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Measures of non-Markovianity: Divisibility versus backflow of information

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We analyze two recently proposed measures of non-Markovianity: one based on the concept of divisibility of the dynamical map and the other one based on distinguishability of quantum states. We provide a model to show that these two measures need not agree. In addition, we discuss possible generalizations and intricate relations between these measures.

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I. INTRODUCTION

The dynamics of open quantum systems are now attracting increasing attention [1–3]. It is relevant not only for better understanding of quantum theory but also it is fundamental in various modern applications of quantum mechanics. Since the system-environment interaction causes dissipation, decay, and decoherence, it is clear that dynamics of open systems is fundamental in modern quantum technologies, such as quantum communication, cryptography, and computation [4]. The usual approach to the dynamics of an open quantum system consists in applying an appropriate Born-Markov approximation, leading to the celebrated quantum Markov semigroup [5,6], which neglects all memory effects. However, recent theoretical studies and technological progress call for a more refined approach based on non-Markovian evolution.

Non-Markovian systems appear in many branches of physics, such as quantum optics [1,7], solid-state physics [8], quantum chemistry [9], and quantum information processing [10]. Since non-Markovian dynamics modifies monotonic decay of quantum coherence, it turns out that when applied to composite systems it may protect quantum entanglement for longer times than standard Markovian evolution [11]. In particular, it may protect the system against the sudden death of entanglement [12]. It is therefore not surprising that non-Markovian dynamics has been intensively studied recently [13].

Surprisingly, it turns out the concept of (non)Markovianity is not uniquely defined. One approach is based on the idea of the composition law, which is essentially equivalent to the idea of divisibility [14]. This approach was used recently by Rivas, Huelga, and Plenio (RHP) [15] to construct the corresponding measure of non-Markovianity, that is, RHP measured the deviation from divisibility. A different approach is advocated by Breuer, Laine, and Piilo (BLP) in Ref. [16]. BLP defined non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system and manifests itself as an increase in the distinguishability of pairs of evolving quantum states. It is clear that RHP characterized a mathematical property of the dynamical map whereas the idea of BLP is based on physical features of the system-reservoir interaction, rather than the mathematical properties of the dynamical map of the open system. In a recent paper [17], Haikka, Cresser, and Maniscalco performed detailed analysis of these approaches, studying the dynamics of the driven qubit in a structured environment. It is indicated [17] that the concepts of RHP and BLP need not agree, as was also conjectured in [18] and checked in [19]. The BLP measure was recently analyzed for the dynamics of a qubit coupled to a spin environment via an energy-exchange mechanism [20]. In the present paper, we perform further analysis of this problem. In particular, we provide a simple model showing that Markovian evolution à la BLP may be indivisible and hence non-Markovian according to RHP. Actually, one may feel the relation of BLP versus RHP analogous to the relation between separable and positive partial transpose (PPT) states in entanglement theory, where separable states define the proper subset of PPT states. States which are PPT but entangled are bound entangled. Using this analogy, one may call Markovian evolution à la BLP indivisible (non-Markovian according to RHP): "bound non-Markovian." Finally, we discuss possible generalizations of BLP using tensor product structures and show that a slight modification of BLP reduces their concept of Markovianity to divisibility (the bound non-Markovianity can be washed out). The main aims of this paper are to expose the relation between two different concepts on a simple example and to present the way to unify them, which we hope will clarify the issue.

II. DIVISIBILITY VERSUS BACKFLOW OF INFORMATION

The measure for non-Markovianity proposed by RHP [15] is based on a notion of divisibility: a trace-preserving completely positive map $\Lambda(t,0)$ is divisible if it can be written as

$$\Lambda(t+\tau,0) = \Lambda(t+\tau,t)\Lambda(t,0), \tag{1}$$

and $\Lambda(t+\tau,t)$ is completely positive for any $t,\tau > 0$. RHP defined a map to be Markovian exactly when it is divisible. Note that

$$\Lambda(t+\tau,t) = \Lambda(t+\tau,0)\Lambda^{-1}(t,0) \tag{2}$$

satisfies composition law

$$\Lambda(s,t) = \Lambda(s,u)\Lambda(u,t) \tag{3}$$

for any $s \geqslant u \geqslant t$, which is usually attributed for Markovian evolution. It is shown [15] that the quantity

$$g(t) = \lim_{\epsilon \to 0+} \frac{||\mathbb{1}_d \otimes \Lambda(t+\epsilon,t) P_d^+||_1 - 1}{\epsilon} \tag{4}$$

enjoys $g(t) > 0$ if and only if the original map $\Lambda(t,0)$ is indivisible ($||\cdot||_1$ is a trace norm). As usual, $\mathbb{1}_d$ denotes an identity map in $M_d(\mathbb{C})$, and P_d^+ denotes the maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$ (we consider the evolution of a d-level system). Now, using the fact that any divisible (and differentiable) completely positive map satisfies a local in the time master equation,

$$\frac{d}{dt}\Lambda(t,0) = L(t)\Lambda(t,0), \quad \Lambda(0,0) = \mathbb{1}_d, \tag{5}$$

where the local generator $L(t)$ is a legitimate Markovian generator for any $t \geqslant 0$. The formula (4) may be equivalently rewritten in terms of $L(t)$:

$$g(t) = \lim_{\epsilon \to 0+} \frac{||\{\mathbb{1}_d \otimes [\mathbb{1}_d + \epsilon L(t)]\} P_d^+||_1 - 1}{\epsilon}. \tag{6}$$

A second criterion of non-Markovianity was proposed by BLP in Ref. [16]. The BLP criterion identifies non-Markovian dynamics with certain physical features of the system-reservoir interaction. They defined non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system. This backflow of information may manifest itself as an increase in the distinguishability of pairs of evolving quantum states. Hence, according to BLP, the dynamical map $\Lambda(t,0)$ is non-Markovian if there exists a pair of initial states ρ_1 and ρ_2 such that for some time $t > 0$ the distinguishability of ρ_1 and ρ_2 increases; that is,

$$\sigma(\rho_1,\rho_2;t) = \frac{d}{dt} D[\rho_1(t),\rho_2(t)], \tag{7}$$

where $D(\rho_1,\rho_2) = \frac{1}{2}||\rho_1 - \rho_2||_1$ is the distinguishability of ρ_1 and ρ_2, and $\rho_k(t) = \Lambda(t,0)\rho_k$.

Using these two criteria, one easily defines the corresponding non-Markovianity measures of the dynamical map Λ:

$$\mathcal{N}_{\rm RHP}(\Lambda) = \frac{\mathcal{I}}{\mathcal{I}+1}, \tag{8}$$

where $\mathcal{I} = \int_0^\infty g(t)dt$. Similarly, one has

$$\mathcal{N}_{\rm BLP}(\Lambda) = \sup_{\rho_1,\rho_2} \int_{\sigma>0} \sigma(\rho_1,\rho_2;t)dt. \tag{9}$$

It is clear [16] that $\mathcal{N}_{\rm RHP}(\Lambda) = 0$ implies $\mathcal{N}_{\rm BLP}(\Lambda) = 0$. Hence, all divisible maps are Markovian according to BLP. In general, the converse is not true. This problem was carefully analyzed in [19] for phenomenological integro-differential master equations and in [17], where the authors have studied the non-Markovian character of a driven qubit in a structured reservoir in different dynamical regime using $\mathcal{N}_{\rm RHP}(\Lambda)$ and $\mathcal{N}_{\rm BLP}(\Lambda)$.

III. MODEL: CLASSICAL STOCHASTIC DYNAMICS

It should be clear that the above discussion applies for a classical stochastic systems as well. Consider stochastic dynamics of d-state classical system described by the probability vector $\mathbf{p} = (p_1,\ldots,p_d)$. Its evolution $\mathbf{p}(0) \longrightarrow \mathbf{p}(t) = \Lambda(t)\cdot\mathbf{p}(0)$ is defined by the family of stochastic matrices $\Lambda(t)$; that is, $\Lambda_{ij}(t) \geqslant 0$ for all $i,j = 1,\ldots,d$ and

$$\sum_{i=1}^d \Lambda_{ij}(t) = 1, \tag{10}$$

for each $j = 1,\ldots,d$ and $t \geqslant 0$. These conditions guarantee that $\mathbf{p}(t)$ is a legitimate probability vector for all $t \geqslant 0$. The corresponding local-in-time master equation

$$\dot{p}_i(t) = \sum_{j=1}^d L_{ij}(t)\, p_j(t) \tag{11}$$

is defined in terms of the local generator $L(t)$, which is defined in by the classical dynamical map $\Lambda(t)$ as follows:

$$L(t) = \dot{\Lambda}(t)\cdot\Lambda^{-1}(t). \tag{12}$$

In particular, if L does not depend on time, $\Lambda(t)$ defines a one-parameter semigroup of stochastic matrices $\Lambda(t) := \Lambda(t,0) = e^{tL}$. Let us recall that Λ is stochastic if and only if it satisfies well-known Kolmogorov conditions:

$$L_{ij} \geqslant 0,\ \ i \neq j, \quad \text{and} \quad \sum_{i=1}^d L_{ij} = 0, \tag{13}$$

for each $j = 1,\ldots,d$. Now, a stochastic evolution $\Lambda(t,0)$ is divisible if it can be written as a composition of two stochastic maps $\Lambda(t+\tau,t)$ and the original $\Lambda(t,0)$

$$\Lambda(t+\tau,0) = \Lambda(t+\tau,t)\cdot\Lambda(t,0), \tag{14}$$

for any $\tau > 0$. Clearly, $\Lambda(t,0)$ is divisible if and only if the corresponding local generator $L(t)$ satisfies Kolmogorov conditions for any $t \geqslant 0$. Now, we compare two non-Markovianity measures due to RHP and BLP.

It is clear that classical evolution may be rewritten in the quantum framework as follows: Any probability vector $\mathbf{p}$ gives rise to the diagonal density matrix $\rho = \sum_k p_k |k\rangle\langle k|$, and hence the stochastic map—classical channel—Λ gives rise to the following Kraus representation:

$$\Lambda\,\rho = \sum_{i,j=1}^d \Lambda_{ji}\, |i\rangle\langle j|\,\rho\,|j\rangle\langle i|. \tag{15}$$

Now, to apply (4), one needs the classical analog of $(\mathbb{1}\otimes\Lambda)P^+$. Let us define $(\mathbb{1}_{\rm cl}\otimes\Lambda)P^+_{\rm cl}$, where the classical identity map is defined by

$$\mathbb{1}_{\rm cl}\,\rho = \sum_{i=1}^d |i\rangle\langle i|\,\rho\,|i\rangle\langle i| \tag{16}$$

and the classical analog of the maximally entangled states reads as follows:

$$P^+_{\rm cl} = \frac{1}{d}\sum_{i=1}^d |i\rangle\langle i| \otimes |i\rangle\langle i|. \tag{17}$$

One finds therefore

$$(\mathbb{1}_{\rm cl} \otimes \Lambda) P^+_{\rm cl} = \frac{1}{d} \sum_{i,j=1}^{d} \Lambda_{ji}\, |i\rangle\langle i| \otimes |j\rangle\langle j|. \tag{18}$$

Now, the "classical" quantity

$$g(t) = \lim_{\epsilon \to 0+} \frac{||\mathbb{1}_{\rm cl} \otimes \Lambda(t+\epsilon,t) P^+_{\rm cl}||_1 - 1}{\epsilon} \tag{19}$$

is strictly positive if and only if the original stochastic map $\Lambda(t,0)$ is indivisible, and hence $g(t)$ identifies non-Markovianity of the classical stochastic process. Equivalently, it may be rewritten using local generator $L(t)$

$$g(t) = \lim_{\epsilon \to 0} \frac{||[\mathbb{1}_{\rm cl} \otimes (\mathbb{1}_{\rm cl} - \epsilon L(t))] P^+_{\rm cl}||_1 - 1}{\epsilon}. \tag{20}$$

Consider now the following model of stochastic dynamics of a two-state system

$$\Lambda(t,0) = \begin{pmatrix} 1 - x_0(t) & x_1(t) \\ x_0(t) & 1 - x_1(t) \end{pmatrix}, \tag{21}$$

where $x_0(t), x_1(t) \in [0,1]$ for all $t \geqslant 0$. One finds for the local generator

$$L(t) = \begin{pmatrix} -a_0(t) & a_1(t) \\ a_0(t) & -a_1(t) \end{pmatrix}, \tag{22}$$

where

$$a_0 = \frac{\dot{x}_0(1 - x_1) + \dot{x}_1 x_0}{1 - x_0 - x_1}, \tag{23}$$

$$a_1 = \frac{\dot{x}_0 x_1 + \dot{x}_1 (1 - x_0)}{1 - x_0 - x_1}. \tag{24}$$

Now, $\Lambda(t,0)$ is divisible if and only if $a_0(t), a_1(t) \geqslant 0$ for all $t \geqslant 0$. On the other hand, one easily finds

$$\sigma(t) = -2[a_0(t) + a_1(t)]|\Delta_0|, \tag{25}$$

where $\Delta_k = (\mathbf{p}_1 - \mathbf{p}_2)_k$. Hence, contrary to $g(t)$, the quantity $\sigma(t)$ controls only the sum of a_0 and a_1. In principle, one may have $\sigma(t) \leqslant 0$ even if $a_0(t) < 0$ or $a_1(t) < 0$. Indeed, let us define

$$x_0(t) = \int_0^t f_0(\tau) d\tau\ , \quad x_1(t) = \int_0^t f_1(\tau) d\tau \tag{26}$$

such that $0 \leqslant \int_0^t f_k(\tau) d\tau \leqslant 1$, for all $t \geqslant 0$. Hence, our model is fully controlled by functions f_0 and f_1. Let

$$f_0(t) = \kappa \sin t, \quad t \geqslant 0 \tag{27}$$

and $f_1(t) = 0$ for $t \in [0,\pi]$ together with

$$f_1(t) = -\kappa \sin t, \quad t \geqslant \pi, \tag{28}$$

where $0 < \kappa < 1/2$. One finds

$$a_0(t) + a_1(t) = \frac{\kappa \sin t}{1 - \kappa + \kappa \cos t}, \quad t \in [0,\pi], \tag{29}$$

and $a_0(t) + a_1(t) = 0$ for $t \geqslant \pi$. Note, that for $t \geqslant \pi$ one has

$$a_0(t) = -a_1(t) = \kappa \sin t, \tag{30}$$

which proves that $\Lambda(t,0)$ is not divisible. However, this map is "classically" Markovian according to BLP [16] due to $a_0(t) + a_1(t) \geqslant 0$ for all $t \geqslant 0$.

IV. QUBIT DYNAMICS

Consider now the following dynamics of a qubit:

$$\begin{aligned} \rho_{00}(t) &= \rho_{00}\, x_0(t) + \rho_{11}\, [1 - x_1(t)], \\ \rho_{11}(t) &= \rho_{00}\, [1 - x_0(t)] + \rho_{11}\, x_1(t), \\ \rho_{01}(t) &= \rho_{01}\, \gamma(t), \end{aligned} \tag{31}$$

where $x_0(t), x_1(t) \in [0,1]$, and

$$|\gamma(t)|^2 \leqslant x_0(t) x_1(t). \tag{32}$$

The above conditions for $x_k(t)$ and $\gamma(t)$ guarantee that the dynamics is completely positive. One easily finds for the corresponding local generator

$$L\,\rho = -i\frac{\Omega}{2}[\sigma_z,\rho] + \sum_{k=0}^{1} a_k L_k \rho + \frac{\Gamma}{2} L_z \rho, \tag{33}$$

where

$$\begin{aligned} L_0 \rho &= \sigma_+ \rho \sigma_- - \tfrac{1}{2}\{\sigma_- \sigma_+, \rho\}, \\ L_1 \rho &= \sigma_- \rho \sigma_+ - \tfrac{1}{2}\{\sigma_+ \sigma_-, \rho\}, \\ L_z \rho &= \sigma_z \rho \sigma_z - \rho. \end{aligned} \tag{34}$$

The time-dependent coefficients a_0 and a_1 are defined in (23) and (24), whereas $\Gamma(t)$ and $\Omega(t)$ read as follows:

$$\Gamma(t) = -\frac{a_0(t) + a_1(t)}{2} - \mathrm{Re}\, \frac{\dot{\gamma}(t)}{\gamma(t)}, \tag{35}$$

$$\Omega(t) = \mathrm{Im}\, \frac{\dot{\gamma}(t)}{\gamma(t)}. \tag{36}$$

The corresponding dynamical map $\Lambda(t,0)$ is divisible if and only if $a_k(t), \Gamma(t) \geqslant 0$ for all $t \geqslant 0$. On the other hand, one finds the following formula:

$$\sigma(t) = -\frac{2A(t)\Delta_{00}^2 + [A(t) + 4\Gamma(t)]|\Delta_{01}|^2}{\sqrt{\Delta_{00}^2 + |\Delta_{01}|^2}}, \tag{37}$$

where $A(t) = a_0(t) + a_1(t)$, and a 2×2 matrix Δ reads

$$\Delta = \rho_1(0) - \rho_2(0). \tag{38}$$

It is therefore clear that using the same arguments as in the classical model we may have $\sigma(t) \leqslant 0$ but the dynamical map is not divisible.

V. HOW TO RECONCILE TWO MEASURES

We have studied so far the non-Markovian character of simple classical and quantum models and compared two non-Markovianity measures due to RHP and BLP. We performed explicit construction showing that in general these two measures do not agree, supporting [17–19].

Now, we show how these two measures may be reconciled. Let us observe that the approach of BLP requires only that $\Lambda(t,0)$ is a positive map for all $t \geqslant 0$, that is, it maps a density operator at $t = 0$ into a density operator at time t. We stress that a concept of complete positivity does not enter the definition of $\sigma(\rho_1,\rho_2,t)$. On the other hand, the notion of complete positivity is crucial for RHP. Hence, to reconcile these measures one has to incorporate complete positivity into BLP approach. It is clear that if $\Lambda(t,0)$ is completely positive then $\mathbb{1}_d \otimes \Lambda(t,0)$ is positive, and one may introduce

$$\widetilde{\sigma}(\widetilde{\rho}_1,\widetilde{\rho}_2;t) = \frac{d}{dt} D[\widetilde{\rho}_1(t),\widetilde{\rho}_2(t)], \tag{39}$$

where

$$\widetilde{\rho}_k(t) = [\mathbb{1}_d \otimes \Lambda(t,0)]\widetilde{\rho}_k,\ \ k = 1,2. \tag{40}$$

It is clear that if $\widetilde{\sigma}(\widetilde{\rho}_1,\widetilde{\rho}_2;t) \leqslant 0$ for all initial states $\widetilde{\rho}_k$, then $\sigma(\rho_1,\rho_2;t) \leqslant 0$ for all initial states ρ_k. Note, however, that the converse needs not be true. Again, one may show that there exists indivisible maps $\Lambda(t,0)$ for which $\widetilde{\sigma}(\widetilde{\rho}_1,\widetilde{\rho}_2;t) \leqslant 0$ for all initial states $\widetilde{\rho}_k$. Note that condition $\widetilde{\sigma}(\widetilde{\rho}_1,\widetilde{\rho}_2;t) \leqslant 0$ may be reformulated as

$$\widetilde{\sigma}(\Delta;t) = \frac{d}{dt}||\Delta(t)||_1 \leqslant 0, \tag{41}$$

where $\Delta(t) = [\mathbb{1}_d \otimes \Lambda(t,0)]\Delta$, and $\Delta^\dagger = \Delta$, together with $\mathrm{Tr}\Delta = 0$, which follows from $\Delta = (\widetilde{\rho}_1 - \widetilde{\rho}_2)/2$ and $\widetilde{\rho}_k$, are true states. It should be stressed that condition $\mathrm{Tr}\Delta = 0$ is very restrictive. Let us recall that if Φ is trace preserving then Φ is positive if and only if [21]

$$||\Phi\, a||_1 \leqslant ||a||_1 \tag{42}$$

for all $a^\dagger = a$. Note, however, that if one is restricted to a enjoying $\mathrm{Tr}a = 0$, then $||\Phi\, a||_1 \leqslant ||a||_1$ does not imply positivity of Φ. If we relax $\mathrm{Tr}\Delta = 0$, then the condition $\widetilde{\sigma}(\Delta;t) \leqslant 0$ for all Hermitian Δ implies that $\Lambda(t,0)$ is (completely) divisible and hence the definition of Markovianity due to BLP reduces to divisibility. More explicitly, this fact can be formulated as

Theorem. For a bijective evolution $\Lambda(t,0)$, $\tilde{\sigma}(\Delta;t) \leqslant 0$ for all $\Delta^\dagger = \Delta$, if and only if it is divisible.

Proof. The "if" part is straightforward because the trace norm is monotonically decreasing given the complete positivity of the $\Lambda(t+\tau,t)$,

$$\|\Delta(t+\tau)\|_1 = \|[\mathbb{1}_d \otimes \Lambda(t+\tau,t)]\Delta(t)\|_1 \leqslant \|\Delta(t)\|_1,$$

for every t and $\tau > 0$. Conversely, if the evolution $\Lambda(t,0)$ is bijective, the partitions $\Lambda(t+\tau,t) = \Lambda(t+\tau,0)\Lambda^{-1}(t,0)$ are well defined. Then $\tilde{\sigma}(\Delta;t) > 0$ for some t implies that for some small τ

$$\|\Delta(t+\tau)\|_1 = \|[\mathbb{1}_d \otimes \Lambda(t+\tau,t)]\Delta(t)\|_1 > \|\Delta(t)\|_1.$$

So there exists some partition $\Lambda(t+\tau,t)$ which is not completely positive, and therefore the dynamics is not divisible. ∎

VI. OPERATIONAL MEANING OF $\tilde{\sigma}(\Delta;t)$

One may think that by relaxing the condition $\mathrm{Tr}\Delta = 0$ the interpretation of an increase of $\tilde{\sigma}(\Delta;t)$ in terms of backflow of information from environment to system is lost; however, that is not the case. To explain this more carefully, we have to recall here some results from quantum hypothesis testing, particularly the one-shot, two-state discrimination problem [22].

Consider a quantum system whose state is represented by the density matrix ρ_1 with probability p and ρ_2 with probability $(1-p)$. We want to determine the density matrix that describes the true state of the quantum system by performing a measurement. If we consider some general positive operator valued measure (POVM) $\{M_j\}$, where $j \in \Omega$ is the set of possible outcomes, we may split this set in two cases. If the outcome of the measurement is inside of some subset $A \subset \Omega$, then we say that the state is ρ_1. Conversely, if the result of the measurement belongs to the complementary set A^c such that $A \cup A^c = \Omega$, we say that the state is ρ_2. Let us group the results of this measurement in another POVM given by the couple $\{T,\mathbb{I}-T\}$, with $T = \sum_{j\in A} M_j$.

Thus, when the true state is ρ_1 (which happens with probability p), we erroneously conclude that the state is ρ_2 with probability

$$\begin{aligned}(1-p)\sum_{j\in A^c}\mathrm{Tr}[\rho_1 M_j] &= (1-p)\mathrm{Tr}\Big[\rho_1\Big(\sum_{j\in A^c} M_j\Big)\Big]\\ &= (1-p)\mathrm{Tr}[\rho_1(\mathbb{I}-T)].\end{aligned}$$

On the other hand, when the true state is ρ_2, we erroneously conclude that the state is ρ_1 with probability

$$p\sum_{j\in A}\mathrm{Tr}[\rho_2 M_j] = p\mathrm{Tr}\Big[\rho_2\Big(\sum_{j\in A} M_j\Big)\Big] = p\mathrm{Tr}[\rho_2 T].$$

Note that when $p = 0$ or $p = 1$ we immediately obtain zero probability to identify wrongly the true state. The problem in one-shot, two-state discrimination is to examine the tradeoff between the two error probabilities $p\mathrm{Tr}[\rho_2 T]$ and $(1-p)\mathrm{Tr}[\rho_1(\mathbb{I}-T)]$. Thus, consider the best choice of T that minimizes the total error probability

$$\begin{aligned}&\min_{0\leqslant T\leqslant \mathbb{I}}\{p\mathrm{Tr}[\rho_2 T] + (1-p)\mathrm{Tr}[\rho_1(\mathbb{I}-T)]\}\\ &\quad= \min_{0\leqslant T\leqslant \mathbb{I}}\{(1-p) + \mathrm{Tr}[p\rho_2 T - (1-p)\rho_1 T]\}\\ &\quad= (1-p) - \max_{0\leqslant T\leqslant \mathbb{I}}[\mathrm{Tr}(\Delta T)],\end{aligned}$$

where $\Delta = (1-p)\rho_1 - p\rho_2$ is a self-adjoint operator (sometimes called a Helstrom matrix [23]) with trace $\mathrm{Tr}\Delta = 1-2p$, which vanishes only for the unbiased case $p = 1/2$. By using the same arguments as for $p = 1/2$ (see [4,22]) the result of the optimization process turns out to be

$$\min_{0\leqslant T\leqslant \mathbb{I}}\{p\mathrm{Tr}[\rho_2 T] + (1-p)\mathrm{Tr}[\rho_1(\mathbb{I}-T)]\} = \frac{1-\|\Delta\|_1}{2}. \tag{43}$$

Thus, the trace norm of $\Delta = (1-p)\rho_1 - p\rho_2$ gives our capability to distinguish correctly between ρ_1 and ρ_2 in the one-shot, two-state discrimination problem. Since that is only a function of the information we gather by the prior probability p and by the measurement T, $\|\Delta\|_1$ is a measure of the information we have.

Consider again a trace-preserving map $\Phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$. If it increases the trace norm of Δ, $\|\Phi(\Delta)\|_1 > \|\Delta\|_1$, we can assume that Φ carries information about the correct state of the system. Otherwise, it cannot decrease the probability to identify wrongly the true state. For that reason, if we process data before making a measurement, which means applying some Φ over the states, we cannot increase the trace norm of Δ since we cannot gain more information about some data just by processing it. This is the so-called data

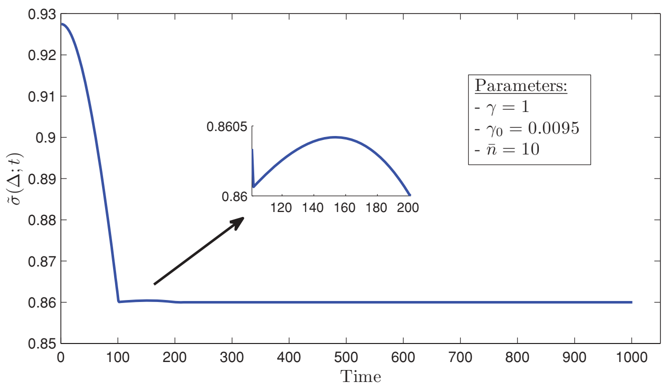

FIG. 1. (Color online) Behavior of $\tilde{\sigma}(\Delta;t)$ for the integro-differential model (44). The initial Helstrom matrix is $\Delta = 0.93\rho_1 - 0.07\rho_2$, where ρ_1 and ρ_2 are given in (46) and (47). There is a time period between 100 and 150 where $\tilde{\sigma}(\Delta;t)$ increases (the units of γ and γ_0 are [Time]$^{-1}$).

processing inequality [22], and it implies that Φ is contractive (i.e., positive). In addition, we can never dismiss that the states ρ_1 and ρ_2 are part of the larger system "SA" in such a way that they are the result of tracing out the ancillary systems $\rho_{1,2} = \mathrm{Tr}_A[\rho_{1A,2A}]$. By making the same analysis as before, in order not to gain information we then require the map $\mathbb{1}_d \otimes \Phi$ to be contractive (i.e., Φ completely contractive) and thus completely positive.

Going back to the problem of Markovianity, one may now realize that $\tilde{\sigma}(\Delta;t)$ is a measure of the information gained by the system for some initial Helstrom matrix Δ. This interpretation puts together the two concepts of Markovianity based upon divisibility and backflows of information. Particularly if one ignores the potential existence of an ancilla, under the case of unbiased discrimination problem, $\|\Delta\|_1 = \frac{1}{2}\|\rho_1 - \rho_2\|_1$, which is the definition of the trace distance between ρ_1 and ρ_2, recovering the approach of BLP. Nevertheless, in principle there is no reason to presuppose the fulfillment of these particular assumptions.

One can define a measure of non-Markovianity as (9), with $\tilde{\sigma}(\Delta;t)$ in the place of $\sigma(\rho_1,\rho_2;t)$. That will be zero if and only if (8) is; however, (8) is easier to compute because it avoids the complicated optimization procedure in (9) now upon every Helstrom matrix Δ.

We have already proposed examples of nondivisible dynamics with $\mathcal{N}_{\mathrm{BLP}}(\Lambda) = 0$, but for completeness we have computed $\tilde{\sigma}(\Delta;t)$ for the phenomenological integro-differential master equation [24]:

$$\frac{d\rho(t)}{dt} = \int_0^t k(t')L\rho(t-t')dt', \tag{44}$$

where $k(t) = \gamma e^{-\gamma t}$, and

$$L\rho = \gamma_0(\bar{n}+1)L_0\rho + \gamma_0\bar{n}L_1\rho, \tag{45}$$

with $\gamma,\gamma_0,\bar{n} \geqslant 0$, and L_0 and L_1 were given in (34). Despite that this kind of equation is used as a simple model of non-Markovian dynamics, it was shown in [19] that $\mathcal{N}_{\mathrm{BLP}}(\Lambda) = 0$. However, $\tilde{\sigma}(\Delta;t)$ is not always decreasing for any Helstrom matrix. For example, by taking $p = 0.07$ and

$$\rho_1 = \begin{pmatrix} 0.5 & 0 & 0 & 0.48 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.019 & 0 \\ 0.48 & 0 & 0 & 0.48 \end{pmatrix}, \tag{46}$$

$$\rho_2 = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

there is a small but nonzero period where $\tilde{\sigma}(\Delta;t)$ grows as we have represented in Fig. 1. This denotes the existence of a backflow of information, which was expected from the nondivisible character of this dynamics. One may look for larger increments of $\tilde{\sigma}(\Delta;t)$; however, it is not easy to find such an example. To actually solve the optimization problem on Helstrom matrices is quite hard, as we have already pointed out.

VII. CONCLUSIONS

We have analyzed two concepts of Markovianity, one based on the divisibility property of the dynamical map and the other based upon the distinguishability of quantum states. We have given very simple examples where these two criteria do not coincide. Furthermore, we have proposed a way to make them equivalent, in the sense that Markovianity would be identified by divisibility, but keeping the interpretation in terms of flows of information. For that we resort to the results in the one-shot, two-state discrimination problem and point out that an increase of the trace norm of any Hermitian matrix during the dynamics can also be associated with a backflow of information from the environment to the system.

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