expression for the integrated current is also obtained for any value of the magnetic field strength. We have also performed numerical computations of the density profile in Fourier space where shell oscillations are produced at all considered field strengths.

Such expressions should be useful in a wide range of physical problems. This is, in particular, the case of light scattered from harmonically confined quantum gases [12]. Using the exact expression of the orbital current density obtained in Eqs. (39), (40), and (47), one can easily derive a simple form for the total magnetization of an electron system in the presence of a magnetic field or for the moment of inertia in the case of fermions in a rotating trap. In the latter case, the moment of inertia would be determined exactly in our approach, thus complementing the semiclassical approximation estimate by Durand and coworkers using the partial $\hbar$-resummation method [35].

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APPENDIX

As in the case of a homogeneous [36] or harmonically confined quantum gas [27], the chemical potential μ in Eq. (29) is determined by the relation

$$N = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{e^{\frac{\varepsilon_{n,m}-\mu}{k_BT}}-1}, \tag{A1}$$

which may be separated into

$$N = N_o(T) + {\sum_n}'{\sum_m}'\frac{1}{e^{\frac{\varepsilon_{n,m}-\mu}{k_BT}}-1}. \tag{A2}$$

Here N_o is the number of bosons populating the lowest energy state with a single-particle energy $E_o = \varepsilon_{0,0} = \hbar\Omega$ and where the primed double summation is carried out over all the excited single-particle states (i.e., the states with quantum numbers $n+m>0$). For the finite-temperature single-particle density $\rho_T^B(\vec{r}\,) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\phi_{n,m}(\vec{r}\,)|^2(e^{\frac{\varepsilon_{n,m}-\mu}{k_BT}}-1)^{-1}$, a similar separation, as above, between ground and excited states can be used to write [27]

$$\rho_T^B(\vec{r}\,) = N_o|\phi_{0,0}(\vec{r}\,)|^2 + {\sum_n}'{\sum_m}'|\phi_{n,m}(\vec{r}\,)|^2\,\frac{1}{e^{\frac{\varepsilon_{n,m}-\mu}{k_BT}}-1}. \tag{A3}$$

Taking the Fourier transform of Eq. (A3) and using the result of Eq. (29), one obtains, using the identity $L_n(0) = L_m(0) = 1$,

$$n_T^B(\vec{k}) = \left[N_o + {\sum_n}'{\sum_m}' L_n\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)L_m\left(\frac{\hbar\vec{k}^2}{4m^*\Omega}\right)\frac{1}{e^{\frac{\varepsilon_{n,m}-\mu}{k_BT}}-1}\right]\times\exp\left(-\frac{\hbar\vec{k}^2}{4m^*\Omega}\right). \tag{A4}$$

For temperatures $T > T_c$ with some critical temperature T_c, we have $\mu < E_o$ and N_o is very small. As the gas is cooled to small temperature $T < T_c$, we have $\mu = \mu_c = E_o$ and the number of bosons in the ground state N_o becomes of the order of N. Following Wang [27], the critical temperature T_c can be defined as the temperature at which a macroscopic occupation of the lowest energy level appears. Evaluating Eq. (A2) for $T = T_c$ and $\mu = \hbar\Omega$, one obtains

$$N_o(T = T_c) = N - \sum_n{}' \sum_m{}' \frac{1}{e^{\frac{\varepsilon_{n,m}-\hbar\Omega}{k_B T_c}} - 1} = 0. \tag{A5}$$

To summarize, for $T > T_c$, and since N_o is very small, the density profile of Eq. (A4) becomes

$$n_T^B(\vec{k}\,) = \left[\sum_n{}' \sum_m{}' L_n\left(\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right) L_m\left(\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right) \frac{1}{e^{\frac{\varepsilon_{n,m}-\mu}{k_B T}} - 1} \right] \times \exp\left(-\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right), \tag{A6}$$

where μ is determined through the particle-number condition $N = \sum_n{}' \sum_m{}' [e^{\frac{\varepsilon_{n,m}-\mu}{k_B T}} - 1]^{-1}$.

For $T < T_c$, Eq. (A4) becomes

$$n_T^B(\vec{k}\,) = \left[N_o + \sum_n{}' \sum_m{}' L_n\left(\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right) L_m\left(\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right) \times \frac{1}{e^{\frac{\varepsilon_{n,m}-\hbar\Omega}{k_B T}} - 1} \right] \exp\left(-\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right). \tag{A7}$$

As $T \to 0$, the number of bosons located in the ground state, N_o, is of the order of the total number of bosons N, and Eq. (A7) reduces to the Bose-Einstein condensate density

$$n^B(\vec{k}\,) = N \exp\left(-\frac{\hbar\vec{k}^{\,2}}{4m^*\Omega}\right). \tag{A8}$$

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