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Controlling directed transport of matter-wave solitons using the ratchet effect

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We demonstrate that directed transport of bright solitons formed in a quasi-one-dimensional Bose-Einstein condensate can be reliably controlled by tailoring a weak optical lattice potential, biharmonic in both space and time, in accordance with the degree of symmetry breaking mechanism. By considering the regime where matter-wave solitons are narrow compared to the lattice period, (i) we propose an analytical estimate for the dependence of the directed soliton current on the biharmonic potential parameters that is in good agreement with numerical experiments, and (ii) we show that the dependence of the directed soliton current on the number of atoms is a consequence of the ratchet universality.

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I. INTRODUCTION

The ratchet effect, that is, directed transport without any net external force [1,2], has attracted enormous interest in the last few decades owing to its wide range of potential technological applications, including micro- and nanotechnology, as well as its relevance in biology, where ratchet mechanisms are found to underlie the working principles of molecular motors [3,4]. Directed ratchet transport (DRT) is today understood to be a result of the interplay of nonlinearity, symmetry breaking [5], and nonequilibrium fluctuations, in which these fluctuations include temporal noise [2], spatial disorder [6], and quenched temporal disorder [7]. While particle ratchets have been extensively considered in different physical contexts, such as Josephson junctions [8] and particle separation devices [9], the study of the directed transport of particle-like excitations, the so-called soliton ratchets, is a relatively recent subfield. Previous studies on soliton ratchets have focused mainly on topological solitons [7,10] and, only very recently, on matter-wave solitons in Bose-Einstein condensates (BECs) [11], where the average velocity of the soliton was shown to depend on the number of atoms in the soliton. Also, resonant ratcheting of a BEC due to the action of an external ac biharmonic driving was studied in Ref. [12], while the DRT of atoms in BECs subjected to spatiotemporal potentials, biharmonic in both space and time, was recently considered in Refs. [13] and [14].

In this work we study the DRT of nondissipative nontopological bright matter-wave solitons and provide additional confirmation of *ratchet universality* [15,16], that is, that the maximum strength of the directed soliton current is a consequence of the existence of a universal force waveform that optimally enhances DRT. Importantly, this was shown also to be the case for topological solitons in the context of superconducting Josephson arrays [7]. Since we wish the soliton shape to remain unaffected to justify its consideration as an effective quasiparticle, we assume a weak potential for matter waves in the simple case of a quasi-one-dimensional, cigar-shaped BEC [17]. At low enough temperatures (well below the critical temperature for condensation), the macroscopic (mean-field) wave function $\Psi(x,t)$ can be accurately modeled by the one-dimensional Gross-Pitaevskii (GP) equation, which, expressed in nondimensional units, reads [17]

$$i\Psi_t + \tfrac{1}{2}\Psi_{xx} + |\Psi|^2\,\Psi = V(x,t)\Psi, \tag{1}$$

where subscripts denote partial derivatives and

$$\begin{aligned} V(x,t) &= V_0 W_\omega(t)[\eta_2 \sin x + (1-\eta_2)\sin(2x+\phi_2)],\\ W_\omega(t) &\equiv [\eta_1 \sin(\omega t) + (1-\eta_1)\sin(2\omega t+\phi_1)] \end{aligned} \tag{2}$$

is the Fourier-synthesized lattice potential, where $\eta_{1,2} \in [0,1]$ and $\phi_{1,2}$ account for the relative amplitudes and phase differences of the two harmonics, respectively, while V_0 is a measure of the intensity of the laser beams forming the lattice. To facilitate the comparison with previous results (cf. Ref. [11], where the particular case $\eta_1 = 1/2$, $\phi_1 = 0$ is considered), we shall fix the frequency $\omega = 10$ throughout this work. We show below that this choice has remarkable consequences on the soliton current due to the time and space scales of the corresponding lattice potential being very different [cf. Eq. (2)].

Here we demonstrate that the relative amplitudes and the phase differences *together* play an essential role in optimally controlling the strength and direction of the soliton current according to the degree of symmetry breaking (DSB) mechanism [15]. This mechanism has led to the unveiling of a criticality scenario for DRT. Indeed, it has been shown that optimal enhancement of DRT is achieved when maximal effective (i.e., *critical*) symmetry breaking occurs, which is in turn a consequence of two reshaping-induced competing effects: the increase in the DSB and the decrease in the (normalized) maximal transmitted impulse over a half-period, thus implying the existence of a particular force waveform that optimally enhances DRT (see Ref. [15] for more details). To connect with physical experiments, note that the number of atoms in the BEC is proportional to $N = \int |\Psi|^2 dx$,

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that is, the norm of the dimensionless wave function, and that the proportionality constant depends on the particular experimental conditions [17]. The rest of the paper is organized as follows. Section II provides an effective particle model governing the dynamics of the soliton center of mass. After identifying its relevant symmetries, we propose an analytical estimate for the dependence of the soliton current on the number of atoms and the biharmonic potential parameters according to the DSB mechanism. Section III compares the analytical predictions of Sec. II with numerical results from both the GP equation (1), and the effective particle model. Finally, Sec. IV presents some concluding remarks.

II. THEORETICAL APPROACH

We consider a standard variational approach [18] to derive an ordinary differential equation governing the dynamics of the soliton center of mass by making the collective variable ansatz

$$\Psi(x,t) = \frac{N}{2}\text{sech}\left[\frac{N}{2}(x-\xi)\right]e^{i[c(x-\xi)+b]}, \tag{3}$$

where the soliton center of mass $\xi(t)$ and $c(t)$ are functions to be determined from the Euler-Lagrange equations $d(\partial L_A/\partial \dot{q})/dt = \partial L_A/\partial q$, where L_A is the effective Lagrangian evaluated at ansatz (3) and q corresponds to each variational parameter (i.e., ξ and c). Defining the soliton phase as $\Phi = c(x-\xi)$, one has that $L_A = \int_{-\infty}^{\infty} L dx$, where

$$L = -\Psi^2\Phi_t - \tfrac{1}{2}\left[\Psi_x^2 + \Psi^2\Phi_x^2\right] + \tfrac{1}{2}\Psi^4 - \Psi^2 V(x,t) \tag{4}$$

is the Lagrangian density corresponding to the GP equation (1). The Euler-Lagrange equations yield the soliton velocity $c = \dot{\xi}$ and a Newtonian equation for the soliton center of mass:

$$\frac{d^2\xi}{d\tau^2} + W_\Omega(\tau)[\eta_2\cos\xi + (1-\eta_2)\beta\cos(2\xi+\phi_2)] = 0, \tag{5}$$

where $\tau \equiv \omega t/\Omega$, $\Omega^2 \equiv \omega^2 N\sinh(\pi/N)/(\pi V_0)$, and $\beta \equiv 2\text{sech}(\pi/N)$. Equation (5) is an effective model that describes the dynamics of matter-wave solitons as point particles through the Newtonian equation

$$\frac{d^2\xi}{d\tau^2} = -\frac{dV_{\text{eff}}(\xi)}{d\xi},$$

under the action of the effective potential

$$V_{\text{eff}}(\xi) \equiv W_\Omega(\tau)\left[\eta_2\sin\xi + \frac{1-\eta_2}{2}\beta\sin(2\xi+\phi_2)\right]. \tag{6}$$

Since the DSB mechanism establishes that there exists a quantitative relationship between symmetry breaking (cause) and DRT (effect), the fact that the breakages of the space and time shift symmetries of the force derived from the lattice potential are *uncoupled* [see Eqs. (2) and (5)] means that both breakages should contribute independently (but not separately; see below) to the DRT. Thus, we propose the following scaling for the average velocity when the two kinds of symmetry are broken simultaneously:

$$\begin{aligned}\langle\overline{v}\rangle &\sim CF(\eta_1,\phi_1)G(\eta_2,\phi_2),\\ F(\eta_1,\phi_1) &\equiv \frac{\eta_1}{\omega} + \frac{1-\eta_1}{2\omega}\cos\phi_1,\\ G(\eta_2,\phi_2) &\equiv \frac{2-\eta_2+P(\eta_2)}{2-\eta_2-P(\eta_2)}\cos\phi_2,\end{aligned} \tag{7}$$

where $v \equiv d\xi/dt$, $\langle\overline{\cdot}\rangle$ indicates the average over time and initial conditions, and

$$P(\eta_2) \equiv \begin{cases}\frac{33\eta_2^2-64\eta_2+32}{16\eta_2-16} & \text{for } 0 \leqslant \eta_2 \leqslant 8/9,\\ 2-3\eta_2 & \text{for } 8/9 \leqslant \eta_2 \leqslant 1,\end{cases} \tag{8}$$

while C is a fitting constant that depends on the remaining system parameters. The scaling laws (7), come from the following considerations.

First, it has been shown [15] that there exists a universal force waveform that optimally enhances directed transport by symmetry breaking. Specifically, such a particular waveform has been shown to be unique for both temporal and spatial biharmonic forces. This universal waveform is a direct consequence of the DSB mechanism: It is possible to define a quantitative measure of the DSB on which the strength of DRT must depend. The interested reader is encouraged to find the details on the derivation of Eq. (8) in Ref. [15].

Second, analysis of the symmetries in Eq. (5) indicates that the mere breakage of the spatial shift symmetry (i.e., $0 < \eta_2 < 1$, $\phi_2 \neq 0, \pi$, $\eta_1 = 0,1$) is insufficient to yield a nonzero asymptotic soliton current [19]. However, our extensive numerical simulations (see Sec. III) showed that the soliton current does not decay noticeably after integration times $\sim 10^7 \times 2\pi/\omega$, that is, one can take such a long transient to be an asymptotic soliton current for any practical (experimental) purposes [20]. Accordingly, one could expect from the DSB mechanism that the average velocity also may *effectively* depend on the degree of breakage of the spatial shift symmetry over such an extremely long time scale. Thus, the dependence of the spatial factor $G(\eta_2,\phi_2)$ on η_2 is obtained by computing the maximal impulse (integral of the spatial force) transmitted by the (normalized; see the second paper in Ref. [15] for details) optical lattice potential at $\phi_2 = \phi_{2,\text{opt}} = 0$ over a half-period. Also, we used a first-order approximation for the dependence of the spatial factor $G(\eta_2,\phi_2)$ on ϕ_2 since a scaling in closed form is not readily available for arbitrary ϕ_2. Nonetheless, it is worth noting that for the usual case where the two spatial harmonics are in phase or in opposition, that is, $\phi_2 = 0$ or π, respectively, such a first-order approximation is exact. It is important to stress at this point that the specific form of the measure of the DSB is not crucial to obtain approximate scaling laws like Eqs. (7). What is most important from the ratchet universality application is that one can obtain a good approximation for the *optimal* parameter values that *maximize* ratcheting [cf. Eq. (9)].

Third, the temporal factor $F(\eta_1,\phi_1)$ is obtained by noting that the matter-wave soliton as a point particle undergoes an adiabatically varying spatial optical lattice potential on average over the relevant time scale $2\pi/\omega$. This means that the dependence of the average velocity on the breakage of the symmetries associated with the temporal force should

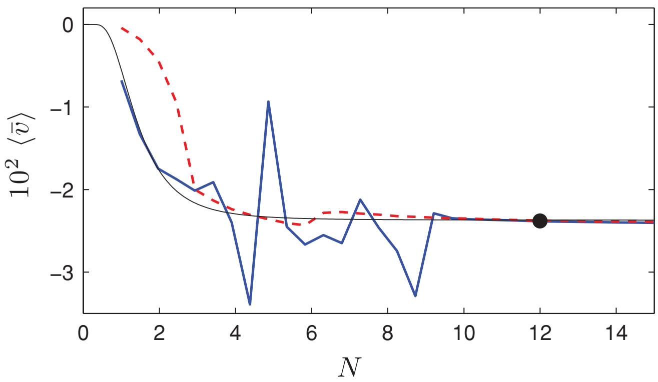

FIG. 1. (Color online) Average velocity $\langle \overline{v} \rangle$ versus soliton effective mass N calculated using the GP equation (1) [thick solid (blue) line] and the effective model of Eq. (5) [dashed (red) line]. Also plotted is the scaling law (7) with $\eta_2 = \eta_{2,\mathrm{opt}}(N)$ [cf. Eq. (9); thin solid (black) line]. Fixed parameters: $V_0 = 1$, $\omega = 10$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\phi_{1,2} = 0$, and $C = -0.79$.

be fairly *insensitive* to the specific spatial dependence of the potential. Thus, one can consider a linear (local) spatial dependence instead, $V_{\mathrm{eff}}(\xi) \equiv W_\Omega(\tau)\xi$, to readily obtain an analytical estimate accounting for the effect of the temporal force, which is the function $F(\eta_1,\phi_1)$ in Eq. (7).

It is worth noting that the ratchet universality [15] predicts that the optimal value of the relative amplitude η_2 comes from the condition that the amplitude of $\cos\xi$ must be twice as large as that of $\cos(2\xi + \phi_2)$ in Eq. (5) and, hence, that it *depends* on the soliton's size:

$$\eta_{2,\mathrm{opt}} = \left[1 + \frac{1}{2\beta}\right]^{-1} = \left[1 + \frac{\cosh(\pi/N)}{4}\right]^{-1}, \qquad (9)$$

which satisfies $\lim_{N\to\infty} \eta_{2,\mathrm{opt}} = 4/5$ and also $\lim_{N\to 0} \eta_{2,\mathrm{opt}} = 0$. Interestingly, this optimal ratcheting value for η_2 induces a ratcheting where (i) larger solitons (for sufficiently large N) move faster and (ii) the soliton current vanishes for sufficiently small N [11]—this is confirmed by our own simulations (see Fig. 1). Indeed, from the scaling law (7), we see that the spatial factor $G(\eta_2,\phi_2)$ (and hence the average velocity) presents, as a function of η_2, a single maximum at $\eta_2 = 4/5$ [i.e., $N \to \infty$; cf. Eq. (9)] that vanishes at $\eta_2 = 0$ [i.e., $N = 0$; cf. Eq. (9)]. Furthermore, as demonstrated in Fig. 1, the average velocity tends to saturate rather quickly as N is increased. We therefore assume a clearly localized soliton in the scaling (7), for which the ratchet dynamics is well described by the effective particle model (5).

III. NUMERICAL RESULTS

We found good agreement between the scaling, Eq. (7), and extensive numerical results from both the GP equation (1), and the effective particle model, Eq. (5). Some illustrative examples are depicted in Figs. 2–4, where the average velocity $\langle \overline{v} \rangle$ is plotted versus different lattice potential parameters for $V_0 = 1$, $\omega = 10$, and $N = 12$. This value for N ensures the convergence of the effective model to the original GP equation (1). This, in turn, allows us to efficiently model the matter-wave soliton as an effective quasiparticle—as long as the quasi-one-dimensional mean-field model (1), remains

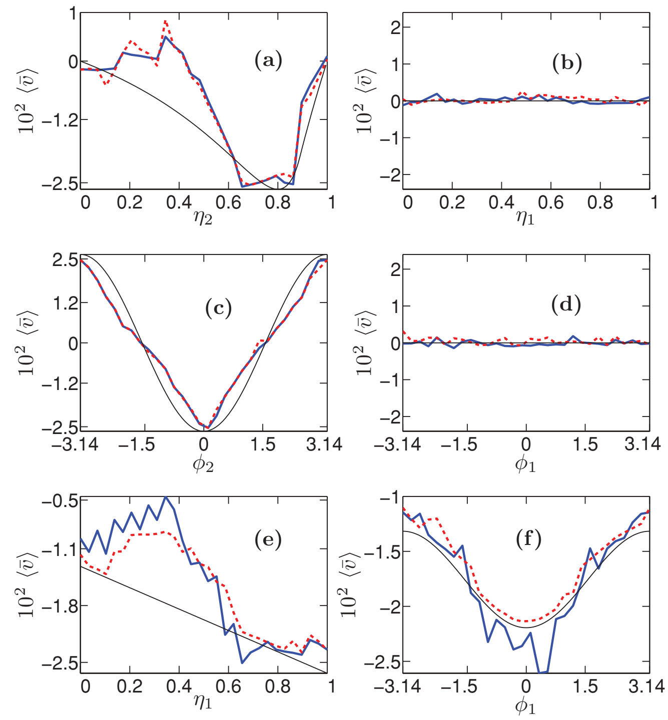

FIG. 2. (Color online) Average velocity $\langle \overline{v} \rangle$ versus (a) relative amplitude η_2 for $\phi_2 = \pi$ and $\eta_1 = 1$ (unbroken time shift symmetry), (b) relative amplitude η_1 for $\phi_1 = 0$ and $\eta_2 = 1$ (unbroken space shift symmetry), (c) phase difference ϕ_2 for $\eta_2 = 4/5$ and $\eta_1 = 1$ (unbroken time shift symmetry), (d) phase difference ϕ_1 for $\eta_1 = 2/3$ and $\eta_2 = 1$ (unbroken space shift symmetry), (e) relative amplitude η_1 for $\phi_1 = 0$, $\eta_2 = 4/5$, and $\phi_2 = \pi$, and (f) phase difference ϕ_1 for $\eta_1 = 2/3$, $\eta_2 = 4/5$, and $\phi_2 = \pi$. Results were calculated using the GP equation (1) [thick (blue) solid lines] and the effective particle model (5) [dashed (red) lines]. Also plotted is the scaling law (7) [thin (black) solid lines]. Fixed parameters: $V_0 = 1$, $N = 12$, $\omega = 10$, and $C = -0.79$.

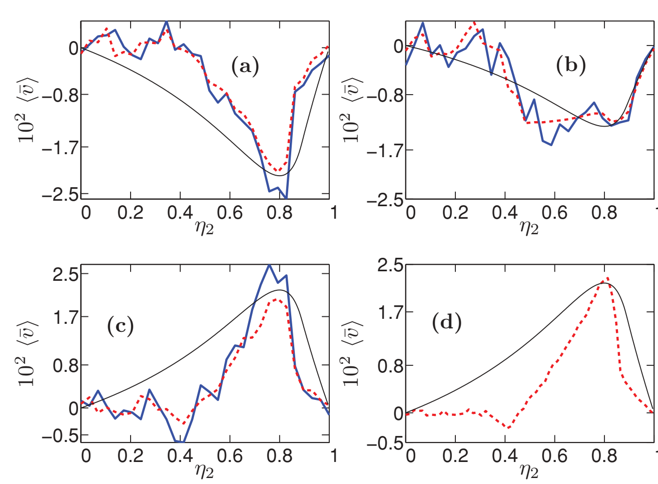

FIG. 3. (Color online) Average velocity $\langle \overline{v} \rangle$ versus relative amplitude η_2 for (a) $\eta_1 = 2/3$, $\phi_1 = \phi_2 = 0$, and $N = 12$; (b) $\eta_1 = 2/3$, $\phi_1 = \pi$, $\phi_2 = 0$, and $N = 12$; (c) $\eta_1 = 2/3$, $\phi_1 = 0$, $\phi_2 = \pi$, and $N = 12$; and (d) $\eta_1 = 2/3$, $\phi_1 = 0$, $\phi_2 = \pi$, and $N \to \infty$. Results presented as in Fig. 2 for fixed parameters: $V_0 = 1$, $\omega = 10$, and $C = -0.79$.

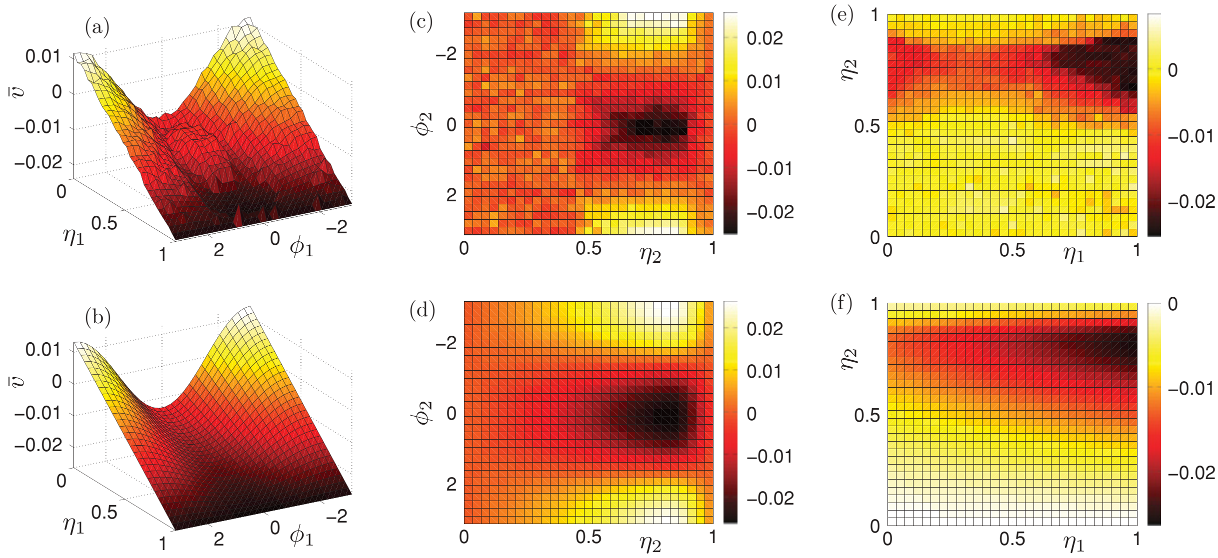

FIG. 4. (Color online) Average velocity $\langle\overline{v}\rangle$ versus (a, b) η_1 and ϕ_1 for $\eta_2 = 4/5$, and $\phi_2 = 0$; (c, d) η_2 and ϕ_2 for $\eta_1 = 0$, and $\phi_1 = 0$; (e, f) η_1 and η_2 for $\phi_1 = \phi_2 = 0$. (a, c, e) Results calculated using the effective particle model (5); (b, d, f) corresponding results from the scaling law (7). Fixed parameters: $V_0 = 1$, $N = 12$, $\omega = 10$, and $C = -0.79$.

valid, that is, for low enough temperatures. As predicted from the ratchet universality, a single value of the fitting constant C $(= -0.79)$ is enough to fit the scaling law to *all* numerical results. Note that this in turn provides strong support for the effective particle model (5).

Figures 2(b) and 2(d) confirm that the breakage of the space shift symmetry is a necessary condition to have a nonzero soliton current as expected [19], while Figs. 2(a) and 2(c) show that even transient soliton currents for *unbroken* time shift symmetry follow the ratchet universality, which was an unanticipated result. Indeed, for the case of a symmetric temporal force ($\eta_1 = 0$) and an asymmetric spatial force, one typically finds that the maximum average velocity is achieved over a certain range around the optimal relative amplitude $\eta_{2,\mathrm{opt}} \equiv 4/5$ and at the optimal phase differences $\phi_{2,\mathrm{opt}} \equiv 0,\pi$ [cf. Eq. (7)], where the two values of ϕ_2 correspond to soliton currents in opposite directions [see Figs. 2(a) and 2(c)]. For the case of asymmetric spatial and temporal forces, one obtains a similarly good agreement between the theoretical scaling and the numerics. In particular, the dependence of the average velocity on the temporal force parameters confirms reasonably well the implications of having very different space and time scales in the lattice potential, as can be appreciated in Figs. 2(e) and 2(f).

Additional confirmation of the scale analysis described by Eq. (7) is provided in Fig. 4, where the results from the effective particle model (5) [Figs. 4(a), 4(c), and 4(e)] are in excellent agreement with those corresponding to the scaling (7) [Figs. 4(b), 4(d), and 4(f)]. Remarkably, for the case of asymmetric spatial and temporal forces, the dependence of the average velocity on the relative amplitude of the spatial force provides additional unanticipated support for the DSB mechanism. Indeed, when the breakages of the shift symmetries of both kinds of force induce soliton currents in the *same* direction [as in Fig. 3(a), where $\phi_1 = \phi_2 = 0$], the maximum of the average velocity is larger than that corresponding to the case where the induced soliton currents are *in opposition* [as in Fig. 3(b), where $\phi_1 = \pi, \phi_2 = 0$]. It is worth noting that both maxima occur over certain ranges around the predicted optimal value $\eta_{2,\mathrm{opt}} = 4/5$, the range in the former case being much smaller than in the latter case, while an intermediate range appears in the case of a symmetric temporal force [$\eta_1 = 0$; cf. Fig. 2(a)]. Finally, a comparison of the average velocities of two solitons subjected to the same lattice potential but with different sizes [$N = 12$, Fig. 3(c), and $N \to \infty$, Fig. 3(d)] provides additional confirmation that larger solitons move faster. The corresponding optimal values of the relative amplitudes for $N = 12$ and $N \to \infty$ are, respectively, $\eta_{2,\mathrm{opt}} = 0.794523$ and $\eta_{2,\mathrm{opt}} = 4/5$ [cf. Eq. (9)], confirming in turn the ratchet universality. Finally, let us stress, as pointed out before, that the theoretical optimal parameter values match extremely well the maximal ratcheting scenarios in Figs. 2 and 3; specifically, the extrema in Figs. 2(a), 3(b), 3(c), and 3(d) are reached very close to the optimal value $\eta_{2,\mathrm{opt}} = 4/5$.

IV. CONCLUSIONS

In summary, we have demonstrated that optimal control of directed transport of matter-wave solitons can be achieved by suitably tailoring a weak optical lattice potential, biharmonic in both space and time, in accordance with the degree-of-symmetry-breaking mechanism. This mechanism has allowed us to propose an analytical scaling law for the strength of the directed soliton current that is fully confirmed by numerics. Additionally, it explains why the directed soliton current is dependent on the number of atoms, a result previously found numerically [11]. We expect that the present findings should be readily testable by experiment on the basis of experimental advances in implementing ratchets for BECs [13] and could find applications in coherent atom optics, atom interferometry, and atom transport [20]. In particular, application of the ratchet universality to atomic soliton lasers [21] could be possible, in principle, by considering a suitable temporal variation of Feshbach resonances so that the relevant symmetries of the corresponding effective equation for the soliton center of mass may be broken. Additional applications to soliton accelerators [22] would deserve further exploration following the same line of thinking. Finally, another interesting point that warrants further investigation is the apparent lack of decay in the soliton current that we observed when the temporal symmetry is not broken.

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