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Discrimination of the change point in a quantum setting

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In the change point problem, we determine when the observed distribution has changed to another one. We expand this problem to a quantum case where copies of an unknown pure state are being distributed. That is, we estimate when the distributed quantum pure state is changed. As the most fundamental case, we treat the problem of deciding the true change point t_c between the two given candidates t_1 and t_2. Our problem is mathematically equal to identifying a given state with one of the two unknown states when multiple copies of the states are provided. The minimum of the averaged error probability is given and the optimal positive operator-valued measure (POVM) is given to obtain it when the initial and final quantum pure states are subject to the invariant prior. We also compute the error probability for deciding the change point under the above POVM when the initial and final quantum pure states are fixed. These analytical results allow us to calculate the value in the asymptotic case.

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I. INTRODUCTION

The change point problem, which is studied in many fields (e.g., statistics [1]), originally arose out of considerations of quality control. When a process is "in control," products are produced according to some rule. At an unknown point, the process jumps "out of control" and ensuing products are produced according to another rule. It is necessary to determine the change point.

In the classical case, we observe sequentially a discrete series of independent observations $X_0, X_1, \ldots$ whose distribution possibly changes at an unknown point in time. It is assumed that independent random variables $X_0, X_1, \ldots, X_{\nu-1}$ are each distributed according to some distribution and the remaining independent random variables $X_\nu, X_{\nu+1}, \ldots$ are each distributed according to another distribution. Our purpose is to detect the change point ν [1].

We extend this problem to a quantum setting where copies of an unknown pure state are being distributed in discrete time. We now consider the device distributing copies of an unknown pure state. This device has an unknown change point and distributes copies of another unknown pure state after the change point. In this setting, the device distributes unknown pure states $\hat{\rho}_t$ on the d-dimensional space for the discrete time $t = 0,1,2,\ldots,t_3$. The state $\hat{\rho}_t$ is changed at the change point t_c. That is, the states $\hat{\rho}_0, \ldots, \hat{\rho}_{t_c-1}$ are identical, and the other states $\hat{\rho}_{t_c}, \ldots, \hat{\rho}_{t_3}$ are also identical.

In this paper, we deal with the most fundamental case; that is, we have only two candidates, t_1 and t_2, for the change point t_c. Our goal is to determine whether the true change point t_c is t_1 or t_2. In order to analyze this problem, we introduce systems 1, 0, and 2 to denote the composite systems corresponding to the time periods $0 \leqslant t < t_1$, $t_1 \leqslant t < t_2$, and $t_2 \leqslant t \leqslant t_3$, respectively, as explained in Fig. 1. Our task is then to choose the correct change point t_c between the two candidates t_1 and t_2 by using all three systems. This problem is equal to deciding whether the state in system 0 coincides with the state in system 1 or the state in system 2.

When we denote the unknown initial pure state $\hat{\rho}_0$ and the unknown final pure state $\hat{\rho}_{t_3}$ by ρ_1 and ρ_2, we have t_1 copies of ρ_1 in system 1 and $t_3 - t_2 + 1$ copies of ρ_2 in system 2. Then, our problem is to decide whether the state in system 0 is ρ_1 or ρ_2. Thus, when we choose the parameters M, N_1, and N_2 by

$$M := t_2 - t_1, \quad N_1 := t_1, \quad N_2 := t_3 - t_2 + 1, \tag{1}$$

this problem is mathematically equivalent to quantum discrimination for programmable devices for pure qudit states with $N_1 \times M \times N_2$. In this problem, it is important to derive the optimal positive operator-valued measure (POVM) and the minimum of the averaged error probability with the group invariant prior. Hayashi *et al.* derived them when $N_1 = N_2$ and $M = 1$, i.e., $t_1 = t_3 - t_2 + 1$ and $t_2 - t_1 = 1$. However, Sentis *et al.* [2] derived the correct optimal POVM and the correct minimum of the averaged error probability in the symmetric case (i.e., the case of $N_1 = N_2$), i.e., $t_1 = t_3 - t_2 + 1$. As described in the Erratum by Sentis *et al.* [2],[1] they have an error for the derivation in the asymmetric case (i.e., the case of $N_1 \neq N_2$), i.e., $t_1 \neq t_3 - t_2 + 1$. It is natural to restrict our analysis to the symmetric case from the viewpoint of quantum discrimination for programmable devices; however, this restriction is not natural from the viewpoint of discrimination of the change point. So, it is important to treat the asymmetric case from the latter viewpoint. The first and our main result is to derive the optimal POVM and the minimum of the averaged error probability in the general setting with the qudit system including the asymmetric case.

In order to consider the performance of our optimal POVM under an arbitrary prior, we need to compute the nonaveraged error probability under the application of our optimal POVM, which depends on the inner product of two states in system 1 and system 2. The second result is to calculate this value, which has not been calculated in the existing results [2,3].

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[1]The Erratum [2] reflects the indication of the preprint version of this paper.

As the numbers of copies in three systems approach infinity, this error probability clearly exponentially approaches 0 unless the inner product between the density matrices ρ_1 and ρ_2 is 1. Hence, the convergence speed can be measured by the exponentially decreasing rate. When we have perfect knowledge of the density matrices ρ_1 and ρ_2, our problem is to determine whether the state in system 0 is ρ_1 or ρ_2. Then, the exponentially decreasing rate coincides with the quantum Chernoff bound [4]. Since we have only N_1 copies of ρ_1 and N_2 copies of ρ_2 as quantum states in our problem, our optimal POVM has a smaller exponentially decreasing rate of the nonaveraged error probability than the quantum Chernoff bound. In this paper, for simplicity we assume the symmetric case (i.e., the case of $N_1 = N_2$). As the third result, using the analytical result for the nonaveraged error probability, we derive the above exponentially decreasing rate when $N_1 = N_2$, increasing in proportion with M. We clarify the relationship between the quantum Chernoff bound and our exponentially decreasing rate, which depends on this proportional constant.

The paper is organized as follows. In the next section we give the optimal strategy for the minimum averaged error probability. The optimal POVM is described by using the representation theory for easy calculation in the following sections. In Sec. III we obtain the minimum error probability represented as a function of the number of copies in each of the three systems and the dimension of the state space. Using the optimal POVM, in Sec. IV we compute the error probability which depends on the inner product of two states in system 1 and system 2. In Sec. V we consider the asymptotic behaviors of the minimum averaged error probability in several scenarios. In Sec. VI, we finally compute the convergence speed of the error probability. Some brief conclusions follow and we end with a technical Appendix.

II. THE OPTIMAL POVM

In order to derive the optimal POVM, we treat our problem by using the parameters N_1, M, and N_2 given in Eqs. (1) instead of t_1, t_2, and t_3. In the following, we denote the unknown initial pure state by ρ_1 and the unknown final pure state by ρ_2 on the d-dimensional vector space $\mathbb{C}^d$. That is, we have N_1 (N_2) copies of ρ_1 (ρ_2) in system 1 (2) in Fig. 1. System 0 has M copies of the unknown state ρ that is guaranteed to be one of either ρ_1 or ρ_2. Note that we assume that $N_1 \leqslant N_2$, which creates no loss of generality in this problem. Our purpose is to identify the state ρ with one of the two states by using all systems. This is equal to distinguishing two states $\rho_1^{\otimes N_1} \otimes \rho_1^{\otimes M} \otimes \rho_2^{\otimes N_2}$ and $\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M} \otimes \rho_2^{\otimes N_2}$, which are assumed to occur with equal probability.

In this decision problem, we apply a two-valued POVM $\{E_1, E_2\}$, in which E_1 (E_2) corresponds to the decision

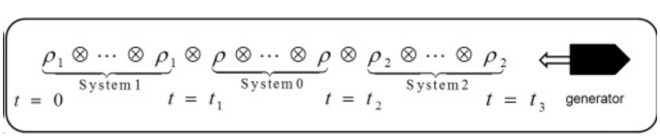

FIG. 1. Discrete time series of quantum systems produced by quantum state generator. When $t_c = t_1$, the state ρ is ρ_2. When $t_c = t_2$, the state ρ is ρ_1.

$t_c = t_1$ ($t_c = t_2$). Since $E_1 = I - E_2$, our POVM can be described by a Hermitian matrix $0 \leqslant E_2 \leqslant I$, where I is the unit matrix. Then, the error probability is given as

$$\begin{aligned} p_{M,N_1,N_2}(\rho_1,\rho_2,E_2) &\equiv \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2}\right)E_2\right] \\ &\quad + \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2}\right)E_1\right] \\ &= \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2}\right)E_2\right] \\ &\quad + \tfrac{1}{2}\,\mathrm{Tr}\left[\left(\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2}\right)(I-E_2)\right]. \end{aligned} \tag{2}$$

Now, we assume that ρ_1 and ρ_2 are independently distributed according to the unitary invariant distribution μ_{Θ_d} on the set Θ_d of pure states on the d-dimensional vector space $\mathbb{C}^d$. By using the POVM $\{I - E_2, E_2\}$, we can define the averaged error probability as

$$\begin{aligned} \overline{p}_{M,N_1,N_2}(E_2) \equiv \int_{\Theta_d}\int_{\Theta_d} & p_{M,N_1,N_2}(\rho_1,\rho_2,E_2)\mu_{\Theta_d} \\ & \times(d\rho_1)\mu_{\Theta_d}(d\rho_2). \end{aligned} \tag{3}$$

Here it is very helpful to use the following formula for the integral of the tensor product of L identically prepared pure states [3]:

$$\int_{\Theta_d} \sigma^{\otimes L}\mu_{\Theta_d}(d\sigma) = \frac{I_L}{\mathrm{Tr}[I_L]}, \tag{4}$$

where σ is a pure state and I_L is the projector onto the totally symmetric subspace of $(\mathbb{C}^d)^{\otimes L}$.

By using this formula, the averaged error probability reads

$$\begin{aligned} \overline{p}_{M,N_1,N_2}(E_2) = \frac{1}{2}\Bigg\{1 + \mathrm{Tr}\Bigg[\Bigg(&\frac{I_{N_1} \otimes I_{M+N_2}}{A_1} \\ &- \frac{I_{M+N_1} \otimes I_{N_2}}{A_2}\Bigg)E_2\Bigg]\Bigg\}, \end{aligned} \tag{5}$$

where A_1 and A_2 are defined as follows:

$$\begin{aligned} A_1 &\equiv \mathrm{Tr}\left[I_{N_1}\right]\mathrm{Tr}\left[I_{M+N_2}\right] \\ &= \binom{N_1+d-1}{d-1}\binom{M+N_2+d-1}{d-1}, \end{aligned} \tag{6}$$

$$\begin{aligned} A_2 &\equiv \mathrm{Tr}\left[I_{M+N_1}\right]\mathrm{Tr}\left[I_{N_2}\right] \\ &= \binom{M+N_1+d-1}{d-1}\binom{N_2+d-1}{d-1}. \end{aligned} \tag{7}$$

Note that $A_1 \leqslant A_2$, and the equation holds if and only if $N_1 = N_2$.

Equation (5) guarantees that the optimal strategy to minimize the averaged error probability is given by the Hermitian matrix

$$E_{M,N_1,N_2} \equiv \left\{\frac{I_{N_1} \otimes I_{M+N_2}}{A_1} - \frac{I_{M+N_1} \otimes I_{N_2}}{A_2} < 0\right\}, \tag{8}$$

where $\{A < 0\}$ represents a projector onto the eigenspaces with negative eigenvalues of A. That is, plugging Eq. (8) into Eq. (5), one obtains the minimum averaged error probability.

In order to compute the minimum averaged error probability, we deform the expression of the optimal POVM by using the tensor product representation of the unitary group $\mathrm{U}(d)$. Any irreducible representation of the unitary group $\mathrm{U}(d)$ is characterized by a Young diagram $\boldsymbol{\lambda} = [\lambda_1,\lambda_2,\ldots,\lambda_d]$ and is denoted by $\mathcal{U}_{\boldsymbol{\lambda}}$. We use the shorthand notations λ_1 and $[\lambda_1,\lambda_2]$ to denote $[\lambda_1,0,0,\ldots,0]$ and $[\lambda_1,\lambda_2,0,0,\ldots,0]$. Note that $\mathcal{U}_L$ means the totally symmetric subspace of $(\mathbb{C}^d)^{\otimes L}$. The dimension of $\mathcal{U}_{[\lambda_1,\lambda_2]}$ is given as

$$\dim \mathcal{U}_{[\lambda_1,\lambda_2]} = \frac{(\lambda_1+d-1)!(\lambda_2+d-2)!(\lambda_1-\lambda_2+1)}{(d-1)!(d-2)!(\lambda_1+1)!\lambda_2!}. \tag{9}$$

In our problem, the total system size is $N \equiv M + N_1 + N_2$, and the total tensor product space $(\mathbb{C}^d)^{\otimes N}$ can be decomposed to

$$(\mathbb{C}^d)^{\otimes N} = \oplus_{\boldsymbol{\lambda}} \mathcal{U}_{\boldsymbol{\lambda}} \otimes \mathcal{V}_{\boldsymbol{\lambda}}, \tag{10}$$

where $\mathcal{V}_{\boldsymbol{\lambda}}$ corresponds to the multiplicity of the irreducible space $\mathcal{U}_{\boldsymbol{\lambda}}$.

Since the tensor product space $(\mathbb{C}^d)^{\otimes N}$ contains the two subspaces $\mathcal{U}_{N_1} \otimes \mathcal{U}_{M+N_2}$ and $\mathcal{U}_{M+N_1} \otimes \mathcal{U}_{N_2}$ without multiplicity, these two subspaces have the form

$$\mathcal{U}_{N_1} \otimes \mathcal{U}_{M+N_2} = \oplus_{k=0}^{N_1} \mathcal{U}_{[N-k,k]} \otimes \mathbb{C}|u_k\rangle, \tag{11}$$

$$\mathcal{U}_{M+N_1} \otimes \mathcal{U}_{N_2} = \oplus_{k=0}^{\min(M+N_1,N_2)} \mathcal{U}_{[N-k,k]} \otimes \mathbb{C}|v_k\rangle, \tag{12}$$

by using two normalized vectors,

$$|u_k\rangle \in \mathcal{V}_{[N-k,k]} (0 \leqslant k \leqslant N_1), \tag{13}$$

$$|v_k\rangle \in \mathcal{V}_{[N-k,k]} (0 \leqslant k \leqslant \min[M+N_1,N_2]), \tag{14}$$

satisfying $\langle u_k|v_k\rangle \geqslant 0$. Since the dimension of $\mathcal{V}_N$ is 1, the relation $|u_0\rangle = |v_0\rangle$ holds.

Letting $I_{\boldsymbol{\lambda}}$ be the projector onto the space $\mathcal{U}_{\boldsymbol{\lambda}}$, from Eqs. (11) and (12) one obtains

$$I_{N_1} \otimes I_{M+N_2} = \sum_{k=0}^{N_1} I_{[N-k,k]} \otimes |u_k\rangle\langle u_k|, \tag{15}$$

$$I_{M+N_1} \otimes I_{N_2} = \sum_{k=0}^{\min(M+N_1,N_2)} I_{[N-k,k]} \otimes |v_k\rangle\langle v_k|. \tag{16}$$

Here, we note that the ranges of summations are different from each other. Using these equations, one has

$$\begin{aligned} \frac{I_{N_1} \otimes I_{M+N_2}}{A_1} - \frac{I_{M+N_1} \otimes I_{N_2}}{A_2} &= \sum_{k=0}^{N_1} I_{[N-k,k]} \otimes \left(\frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2} \right) \\ &\quad - \sum_{k=N_1+1}^{\min(M+N_1,N_2)} I_{[N-k,k]} \otimes \frac{|v_k\rangle\langle v_k|}{A_2}. \end{aligned} \tag{17}$$

Since $|u_k\rangle$ and $|v_k\rangle$ are linearly independent in the range $1 \leqslant k \leqslant N_1$ [shown in Eq. (29)], there exists only one negative eigenvalue of $\frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2}$. Using the normalized eigenvector $|w_k\rangle \in \mathcal{V}_{[N-k,k]}$ with this eigenvalue, we therefore can write the optimal POVM as

$$E_{M,N_1,N_2} = \sum_{k=1}^{N_1} I_{[N-k,k]} \otimes |w_k\rangle\langle w_k| + \sum_{k=N_1+1}^{\min(M+N_1,N_2)} I_{[N-k,k]} \otimes |v_k\rangle\langle v_k|. \tag{18}$$

III. THE MINIMUM AVERAGED ERROR PROBABILITY

In this section, we compute the minimum averaged error probability. Plugging Eqs. (17) and (18) into Eq. (5) as $E_2 = E_{M,N_1,N_2}$, one obtains

$$\begin{aligned} \overline{p}_{M,N_1,N_2}(E_{M,N_1,N_2}) &= \frac{1}{2}\Bigg[\sum_{k=1}^{N_1} \mathrm{Tr}[I_{[N-k,k]}] \langle w_k| \left(\frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2} \right) |w_k\rangle \\ &\quad - \frac{1}{A_2} \sum_{k=N_1+1}^{\min(M+N_1,N_2)} \mathrm{Tr}[I_{[N-k,k]}] + 1 \Bigg]. \end{aligned} \tag{19}$$

When two arbitrary real nonzero constants C_1 and C_2 and two linearly independent normalized vectors $|a\rangle$ and $|b\rangle$ are given, the unique negative eigenvalue of $\frac{|a\rangle\langle a|}{C_1} - \frac{|b\rangle\langle b|}{C_2}$ is given by

$$\frac{C_2 - C_1 - \sqrt{(C_2 - C_1)^2 + 4C_1C_2(1 - |\langle a|b\rangle|^2)}}{2C_1C_2}. \tag{20}$$

Therefore, the eigenvector $|w_k\rangle$ associated with the negative eigenvalue satisfies

$$\langle w_k| \left(\frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2} \right) |w_k\rangle = \frac{A_2 - A_1 - \sqrt{(A_2 - A_1)^2 + 4A_1A_2(1 - |\langle u_k|v_k\rangle|^2)}}{2A_1A_2}. \tag{21}$$

We also obtain the following equations:

$$\mathrm{Tr}[I_{[N-k,k]}] = \dim \mathcal{U}_{[N-k,k]}, \tag{22}$$

$$\sum_{k=0}^{N_1} \mathrm{Tr}[I_{[N-k,k]}] = A_1, \tag{23}$$

$$\sum_{k=0}^{\min(M+N_1,N_2)} \mathrm{Tr}[I_{[N-k,k]}] = A_2. \tag{24}$$

Using these equations, we can write the minimum averaged error probability as

$$\begin{aligned} \overline{p}_{M,N_1,N_2}(E_{M,N_1,N_2}) &= \frac{1}{4}\Bigg[\frac{A_1 + A_2}{A_2} - \sum_{k=0}^{N_1} \frac{\dim \mathcal{U}_{[N-k,k]}}{A_1A_2} \\ &\quad \times \sqrt{(A_2 - A_1)^2 + 4A_1A_2(1 - |\langle u_k|v_k\rangle|^2)} \Bigg]. \end{aligned} \tag{25}$$

Our remaining task is to calculate the inner product $\langle u_k|v_k\rangle$. When we denote the highest weight vector of the space $\mathcal{U}_{[N-k,k]}$ by $|[N-k,k]^d\rangle$, $\langle u_k|v_k\rangle$ is equal to the inner product of $|[N-k,k]^d\rangle|u_k\rangle$ and $|[N-k,k]^d\rangle|u_k\rangle$. We can assume $d=2$ without loss of generality since the inner product does not depend on the dimension. Let us fix some notations as follows:

$$\mu_0 \equiv \frac{M}{2}, \quad \mu_1 \equiv \frac{N_1}{2}, \quad \mu_2 \equiv \frac{N_2}{2},$$
$$\mu_{01} \equiv \frac{M+N_1}{2}, \quad \mu_{02} \equiv \frac{M+N_2}{2}, \quad \mu \equiv \frac{N}{2}-k. \tag{26}$$

Using Wigner's 6-j function [5], we then can write

$$\langle u_k|v_k\rangle = (-1)^{\mu_0+\mu_1+\mu_2+\mu}\sqrt{(2\mu_{01}+1)(2\mu_{02}+1)} \times \begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix}. \tag{27}$$

Moreover, Wigner's 6-j function can be computed as

$$\begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix} = \frac{(-1)^{\mu_0+\mu_1+\mu_2+\mu}}{\sqrt{(2\mu_{01}+1)(2\mu_{02}+1)}}\sqrt{\frac{\binom{N_1}{k}\binom{N_2}{k}}{\binom{M+N_1}{k}\binom{M+N_2}{k}}} \tag{28}$$

(see Appendix). Thus, one obtains

$$\langle u_k|v_k\rangle = \sqrt{\frac{\binom{N_1}{k}\binom{N_2}{k}}{\binom{M+N_1}{k}\binom{M+N_2}{k}}}, \tag{29}$$

and in order to denote this value we use the notation ϕ_k satisfying

$$\cos\phi_k = \sqrt{\frac{\binom{N_1}{k}\binom{N_2}{k}}{\binom{M+N_1}{k}\binom{M+N_2}{k}}} = \langle u_k|v_k\rangle. \tag{30}$$

Therefore, the minimum averaged error probability can be written as

$$\overline{p}_{M,N_1,N_2}(E_{M,N_1,N_2}) = \frac{1}{4}\left[\frac{A_1+A_2}{A_2} - \sum_{k=0}^{N_1}\frac{\dim\mathcal{U}_{[N-k,k]}}{A_1A_2} \times\sqrt{(A_2-A_1)^2+4A_1A_2\sin^2\phi_k}\right]. \tag{31}$$

In the case of $d=2$, since

$$A_1 = (N_1+1)(M+N_2+1),$$
$$A_2 = (M+N_1+1)(N_2+1),$$
$$A_2 - A_1 = M(N_2-N_1),$$
$$\dim\mathcal{U}_{[N-k,k]} = M+N_1+N_2-2k+1,$$

Eq. (31) is calculated to the following way. Our result is different from the original version of Sentis *et al.* [2]; however, their Erratum coincides with our result (31):

$$\overline{p}_{M,N_1,N_2}(E_{M,N_1,N_2}) = \frac{1}{4}\Bigg\{1+\frac{(N_1+1)(M+N_2+1)}{(M+N_1+1)(N_2+1)} - \sum_{k=0}^{N_1}\frac{M+N_1+N_2-2k+1}{(N_1+1)(M+N_2+1)(M+N_1+1)(N_2+1)} \times\sqrt{M^2(N_2-N_1)^2+4(N_1+1)(M+N_2+1)(M+N_1+1)(N_2+1)\left[1-\left(\frac{N_1!(M+N_1-k)!}{(M+N_1)!(N_1-k)!}\right)\left(\frac{N_2!(M+N_2-k)!}{(M+N_2)!(N_2-k)!}\right)\right]}\Bigg\}. \tag{32}$$

When $N_1=N_2$ (i.e., $t_1=t_3-t_2+1$), the equation $A_1=A_2$ holds and this probability is concretely computed as

$$\overline{p}_{M,N_1,N_1}(E_{M,N_1,N_1}) = \frac{1}{2}\left[1-\frac{(d-1)N_1!(M+N_1)!}{(N_1+d-1)!(M+N_1+d-1)!}\sum_{k=0}^{N_1}(M+2N_1-2k+1)\frac{(M+2N_1-k+d-1)!(k+d-2)!}{(M+2N_1-k+1)!k!} \times\sqrt{1-\left(\frac{N_1!(M+N_1-k)!}{(M+N_1)!(N_1-k)!}\right)^2}\right]. \tag{33}$$

Moreover, plugging $d=2$ into this equation, we have

$$\overline{p}_{M,N_1,N_1}(E_{M,N_1,N_1}) = \frac{1}{2}\left[1-\sum_{k=0}^{N_1}\frac{M+2N_1-2k+1}{(N_1+1)(M+N_1+1)} \times\sqrt{1-(\frac{N_1!(M+N_1-k)!}{(M+N_1)!(N_1-k)!})^2}\right]. \tag{34}$$

This result coincides with the result of Sentis *et al.* [2].

IV. THE ERROR PROBABILITY WITH THE OPTIMAL POVM

In the previous section, we obtained the averaged error probability when the initial and final states ρ_1 and ρ_2 are distributed independently. However, the unknown states ρ_1 and ρ_2 do not necessarily obey the uniform distribution. In order to treat the performance of our optimal POVM in a more general setting, we consider the error probability with our optimal POVM when two pure states ρ_1 and ρ_2 are fixed.

This error probability depends on the inner product of the two states, i.e., $q \equiv \mathrm{Tr}[\rho_1\rho_2]$. In the following, we calculate the error probability given by Eq. (2) in the case of the optimal POVM $\{I - E_{M,N_1,N_2}, E_{M,N_1,N_2}\}$.

Theorem. The error probability with the optimal POVM $\{I - E_{M,N_1,N_2}, E_{M,N_1,N_2}\}$ can be written as

$$p_{M,N_1,N_2}(\rho_1,\rho_2,E_{M,N_1,N_2}) = \frac{1}{4}\sum_{k=0}^{N_1}\left[P_k + Q_k - \frac{(A_2-A_1)(P_k-Q_k)+2\sin^2\phi_k(A_1P_k+A_2Q_k)}{\sqrt{(A_2-A_1)^2+4A_1A_2\sin^2\phi_k}}\right], \quad (35)$$

where P_k and Q_k are given as follows:

$$P_k \equiv \frac{(N-2k+1)N_1!(M+N_2)!}{(N-k+1)!k!} \times \sum_{l=0}^{N_1-k}\binom{N_1-k}{l}\binom{M+N_2-k+l}{l}q^l(1-q)^{N_1-l}, \quad (36)$$

$$Q_k \equiv \frac{(N-2k+1)(M+N_1)!N_2!}{(N-k+1)!k!} \times \sum_{l=0}^{N_2-k}\binom{M+N_1-k+l}{l}\binom{N_2-k}{l}q^l(1-q)^{N_2-l}. \quad (37)$$

Here we have defined $0^0 \equiv 1$.

When $N_1 = N_2$, i.e., $t_1 = t_3 - t_2 + 1$, we can write

$$p_{M,N_1,N_1}(\rho_1,\rho_2,E_{M,N_1,N_1}) = \frac{1}{2}\left[1 - \sum_{k=0}^{N_1}\frac{(M+2N_1-2k+1)N_1!(M+N_1)!}{(M+2N_1-k+1)!k!}\sqrt{1-\left(\frac{N_1!(M+N_1-k)!}{(M+N_1)!(N_1-k)!}\right)^2} \times \sum_{l=0}^{N_1-k}\binom{N_1-k}{l}\binom{M+N_1-k+l}{l}q^l(1-q)^{N_1-l}\right]. \quad (38)$$

Proof. Let us start by defining the notation for P_k and Q_k as

$$P_k \equiv \mathrm{Tr}\left[I_{[N-k,k]}\otimes|u_k\rangle\langle u_k|\left(\rho_1^{\otimes N_1}\otimes\rho_2^{\otimes M+N_2}\right)\right] \quad (0 \leqslant k \leqslant N_1), \quad (39)$$

$$Q_k \equiv \mathrm{Tr}\left[I_{[N-k,k]}\otimes|v_k\rangle\langle v_k|\left(\rho_1^{\otimes M+N_1}\otimes\rho_2^{\otimes N_2}\right)\right] \quad (0 \leqslant k \leqslant \min[M+N_1,N_2]). \quad (40)$$

We note that the following equation holds:

$$\sum_{k=0}^{N_1}P_k = \sum_{k=0}^{\min(M+N_1,N_2)}Q_k = 1. \quad (41)$$

Since an arbitrary pure state σ satisfies

$$I_L^d\sigma^{\otimes L}I_L^d = \sigma^{\otimes L}, \quad (42)$$

one obtains

$$\mathrm{Tr}\left[\left(\rho_1^{\otimes N_1}\otimes\rho_2^{\otimes M+N_2}\right)E_{M,N_1,N_2}\right] = \mathrm{Tr}\left[\left(I_{N_1}\otimes I_{M+N_2}\right)\left(\rho_1^{\otimes N_1}\otimes\rho_2^{\otimes M+N_2}\right)\left(I_{N_1}\otimes I_{M+N_2}\right) \times E_{M,N_1,N_2}\right]. \quad (43)$$

Plugging Eqs. (15) and (18) into Eq. (43), one has

$$\mathrm{Tr}\left[\left(\rho_1^{\otimes N_1}\otimes\rho_2^{\otimes M+N_2}\right)E_{M,N_1,N_2}\right] = \sum_{k=1}^{N_1}|\langle v_k|w_k\rangle|^2P_k. \quad (44)$$

In the same way, using Eqs. (16) and (43), we obtain

$$\mathrm{Tr}\left[\left(\rho_1^{\otimes M+N_1}\otimes\rho_2^{\otimes N_2}\right)E_{M,N_1,N_2}\right] = \sum_{k=1}^{N_1}|\langle v_k|w_k\rangle|^2Q_k + \sum_{k=N_1+1}^{\min(M+N_1,N_2)}Q_k. \quad (45)$$

When two arbitrary normalized and linearly independent vectors $|a\rangle,|b\rangle$ and two positive real numbers $C_1,C_2 > 0$ are given, the normalized eigenvector $|-\rangle$ with the unique negative eigenvalue of $\frac{|a\rangle\langle a|}{C_1} - \frac{|b\rangle\langle b|}{C_2}$ satisfies the following equations:

$$|\langle a|-\rangle|^2 = \frac{1}{2}\left[1 - \frac{C_2-C_1+2C_1(1-|\langle a|b\rangle|^2)}{\sqrt{(C_2-C_1)^2+4C_1C_2(1-|\langle a|b\rangle|^2)}}\right], \quad (46)$$

$$|\langle b|-\rangle|^2 = \frac{1}{2}\left[1 - \frac{C_2-C_1-2C_2(1-|\langle a|b\rangle|^2)}{\sqrt{(C_2-C_1)^2+4C_1C_2(1-|\langle a|b\rangle|^2)}}\right]. \quad (47)$$

Applying Eqs. (46) and (47) to the case of $|a\rangle = |u_k\rangle, |b\rangle = |v_k\rangle$, we have

$$|\langle u_k|w_k\rangle|^2 = \frac{1}{2}\left(1 - \frac{A_2-A_1+2A_1\sin^2\phi_k}{\sqrt{(A_2-A_1)^2+4A_1A_2\sin^2\phi_k}}\right), \quad (48)$$

$$|\langle v_k|w_k\rangle|^2 = \frac{1}{2}\left(1 - \frac{A_2-A_1-2A_2\sin^2\phi_k}{\sqrt{(A_2-A_1)^2+4A_1A_2\sin^2\phi_k}}\right), \quad (49)$$

for $1 \leqslant k \leqslant N_1$.

Using these equations, we can write the error probability as

$$p_{M,N_1,N_2}(\rho_1,\rho_2,E_{M,N_1,N_2}) = \frac{1}{2}\Big[1 + \mathrm{Tr}\big[\big(\rho_1^{\otimes N_1}\otimes\rho_2^{\otimes M+N_2}\big)E_{M,N_1,N_2}\big] - \mathrm{Tr}\big[\big(\rho_1^{\otimes M+N_1}\otimes\rho_2^{\otimes N_2}\big)E_{M,N_1,N_2}\big]\Big]$$
$$= \frac{1}{2}\Big[1+\sum_{k=1}^{N_1}|\langle u_k|w_k\rangle|^2P_k - \sum_{k=1}^{N_1}|\langle v_k|w_k\rangle|^2Q_k - \sum_{k=N_1+1}^{\min(M+N_1,N_2)}Q_k\Big]$$
$$= \frac{1}{4}\sum_{k=0}^{N_1}\left[P_k+Q_k - \frac{(A_2-A_1)(P_k-Q_k)+2\sin^2\phi_k(A_1P_k+A_2Q_k)}{\sqrt{(A_2-A_1)^2+4A_1A_2\sin^2\phi_k}}\right]. \quad (50)$$

Now we turn our attention to computing P_k. We can assume $d=2$ since P_k does not depend on the dimension. By using Clebsch-Gordan coefficients and the notations given by Eq. (26), the projector in Eq. (39) can be written as

$$I_{[N-k,k]}\otimes|u_k\rangle\langle u_k|$$
$$= \sum_{l=0}^{N-2k}\sum_{i=0}^{N_1}\sum_{j=0}^{M+N_2}|\langle\mu:\mu-l|\mu_1:\mu_1-i;\mu_{02}:\mu_{02}-j\rangle|^2$$
$$\times|\mu_1:\mu_1-i\rangle\langle\mu_1:\mu_1-i|$$
$$\otimes|\mu_{02}:\mu_{02}-j\rangle\langle\mu_{02}:\mu_{02}-j|. \quad (51)$$

In the following, we fix the notation $|\uparrow\rangle$ ($|\downarrow\rangle$) to denote the vector in the space $\mathbb{C}^2$ whose weight is $\frac{1}{2}$ ($-\frac{1}{2}$). Without loss of generality, we can assume that ρ_2 is $|\uparrow\rangle\langle\uparrow|$. Then, we can write

$$\rho_2^{\otimes M+N_2} = |\mu_{02}:\mu_{02}\rangle\langle\mu_{02}:\mu_{02}| = |\uparrow\rangle\langle\uparrow|^{\otimes M+N_2}. \quad (52)$$

Plugging Eqs. (51) and (52) into Eq. (39), we have

$$P_k = \sum_{l=0}^{N_1-k}\langle\mu_1:\mu_1-k-l|\rho_1^{\otimes N_1}|\mu_1:\mu_1-k-l\rangle$$
$$\times|\langle\mu:\mu-l|\mu_1:\mu_1-k-l;\mu_{02}:\mu_{02}\rangle|^2. \quad (53)$$

Moreover, converting the variable l into N_1-k-l, this can be written as

$$P_k = \sum_{l=0}^{N_1-k}\left\langle\frac{N_1}{2}:-\frac{N_1}{2}+l\middle|\rho_1^{\otimes N_1}\middle|\frac{N_1}{2}:-\frac{N_1}{2}+l\right\rangle\left|\left\langle\frac{N}{2}-k:\right.\right.$$
$$\left.\left.\times\frac{N}{2}-N_1+l\middle|\frac{N_1}{2}:-\frac{N_1}{2}+l;\frac{M+N_2}{2}:\frac{M+N_2}{2}\right\rangle\right|^2. \quad (54)$$

We can calculate the Clebsch-Gordan coefficients [5] as

$$\left|\left\langle\frac{N}{2}-k:\frac{N}{2}-N_1+l\middle|\frac{N_1}{2}:-\frac{N_1}{2}+l;\frac{M+N_2}{2}:\frac{M+N_2}{2}\right\rangle\right|^2$$
$$= \frac{(N-2k+1)(N_1-k)!(N_1-l)!}{(N-k+1)!(N_1-k-l)!} \quad (55)$$
$$\times\frac{(M+N_2)!(M+N_2-k+l)!}{(M+N_2-k)!k!l!}.$$

Denoting $\rho_1=|\phi_1\rangle\langle\phi_1|$, we obtain $|\langle\phi_1|\uparrow\rangle|^2=q$. Thus,

$$\left\langle\frac{N_1}{2}:-\frac{N_1}{2}+l\middle|\rho_1^{\otimes N_1}\middle|\frac{N_1}{2}:-\frac{N_1}{2}+l\right\rangle$$
$$=\binom{N_1}{l}|\langle\phi_1|\uparrow\rangle|^{2l}|\langle\phi_1|\downarrow\rangle|^{2(N_1-l)} \quad (56)$$
$$=\binom{N_1}{l}q^l(1-q)^{N_1-l}.$$

Therefore, we can write P_k as

$$P_k \equiv \frac{(N-2k+1)N_1!(M+N_2)!}{(N-k+1)!k!}$$
$$\times\sum_{l=0}^{N_1-k}\binom{N_1-k}{l}\binom{M+N_2-k+l}{l}q^l(1-q)^{N_1-l}, \quad (57)$$

where we have defined 0^0 as 1.

In the same way, one obtains

$$Q_k \equiv \frac{(N-2k+1)(M+N_1)!N_2!}{(N-k+1)!k!}$$
$$\times\sum_{l=0}^{N_2-k}\binom{M+N_1-k+l}{l}\binom{N_2-k}{l}q^l(1-q)^{N_2-l}. \quad (58)$$

■

V. LIMIT OF THE MINIMUM AVERAGED ERROR PROBABILITY

When the numbers of copies in system 1 and system 2 approach infinitely large values, we have perfect knowledge to determine the states in the two systems. In this limit, by using Eq. (33), the probability $\overline{p}_{M,N_1,N_1}(E_{M,N_1,N_1})$ can be written as

$$\lim_{N_1\to\infty}\overline{p}_{M,N_1,N_1}(E_{M,N_1,N_1})$$
$$=\frac{1}{2}\left[1-2(d-1)\int_0^1 x\sqrt{1-x^{2M}}(1-x^2)^{d-2}dx\right]$$
$$=\frac{1}{2}\left[1-(d-1)\int_0^1\sqrt{1-x^M}(1-x)^{d-2}dx\right], \quad (59)$$

where we have defined $x = \frac{k}{n}$ and used the Euler-McLaurin summation formula. The case of $d = 2$ coincides with Eq. (18) in Ref. [2], and the case of $M = 1$ coincides with Eq. (41) in Ref. [3].

This result could be easily anticipated from the minimum error probability of the discrimination problem [6]. Recall that the minimum error probability, given M identical copies, is $\frac{1}{2}[1-\sqrt{1-(\mathrm{Tr}[\rho_1\rho_2])^M}]$. Assuming that ρ_1 and ρ_2 are distributed according to μ_{Θ_d} independently, the average is given by

$$\begin{aligned}\int_{\Theta_d}\int_{\Theta_d}\frac{1}{2}[1-\sqrt{1-(\mathrm{Tr}[\rho_1\rho_2])^M}]\mu_{\Theta_d}(d\rho_1)\mu_{\Theta_d}(d\rho_2)\\ =\frac{1}{2}\left[1-2(d-1)\int_0^{\frac{\pi}{2}}\sqrt{1-(\cos\theta)^{2M}}(\sin\theta)^{2d-3}\cos\theta d\theta\right]\\ =\frac{1}{2}\left[1-(d-1)\int_0^1\sqrt{1-x^M}(1-x)^{d-2}dx\right].\end{aligned} \tag{60}$$

Therefore, our optimal measurement can achieve the average performance of two-state discrimination under the limit $N_1 = N_2 \to \infty$.

Next, we turn our attention to the complementary case; that is, the number of copies in system 0 is infinitely large. By using Eq. (31), the minimum averaged error probability in this limit can be computed as

$$\lim_{M\to\infty}\overline{p}_{M,N_1,N_2}(E_{M,N_1,N_2}) = \frac{1}{2\binom{N_2+d-1}{d-1}}. \tag{61}$$

Note that this result is independent of N_1.

In this limit, we have perfect knowledge of the pure state ρ in system 0 and this problem is equal to distinguishing two states $\rho^{\otimes N_1}\otimes\hat{\rho}^{\otimes N_2}$ and $\hat{\rho}^{\otimes N_1}\otimes\rho^{\otimes N_2}$ in the composite system 12. This problem can be regarded as a generalization of state comparison [7]. As shown in the following, the minimum error probability for these two states can be obtained with a POVM whose elements are $\{E_1 = I_1\otimes(I_2-\rho^{\otimes N_2}), E_2 = I_1\otimes\rho^{\otimes N_2}\}$, where I_1 (I_2) is the unit matrix on $(\mathbb{C}^d)^{\otimes N_1}$ $[(\mathbb{C}^d)^{\otimes N_2}]$. Here, E_1 (E_2) corresponds to the guess $\rho^{\otimes N_1}\otimes\hat{\rho}^{\otimes N_2}$ ($\hat{\rho}^{\otimes N_1}\otimes\rho^{\otimes N_2}$). We then can write the error probability as

$$\frac{1}{2}\mathrm{Tr}[\rho^{\otimes N_1}\otimes\hat{\rho}^{\otimes N_2}E_2]+\frac{1}{2}\mathrm{Tr}[\hat{\rho}^{\otimes N_1}\otimes\rho^{\otimes N_2}E_1]=\frac{1}{2}(\mathrm{Tr}[\rho\hat{\rho}])^{N_2}. \tag{62}$$

Thus, the average is computed as follows:

$$\begin{aligned}\int_{\Theta_d}\int_{\Theta_d}\frac{1}{2}(\mathrm{Tr}[\rho\hat{\rho}])^{N_2}\mu_{\Theta_d}(d\rho)\mu_{\Theta_d}(d\hat{\rho})\\ =(d-1)\int_0^{\frac{\pi}{2}}(\cos\theta)^{2N_2}(\sin\theta)^{2d-3}\cos\theta d\theta\\ =\frac{d-1}{2}\int_0^1 x^{N_2}(1-x)^{d-2}dx = \frac{1}{2\binom{N_2+d-1}{d-1}}.\end{aligned} \tag{63}$$

Since this value is the same as the expression in Eq. (61), the POVM $\{E_1 = I_1\otimes(I_2-\rho^{\otimes N_2}), E_2 = I_1\otimes\rho^{\otimes N_2}\}$ realizes the optimalperformance.

VI. EXPONENTIALLY DECREASING RATE OF THE ERROR PROBABILITY

When the numbers of copies N_1, N_2, and M are infinitely large, that is, the states in three systems are perfectly known, the error probability approaches zero unless $q = 1$. In order to treat the convergence speed, we focus on the exponentially decreasing rate of the error probability when the optimal POVM E_{M,N_1,N_1} is applied. For simplicity, we assume that the numbers N_1 and N_2 of copies in systems 1 and 2, respectively, increase in proportion to the number M of copies in system 0. When the proportional constant is given to be $\alpha > 0$, we have $N_1 = N_2 = \alpha M$. Thus, using the real numbers

$$C_k \equiv \frac{(M+2\alpha M-2k+1)(\alpha M)!(M+\alpha M)!}{2(M+2\alpha M-k+1)!k!} \times\left[1-\sqrt{1-\left(\frac{(\alpha M)!(M+\alpha M-k)!}{(M+\alpha M)!(\alpha M-k)!}\right)^2}\right], \tag{64}$$

$$D_{k,l} \equiv \binom{\alpha M-k}{l}\binom{M+\alpha M-k+l}{l}q^l(1-q)^{\alpha M-l}, \tag{65}$$

we can write from Eq. (38)

$$p_{M,\alpha M,\alpha M}(\rho_1,\rho_2,E_{M,\alpha M,\alpha M}) = \sum_{k=0}^{\alpha M}\sum_{l=0}^{\alpha M-k}C_kD_{k,l}. \tag{66}$$

The convergence speed is represented as $\lim_{M\to\infty}\frac{-1}{M}\ln p_{M,\alpha M,\alpha M}(\rho_1,\rho_2,E_{M,\alpha M,\alpha M})$, which can be deformed as

$$\begin{aligned}\lim_{M\to\infty}\frac{-1}{M}\ln p_{M,\alpha M,\alpha M}(\rho_1,\rho_2,E_{M,\alpha M,\alpha M})\\ =\lim_{M\to\infty}\frac{-1}{M}\ln\left(\sum_{k=0}^{\alpha M}\sum_{l=0}^{\alpha M-k}C_kD_{k,l}\right)\\ =\lim_{M\to\infty}\frac{-1}{M}\ln\left(\max_{0\leqslant k\leqslant\alpha M}\max_{0\leqslant l\leqslant\alpha M-k}C_kD_{k,l}\right).\end{aligned} \tag{67}$$

In this paper, we adopt the natural logarithm. Moreover, using the approximation formula $\sqrt{1-x}\approx 1-\frac{1}{2}x$ (when $x \ll 1$) and the Stirling approximation $n! \approx n^ne^{-n}\sqrt{2\pi n}$, one obtains

$$\begin{aligned}\lim_{M\to\infty}\frac{-1}{M}\ln p_{M,\alpha M,\alpha M}(\rho_1,\rho_2,E_{M,\alpha M,\alpha M})\\ =\min_{0\leqslant\beta\leqslant\alpha}\min_{0\leqslant\gamma\leqslant\alpha-\beta}h(\beta,\gamma),\end{aligned} \tag{68}$$

where we have defined $\beta \equiv \frac{k}{M}$, $\gamma \equiv \frac{l}{M}$, and

$$\begin{aligned}h(\beta,\gamma) \equiv{}& (\alpha-\beta-\gamma)\ln(\alpha-\beta-\gamma)-(1+\alpha-\beta+\gamma)\\ &\times\ln(1+\alpha-\beta+\gamma)+(\alpha-\beta)\ln(\alpha-\beta)\\ &-(1+\alpha-\beta)\ln(1+\alpha-\beta)-3\alpha\ln\alpha\\ &+(1+\alpha)\ln(1+\alpha)+\beta\ln\beta\\ &+(1+2\alpha-\beta)\ln(1+2\alpha-\beta)\\ &+2\gamma\ln\gamma-\gamma\ln q-(\alpha-\gamma)\ln(1-q).\end{aligned} \tag{69}$$

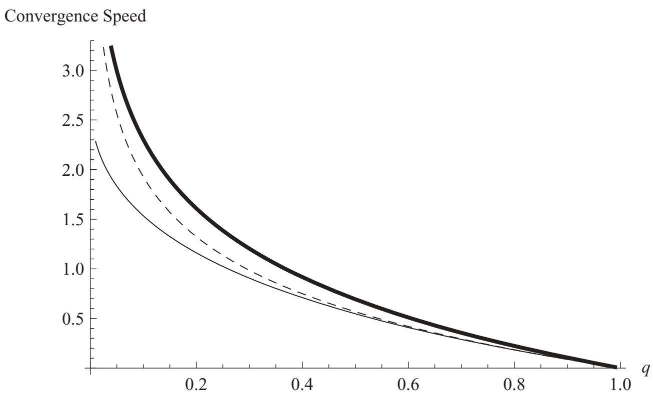

FIG. 2. For $\alpha = 5, h(\beta_1,\gamma_1)$ (thin line), $-\ln q - \frac{1-q}{q(\alpha-1)+2}$ (dashed line), and $-\ln q$ (thick line).

There is the unique root of $\frac{\partial h}{\partial \beta} = \frac{\partial h}{\partial \gamma} = 0$ in the range $0 < \beta < \alpha, 0 < \gamma < \alpha - \beta$ and we use (β_1,γ_1) to denote it. These can be calculated as

$$\gamma_1 = \frac{q(\alpha-1) + \sqrt{q^2(\alpha-1)^2 + 4q\alpha}}{2}, \tag{70}$$

$$\beta_1 = \frac{(2\alpha+1) - \sqrt{(2\alpha+1)^2 - 4(\alpha^2 - \alpha\gamma_1)}}{2}. \tag{71}$$

One can agree that $h(\beta_1,\gamma_1)$ is the minimum of the function h in the range $0 < \beta < \alpha, 0 < \gamma < \alpha - \beta$ due to the following equations:

$$\begin{aligned} &\frac{\partial^2 h}{\partial \beta^2}(\beta,\gamma) > 0, \quad \frac{\partial^2 h}{\partial \gamma^2}(\beta,\gamma) > 0, \\ &\lim_{\beta\to 0} \frac{\partial h}{\partial \beta}(\beta,\gamma) = \lim_{\gamma\to 0} \frac{\partial h}{\partial \gamma}(\beta,\gamma) = -\infty, \\ &\lim_{\beta\to\alpha-\gamma} \frac{\partial h}{\partial \beta}(\beta,\gamma) = \lim_{\gamma\to\alpha-\beta} \frac{\partial h}{\partial \gamma}(\beta,\gamma) = +\infty, \end{aligned} \tag{72}$$

When α is sufficiently large, we can write

$$h(\beta_1,\gamma_1) = -\ln q - \frac{1-q}{q(\alpha-1)+2} + O\left(\frac{1}{\alpha^2}\right). \tag{73}$$

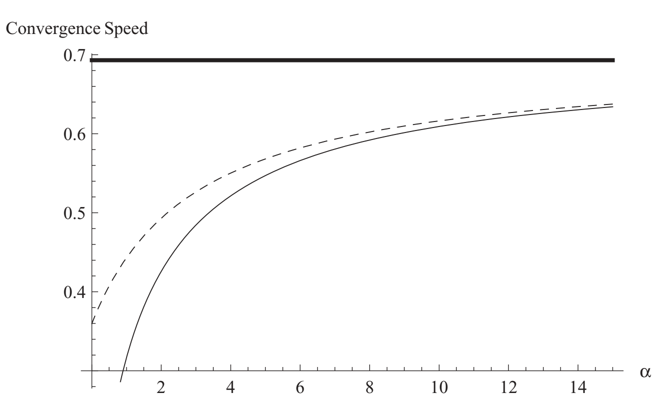

FIG. 3. For $q = 0.5$, $h(\beta_1,\gamma_1)$ (thin line), $-\ln q - \frac{1-q}{q(\alpha-1)+2}$ (dashed line), and $-\ln q$ (thick line).

In fact, as is numerically demonstrated in Figs. 2 and 3, $-\ln q - \frac{1-q}{q(\alpha-1)+2}$ well approximates $h(\beta_1,\gamma_1)$ when α is large. Therefore, we obtain the convergence speed of the error probability,

$$\begin{aligned} &\lim_{M\to\infty} \frac{-1}{M} \ln p_{M,\alpha M,\alpha M}(\rho_1,\rho_2,E_{M,\alpha M,\alpha M}) \\ &\quad = -\ln \mathrm{Tr}[\rho_1\rho_2] - \frac{1-\mathrm{Tr}[\rho_1,\rho_2]}{\mathrm{Tr}[\rho_1\rho_2](\alpha-1)+2} + O\left(\frac{1}{\alpha^2}\right). \end{aligned} \tag{74}$$

In the discrimination problem of two pure states [6], when the number of copies of the state to be identified is infinitely large, the convergence speed is given by

$$\lim_{M\to\infty} \frac{-1}{M} \ln \frac{1}{2}[1-\sqrt{1-(\mathrm{Tr}[\rho_1\rho_2])^M}] = -\ln \mathrm{Tr}[\rho_1\rho_2]. \tag{75}$$

This is called the quantum Chernoff bound [4] and it is equal to the limit of Eq. (74) as $\alpha \to \infty$. This fact means that the performance of our optimal POVM is close to that of the optimal POVM in the sense of quantum state discrimination.

VII. CONCLUSIONS

We have studied the discrimination of the change point problem in a quantum setting when two candidates t_1 and t_2 of the true change point t_c are given. This problem is equal to discriminating two unknown general states when multiple copies of the state are provided. We have obtained the minimum averaged error probability, Eq. (31). Our result of special cases coincides with the results of Refs. [2,3]. However, our result is more general, allowing for arbitrary numbers of copies of general pure states. Moreover, we have calculated the nonaveraged error probability, Eq. (35). This value depends on the inner product between the initial state and the final state. As could be anticipated, when the time lengths t_1 and $t_3 - t_2 + 1$ are infinitely large, we recover the average of the usual discrimination problem. We have also paid attention to the exponential decreasing rate and shown that the convergence rate of the nonaveraged error probability approaches the quantum Chernoff bound.

In discrimination problems one can also consider the unambiguous approach, as was studied in Refs. [2,3]. From the change point perspective, however, the unambiguous approach does not arise in a natural way and it is beyond the scope of this paper. Since it is meaningful from the viewpoint of quantum discrimination for programmable devices, it is an interesting future problem to extend our result to the case of unambiguous discrimination.

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APPENDIX: CALCULATION OF WIGNER'S 6-j FUNCTION

Let us consider the calculation of Wigner's 6-j function $\left\{\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right\}$. Let us define some notations as

$$\alpha_1 \equiv a+b+d+e, \quad \alpha_2 \equiv a+c+d+f, \quad \alpha_3 \equiv b+c+e+f,$$
$$\beta_1 \equiv a+b+c, \quad \beta_2 \equiv a+e+f, \tag{A1}$$
$$\beta_3 \equiv b+d+f, \quad \beta_4 \equiv c+d+e,$$

and let us define A_1, A_2, and A_3 to be the smallest, middle, and largest values of α_1, α_2, and α_3 and define B_1, B_2, B_3, and B_4 to be the smallest, second smallest, second largest, and largest values of β_1, β_2, β_3, and β_4. When $B_4 = A_1$, from the formula in Ref. [8] we can calculate

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{B_4}\left[\prod_{i=1}^{B_4-B_3}\frac{(B_3+1+i)(A_3-A_1+i)}{(A_3-B_1+i)(A_2-B_2+i)}\right]^{\frac{1}{2}} \times \left[\frac{1}{(B_1+1)(B_2+1)} \times \prod_{i=1}^{A_2-A_1}\frac{(A_1-B_1+i)(B_4-B_2+i)(B_4-B_3+i)}{(A_1+B_1-A_2+i)(A_1+B_2-A_2+i)i}\right]^{\frac{1}{2}}. \tag{A2}$$

We compute Wigner's 6-j function $\left\{\begin{smallmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{smallmatrix}\right\}$ in Eq. (27).

1. Case $0 \leqslant k \leqslant M$

In this case, we can write

$$A_1 = N-k, \quad A_2 = N, \quad A_3 = M+N-k,$$
$$B_1 = M+N_1, \quad B_2 = M+N_2, \quad B_3 = N-k, \tag{A3}$$
$$B_4 = N-k.$$

Plugging these into Eq. (A2), one obtains

$$\begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix} = (-1)^{N-k}\left[\frac{1}{(M+N_1+1)(M+N_2+1)} \times \prod_{i=1}^{k}\frac{(N_1-k+i)(N_2-k+i)}{(M+N_1-k+i)(M+N_2-k+i)}\right]^{\frac{1}{2}}. \tag{A4}$$

This is deformed as

$$\begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix} = \frac{(-1)^{\mu_0+\mu_1+\mu_2+\mu}}{\sqrt{(2\mu_{01}+1)(2\mu_{02}+1)}}\sqrt{\frac{\binom{N_1}{k}\binom{N_2}{k}}{\binom{M+N_1}{k}\binom{M+N_2}{k}}}. \tag{A5}$$

2. Case $M+1 \leqslant k \leqslant N_1$

In this case, we can write

$$A_1 = N-k, \quad A_2 = M+N-k, \quad A_3 = N,$$
$$B_1 = M+N_1, \quad B_2 = M+N_2, \quad B_3 = N-k, \tag{A6}$$
$$B_4 = N-k.$$

Plugging these into Eq. (A2), one obtains

$$\begin{Bmatrix} \mu_1 & \mu_0\mu_{01} \\ \mu_2 & \mu\,\mu_{02} \end{Bmatrix} = (-1)^{N-k}\left[\frac{1}{(M+N_1+1)(M+N_2+1)} \times \prod_{i=1}^{M}\frac{(N_1-k+i)(N_2-k+i)}{(N_1+i)(N_2+i)}\right]^{\frac{1}{2}}. \tag{A7}$$

Since $M+1 \leqslant k$, one has

$$\frac{\binom{N_1}{k}\binom{N_2}{k}}{\binom{M+N_1}{k}\binom{M+N_2}{k}} = \prod_{i=1}^{M}\frac{(N_1-k+i)(N_2-k+i)}{(N_1+i)(N_2+i)}. \tag{A8}$$

Thus,

$$\begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix} = \frac{(-1)^{\mu_0+\mu_1+\mu_2+\mu}}{\sqrt{(2\mu_{01}+1)(2\mu_{02}+1)}}\sqrt{\frac{\binom{N_1}{k}\binom{N_2}{k}}{\binom{M+N_1}{k}\binom{M+N_2}{k}}}. \tag{A9}$$

also holds.

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