Entropic uncertainty relations in multidimensional position and momentum spaces

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Commutator-based entropic uncertainty relations in multidimensional position and momentum spaces are derived, twofold generalizing previous entropic uncertainty relations for one-mode states. They provide optimal lower bounds and imply the multidimensional variance-based uncertainty principle. The article concludes with an open conjecture.

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I. INTRODUCTION

Without a classical analog, uncertainty relations are one of the most fundamental ideas of quantum mechanics, underlying many conceptual differences between classical and quantum theories. They reveal by rigorous inequalities that incompatible observables cannot be measured to arbitrarily high precision simultaneously. They are applied widely in areas both related and unrelated to quantum mechanics, such as entanglement detection [1–7] quantum cryptography [8–10], and signal processing [11,12].

We associate a random variable A with an operator $\hat{A}$. The possible values of A are the eigenvalues of $\hat{A}$, and the probability (density) that A takes the value a is the probability (density) that we get a when we measure the operator $\hat{A}$ with respect to a quantum state $|\Psi\rangle$. The variance of $\hat{A}$, denoted $\Delta\hat{A}$, is the variance of A, and the (differential) Shannon entropy of $\hat{A}$, denoted $H(\hat{A},|\Psi\rangle)$, or $H(\hat{A})$, or $H(P(a))$, where $P(a)$ is the distribution of A, is defined as the (differential) Shannon entropy of A.

The famous commutator-based Heisenberg uncertainty principle is formulated by Robertson [13] for observables:

$$\Delta\hat{A}\Delta\hat{B} \geqslant \tfrac{1}{4}|\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle|^2. \tag{1}$$

We set $\hbar = 1$ throughout this article. Denote the n-dimensional position and momentum space H_n. For the position and the momentum operators $\hat{x},\hat{p}$ on H_1, Eq. (1) reduces to

$$\Delta\hat{x}\Delta\hat{p} \geqslant \tfrac{1}{4}. \tag{2}$$

Equation (2) is generalized to multidimensional spaces. The n-dimensional position and momentum space H_n is described by $2n$ operators $\hat{R} = (\hat{x}_1,\hat{p}_1,\hat{x}_2,\ldots,\hat{x}_n,\hat{p}_n)$, which satisfy the canonical commutation relations $[\hat{R}_j,\hat{R}_k] = i\Omega_{jk}$, for $j,k = 1,2,\ldots,2n$, where

$$\Omega = \bigoplus_{j=1}^{n}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

For an n-mode density operator ρ, define the covariance matrix γ as

$$\gamma_{jk} = 2\mathrm{tr}\{\rho[R_j - \mathrm{tr}(\rho R_j)][R_k - \mathrm{tr}(\rho R_k)]\} - i\Omega_{jk}. \tag{4}$$

γ is real and symmetric. The multidimensional variance-based uncertainty relation [14] is given by

$$\gamma + i\Omega \geqslant 0. \tag{5}$$

Reference [15] provides much more detailed backgrounds. There are some other types of uncertainty relations for multimode states (e.g., [6]).

A different approach is to formulate uncertainty relations based on the Shannon entropy, rather than the variance. In the continuous variable case, Refs. [16,17] prove

$$\inf_{|\Psi\rangle}\{H(\hat{x}) + H(\hat{p})\} = 1 + \ln\pi \tag{6}$$

for H_1. Equation (6) implies Eq. (2) [16,17], showing the advantages of entropic uncertainty relations. Lots of entropic uncertainty relations in the discrete variable case are proposed (e.g., [18]). Reference [9] is a recent survey on this topic.

The main contribution of the present work is to twofold generalize Eq. (6). The commutator-based entropic uncertainty relation Eq. (7) holds for more general operators on multidimensional position and momentum spaces. Equation (7) implies the multidimensional variance-based uncertainty relation Eq. (5), so every time we use Eq. (5) in applications, we might think of using Eq. (7) instead to produce better results.

Theorem. We have entropic uncertainty relations in H_n:

$$\inf_{|\Psi\rangle}\{H(\hat{A}) + H(\hat{B})\} = 1 + \ln\pi + \ln|[\hat{A},\hat{B}]|, \tag{7}$$

where

$$\hat{A} = \sum_{i=1}^{n}(a_i\hat{x}_i + a_i'\hat{p}_i), \quad \hat{B} = \sum_{i=1}^{n}(b_i\hat{x}_i + b_i'\hat{p}_i) \tag{8}$$

are linear combinations of the components of $\hat{R}$ (a_i,a_i',b_i,b_i' are real coefficients). Equivalently and more precisely,

$$\begin{aligned}\inf_{|\Psi\rangle}&\left\{H\left(\sum_{i=1}^{n}(a_i\hat{x}_i + a_i'\hat{p}_i)\right) + H\left(\sum_{i=1}^{n}(b_i\hat{x}_i + b_i'\hat{p}_i)\right)\right\}\\ &= 1 + \ln\pi + \ln\left|\left[\sum_{i=1}^{n}(a_i\hat{x}_i + a_i'\hat{p}_i), \sum_{i=1}^{n}(b_i\hat{x}_i + b_i'\hat{p}_i)\right]\right|\\ &= 1 + \ln\pi + \ln\left|\sum_{i=1}^{n}(a_ib_i' - b_ia_i')\right|. \end{aligned}\tag{9}$$

Letting $n = a_1 = b_1' = 1$, $a_1' = b_1 = 0$, Eq. (7) obviously reduces to Eq. (6). Equation (7) strengthens the importance of commutation relations and supports the intuitive idea that commutators quantify the extent of incompatibility of two operators.

The paper is organized as follows. Section II provides the detailed proof of the main theorem, which is arranged in lemmas to help you get the whole picture. Section III proves that Eq. (7) implies the variance-based uncertainty principle Eq. (5). Section IV concludes with an open conjecture.

II. PROOF OF THE MAIN THEOREM

We begin our discussion in H_1.

The fractional Fourier transform [19] $\Phi(\omega)=\hat{F}(\theta)\Psi(x)$ plays an important role in the proof. It is defined as

$$\Phi(\omega)=\sqrt{\frac{\exp\left(i\theta-\frac{\pi}{2}i\right)}{2\pi\sin\theta}}\exp\left(\frac{i\omega^2}{2\tan\theta}\right)\times\int_{-\infty}^{+\infty}\exp\left(-\frac{i\omega x}{\sin\theta}+\frac{ix^2}{2\tan\theta}\right)\Psi(x)dx. \tag{10}$$

Naturally, $\hat{F}(0)$ is the identity map $\hat{I}$, and

$$\hat{F}\left(\frac{\pi}{2}\right)=\hat{\mathcal{F}},\quad \hat{F}\left(-\frac{\pi}{2}\right)=\hat{\mathcal{F}}^{-1}, \tag{11}$$

where $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively. $\hat{F}$ satisfies [19]

$$\hat{F}(\theta_1+\theta_2)=\hat{F}(\theta_1)\circ\hat{F}(\theta_2)=\hat{F}(\theta_2)\circ\hat{F}(\theta_1). \tag{12}$$

The eigenvector of the operator

$$\hat{x}\cos\theta+\hat{p}\sin\theta=\hat{x}\cos\theta-i\sin\theta\frac{d}{dx} \tag{13}$$

corresponding to the eigenvalue ω is

$$\sqrt{\frac{\exp(\frac{\pi}{2}i-i\theta)}{2\pi\sin\theta}}\exp\left(-\frac{i\omega^2}{2\tan\theta}+\frac{i\omega x}{\sin\theta}-\frac{ix^2}{2\tan\theta}\right). \tag{14}$$

Let $\Psi(x)$ be the position wave function of a quantum state $|\Psi\rangle$. Following from the definition of the fractional Fourier transform, the wave function in the $\hat{x}\cos\theta_i+\hat{p}\sin\theta_i$ representation is $\Psi_i=\hat{F}(\theta_i)\Psi(x)$ for $i=1,2$, which implies $\Psi_2=\hat{F}(\theta_2-\theta_1)\Psi_1$ from Eq. (12). Therefore, wave functions of the same quantum state in different representations are related by the fractional Fourier transform.

Lemma 1. For $c\in\mathbf{R}$,

$$H(c\hat{A})=H(\hat{A})+\ln|c| \tag{15}$$

(see Ref. [20]).

Lemma 2.

$$\inf_{|\Psi\rangle}\{H(\hat{x}\cos\theta_1+\hat{p}\sin\theta_1)+H(\hat{x}\cos\theta_2+\hat{p}\sin\theta_2)\} = 1+\ln\pi+\ln|\sin(\theta_2-\theta_1)|, \tag{16}$$

which can be rephrased as

$$\inf_{\Psi(x)}\{H(|\Psi(x)|^2)+H(|\Phi(\omega)|^2)\}=1+\ln\pi+\ln|\sin\theta|, \tag{17}$$

where $\Psi(x)$ runs through all legitimate wave functions and $\Phi(\omega)=\hat{F}(\theta)\Psi(x)$ (see Ref. [12]).

Theorem 1 (Theorem in H_1).

$$\inf_{|\Psi\rangle}\{H(a_1\hat{x}+a_2\hat{p})+H(b_1\hat{x}+b_2\hat{p})\}=1+\ln\pi+\ln|[a_1\hat{x}+a_2\hat{p},b_1\hat{x}+b_2\hat{p}]|=1+\ln\pi+\ln|a_1b_2-a_2b_1| \tag{18}$$

Proof. Using Lemma 1 and then Lemma 2, we have

$$\begin{aligned}&\inf_{|\Psi\rangle}\{H(a_1\hat{x}+a_2\hat{p})+H(b_1\hat{x}+b_2\hat{p})\}\\&=\inf_{|\Psi\rangle}\left\{H\left(\frac{a_1}{\sqrt{a_1^2+a_2^2}}\hat{x}+\frac{a_2}{\sqrt{a_1^2+a_2^2}}\hat{p}\right)+\ln\sqrt{a_1^2+a_2^2}+H\left(\frac{b_1}{\sqrt{b_1^2+b_2^2}}\hat{x}+\frac{b_2}{\sqrt{b_1^2+b_2^2}}\hat{p}\right)+\ln\sqrt{b_1^2+b_2^2}\right\}\\&=1+\ln\pi+\ln|a_1b_2-a_2b_1|=1+\ln\pi+\ln|[a_1\hat{x}+a_2\hat{p},b_1\hat{x}+b_2\hat{p}]|.\end{aligned} \tag{19}$$

We have completed our discussion in H_1. Let us move on to H_2. Define

$$R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{20}$$

Lemma 3.

$$\begin{aligned}&\inf_{|\Psi\rangle}\left\{H\left((a_1\ a_2)\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix}+(a_3\ a_4)\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix},|\Psi\rangle\right)+H\left((b_1\ b_2)\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix}+(b_3\ b_4)\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix},|\Psi\rangle\right)\right\}\\&=\inf_{|\Psi\rangle}\left\{H\left((a_1\ a_2)R_{\theta_1}\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix}+(a_3\ a_4)R_{\theta_2}\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix},\hat{F}(\theta_1)\otimes\hat{F}(\theta_2)|\Psi\rangle\right)\right.\end{aligned}$$

(a) Invariance of infimum under local rotations.

We apply local rotations $(\hat{x}_1,\hat{p}_1)^T\to R_{\theta_1}(\hat{x}_1,\hat{p}_1)^T$ and $(\hat{x}_2,\hat{p}_2)^T\to R_{\theta_2}(\hat{x}_2,\hat{p}_2)^T$. Under this transform, a state $|\Psi\rangle$, whose position wave function is $\Psi(x_1,x_2)$, should become a new state denoted as $\hat{F}(\theta_1)\otimes\hat{F}(\theta_2)|\Psi\rangle$, whose position wave function is $[\hat{F}(\theta_1)\otimes\hat{F}(\theta_2)]\Psi(x_1,x_2)$. Thus,

$$+ H\left((b_1\ b_2)R_{\theta_1}\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (b_3\ b_4)R_{\theta_2}\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}, \hat{F}(\theta_1)\otimes\hat{F}(\theta_2)|\Psi\rangle\right)\Bigg\}$$

$$= \inf_{|\Psi\rangle}\left\{H\left((a_1\ a_2)R_{\theta_1}\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (a_3\ a_4)R_{\theta_2}\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}, |\Psi\rangle\right) + H\left((b_1\ b_2)R_{\theta_1}\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (b_3\ b_4)R_{\theta_2}\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}, |\Psi\rangle\right)\right\}. \quad (21)$$

We pass from the third line to the fourth by using the fact that $\hat{F}(\theta_1)\otimes\hat{F}(\theta_2)|\Psi\rangle$ is a legitimate quantum state if and only if $|\Psi\rangle$ is a legitimate quantum state. The commutator is preserved, which can be verified by direct computation:

$$\left[(a_1\ a_2)\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (a_3\ a_4)\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}, (b_1\ b_2)\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (b_3\ b_4)\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}\right]$$

$$= \left[(a_1\ a_2)R_{\theta_1}\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (a_3\ a_4)R_{\theta_2}\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}, (b_1\ b_2)R_{\theta_1}\begin{pmatrix}\hat{x}_1\\ \hat{p}_1\end{pmatrix} + (b_3\ b_4)R_{\theta_2}\begin{pmatrix}\hat{x}_2\\ \hat{p}_2\end{pmatrix}\right]. \quad (22)$$

(b) Invariance of infimum under global rotations.

For simplicity, the position wave function is denoted $\Psi[(x_1,x_2)^T]$. By change of variables, $(x_1,x_2)^T \to R_\theta(x_1,x_2)^T$, which naturally yields $(\hat{x}_1,\hat{x}_2)^T \to R_\theta(\hat{x}_1,\hat{x}_2)^T$ and $(\hat{p}_1,\hat{p}_2)^T \to R_\theta(\hat{p}_1,\hat{p}_2)^T$, we pass from the first line to the second in the following.

$$\inf_{|\Psi\rangle}\left\{H\left((a_1\ a_2)\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (a_3\ a_4)\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \Psi\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right) + H\left((b_1\ b_2)\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (b_3\ b_4)\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \Psi\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right)\right\}$$

$$= \inf_{|\Psi\rangle}\Bigg\{H\left((a_1\ a_2)R_\theta\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (a_3\ a_4)R_\theta\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \Psi\left[R_\theta\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right]\right)$$

$$+ H\left((b_1\ b_2)R_\theta\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (b_3\ b_4)R_\theta\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \Psi\left[R_\theta\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right]\right)\Bigg\}$$

$$= \inf_{|\Psi\rangle}\left\{H\left((a_1\ a_2)R_\theta\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (a_3\ a_4)R_\theta\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \Psi\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right) + H\left((b_1\ b_2)R_\theta\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (b_3\ b_4)R_\theta\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \Psi\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right)\right\}. \quad (23)$$

We pass from the third line to the fourth by using the fact that $\Psi[(x_1,x_2)^T]$ is a legitimate wave function if and only if $\Psi[R_\theta(x_1,x_2)^T]$ is a legitimate wave function. The commutator is preserved, which can be verified by direct computation:

$$\left[(a_1\ a_2)\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (a_3\ a_4)\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, (b_1\ b_2)\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (b_3\ b_4)\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}\right]$$

$$= \left[(a_1\ a_2)R_\theta\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (a_3\ a_4)R_\theta\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, (b_1\ b_2)R_\theta\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + (b_3\ b_4)R_\theta\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}\right]. \quad (24)$$

Both local and global rotations are symplectic transformations, which preserve commutation relations (Ref. [15] provides detailed relevant backgrounds). This is an alternative argument for the validity of Eqs. (22) and (24).

Lemma 4.

$$\inf_{|\Psi\rangle}\{H(\hat{x}_1) + H(\hat{x}_1\cos\theta + \hat{p}_1\sin\theta)\} = 1 + \ln\pi + \ln|\sin\theta|. \quad (25)$$

Proof. We first show that the right-hand side is a valid lower bound and then prove its optimality. Let $\Psi(x_1,x_2)$ be the position wave function of the quantum state $|\Psi\rangle$ and $\Phi(\omega,x_2)$ be the wave function of the same state in the representation $(\hat{x}_1\cos\theta + \hat{p}_1\sin\theta, \hat{x}_2)$. Thus, $\Phi(\omega,x_2) = [\hat{F}(\theta)\otimes\hat{I}]\Psi(x_1,x_2)$. Define

$$P(x_2) = \int_{-\infty}^{+\infty}|\Psi(x_1,x_2)|^2\,dx_1, \quad (26)$$

which satisfies

$$\int_{-\infty}^{+\infty}P(x_2)dx_2 = 1. \quad (27)$$

According to the definition of $H(\hat{x}_1)$ and due to the concavity of the Shannon entropy,

$$H(\hat{x}_1) = -\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|\Psi(x_1,x_2)|^2dx_2\right)\left(\ln\int_{-\infty}^{+\infty}|\Psi(x_1,x_2)|^2dx_2\right)dx_1 = -\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}P(x_2)\frac{|\Psi(x_1,x_2)|^2}{P(x_2)}dx_2\right)$$

$$\times\left(\ln\int_{-\infty}^{+\infty}P(x_2)\frac{|\Psi(x_1,x_2)|^2}{P(x_2)}dx_2\right)dx_1 \geqslant -\int_{-\infty}^{+\infty}P(x_2)\left(\int_{-\infty}^{+\infty}\frac{|\Psi(x_1,x_2)|^2}{P(x_2)}\ln\frac{|\Psi(x_1,x_2)|^2}{P(x_2)}dx_1\right)dx_2. \quad (28)$$

Similarly,

$$H(\hat{x}_1\cos\theta+\hat{p}_1\sin\theta)\geqslant-\int_{-\infty}^{+\infty}P(x_2)\left(\int_{-\infty}^{+\infty}\frac{|\Phi(\omega,x_2)|^2}{P(x_2)}\ln\frac{|\Phi(\omega,x_2)|^2}{P(x_2)}d\omega\right)dx_2, \tag{29}$$

where

$$\Phi(\omega,x_2)=[\hat{F}(\theta)\otimes\hat{I}]\Psi(x_1,x_2)\Longrightarrow\frac{\Phi(\omega,x_2)}{\sqrt{P(x_2)}}=\hat{F}(\theta)\frac{\Psi(x_1,x_2)}{\sqrt{P(x_2)}}. \tag{30}$$

In the last equation, Ψ and Φ are regarded as functions only of x_1 and ω, respectively. Finally, applying Lemma 2,

$$\begin{aligned}H(\hat{x}_1)+H(\hat{x}_1\cos\theta+\hat{p}_1\sin\theta)&\geqslant-\int_{-\infty}^{\infty}P(x_2)\left(\int_{-\infty}^{\infty}\frac{|\Psi(x_1,x_2)|^2}{P(x_2)}\ln\frac{|\Psi(x_1,x_2)|^2}{P(x_2)}dx_1+\int_{-\infty}^{\infty}\frac{|\Phi(\omega,x_2)|^2}{P(x_2)}\ln\frac{|\Phi(\omega,x_2)|^2}{P(x_2)}d\omega\right)dx_2\\&\geqslant\int_{-\infty}^{\infty}P(x_2)(1+\ln\pi+\ln|\sin\theta|)dx_2=1+\ln\pi+\ln|\sin\theta|.\end{aligned} \tag{31}$$

The lower bound in Eq. (17) can be attained. Suppose it is attained for $\psi(x)$ and $\phi(\omega)$ satisfying $\phi(\omega)=\hat{F}(\theta)\psi(x)$. Let

$$\Psi(x_1,x_2)=\psi(x_1)\varphi(x_2),\quad\Phi(\omega,x_2)=\phi(\omega)\varphi(x_2), \tag{32}$$

where φ is an arbitrary one-dimensional legitimate wave function. In this case, it is easy to verify that the lower bound in Eq. (25) is attained, proving its optimality.

Lemma 5.

$$\inf_{|\Psi\rangle}\{H(\hat{x}_1+a\hat{x}_2)+H(\hat{x}_1\cos\theta+\hat{p}_1\sin\theta)\}=1+\ln\pi+\ln|\sin\theta|. \tag{33}$$

Proof. Let $\Psi(x_1,x_2)$ be the position wave function of the quantum state $|\Psi\rangle$ and $\Phi(\omega,x_2)$ be the wave function of the same state in the representation $(\hat{x}_1\cos\theta+\hat{p}_1\sin\theta,\hat{x}_2)$:

$$\begin{aligned}\Phi(\omega,x_2)=[\hat{F}(\theta)\otimes\hat{I}]\Psi(x_1,x_2)&=\sqrt{\frac{\exp\left(i\theta-\frac{\pi}{2}i\right)}{2\pi\sin\theta}}\exp\left(\frac{i\omega^2}{2\tan\theta}\right)\int_{-\infty}^{+\infty}\exp\left(-\frac{i\omega x_1}{\sin\theta}+\frac{ix_1^2}{2\tan\theta}\right)\Psi(x_1,x_2)\,dx_1\\&=\sqrt{\frac{\exp\left(i\theta-\frac{\pi}{2}i\right)}{2\pi\sin\theta}}\exp\left(\frac{i\omega^2}{2\tan\theta}\right)\int_{-\infty}^{+\infty}\exp\left(-\frac{i\omega(x_1-ax_2)}{\sin\theta}+\frac{i(x_1-ax_2)^2}{2\tan\theta}\right)\Psi(x_1-ax_2,x_2)\,dx_1\end{aligned} \tag{34}$$

implies

$$\begin{aligned}\Phi(\omega,x_2)\exp\left(-\frac{ia\omega x_2}{\sin\theta}\right)&=\sqrt{\frac{\exp\left(i\theta-\frac{\pi}{2}i\right)}{2\pi\sin\theta}}\exp\left(\frac{i\omega^2}{2\tan\theta}\right)\int_{-\infty}^{+\infty}\exp\left(-\frac{i\omega x_1}{\sin\theta}+\frac{ix_1^2}{2\tan\theta}\right)\exp\left(\frac{-2iax_1x_2+ia^2x_2^2}{2\tan\theta}\right)\\&\quad\times\Psi(x_1-ax_2,x_2)dx_1=[\hat{F}(\theta)\otimes\hat{I}]\left[\exp\left(\frac{-2iax_1x_2+ia^2x_2^2}{2\tan\theta}\right)\Psi(x_1-ax_2,x_2)\right].\end{aligned} \tag{35}$$

Finally, applying Lemma 4,

$$\begin{aligned}&\inf_{|\Psi\rangle}\{H(\hat{x}_1+a\hat{x}_2)+H(\hat{x}_1\cos\theta+\hat{p}_1\sin\theta)\}=\inf_{|\Psi\rangle}\left\{H\left(\int_{-\infty}^{\infty}|\Phi(\omega,x_2)|^2dx_2\right)+H\left(\int_{-\infty}^{\infty}|\Psi(x_1-ax_2,x_2)|^2dx_2\right)\right\}\\&\quad=\inf_{|\Psi\rangle}\left\{H\left(\int_{-\infty}^{\infty}\left|\Phi(\omega,x_2)\exp\left(-\frac{ia\omega x_2}{\sin\theta}\right)\right|^2dx_2\right)+H\left(\int_{-\infty}^{\infty}\left|\exp\left(\frac{-2iax_1x_2+ia^2x_2^2}{2\tan\theta}\right)\Psi(x_1-ax_2,x_2)\right|^2dx_2\right)\right\}\\&\quad=1+\ln\pi+\ln|\sin\theta|.\end{aligned} \tag{36}$$

Similarly,

$$\inf_{|\Psi\rangle}\{H(\hat{x}_1)+H(\hat{x}_1\cos\theta+\hat{p}_1\sin\theta+a\hat{x}_2)\}=1+\ln\pi+\ln|\sin\theta|. \tag{37}$$

Theorem 2 (Theorem in H_2).

$$\begin{aligned}&\inf_{|\Psi\rangle}\{H(a_1\hat{x}_1+a_2\hat{p}_1+a_3\hat{x}_2+a_4\hat{p}_2)+H(b_1\hat{x}_1+b_2\hat{p}_1+b_3\hat{x}_2+b_4\hat{p}_2)\}\\&\quad=1+\ln\pi+\ln|[a_1\hat{x}_1+a_2\hat{p}_1+a_3\hat{x}_2+a_4\hat{p}_2,b_1\hat{x}_1+b_2\hat{p}_1+b_3\hat{x}_2+b_4\hat{p}_2]|.\end{aligned} \tag{38}$$

Proof. The idea is to reduce the most general case to more and more simpler cases by using lemmas proved previously. Assume $b_2=b_4=0$ without loss of generality. Otherwise, we apply local rotations [Lemma 3(a)], which preserve the commutator. We only need to prove

$$\inf_{|\Psi\rangle}\{H(a_1\hat{x}_1+a_2\hat{p}_1+a_3\hat{x}_2+a_4\hat{p}_2)+H(b_1\hat{x}_1+b_3\hat{x}_2)\}=1+\ln\pi+\ln|[a_1\hat{x}_1+a_2\hat{p}_1+a_3\hat{x}_2+a_4\hat{p}_2,b_1\hat{x}_1+b_3\hat{x}_2]|. \tag{39}$$

Applying global rotations [Lemma 3(b)], which preserve the commutator, we assume $b_3 = 0$. We only need to prove

$$\inf_{|\Psi\rangle}\{H(a_1\hat{x}_1 + a_2\hat{p}_1 + a_3\hat{x}_2 + a_4\hat{p}_2) + H(b_1\hat{x}_1)\} = 1 + \ln\pi + \ln|[a_1\hat{x}_1 + a_2\hat{p}_1 + a_3\hat{x}_2 + a_4\hat{p}_2, b_1\hat{x}_1]|. \tag{40}$$

Assume $a_4 = 0$ by applying local rotations. It suffices to show

$$\inf_{|\Psi\rangle}\{H(a_1\hat{x}_1 + a_2\hat{p}_1 + a_3\hat{x}_2) + H(b_1\hat{x}_1)\} = 1 + \ln\pi + \ln|[a_1\hat{x}_1 + a_2\hat{p}_1 + a_3\hat{x}_2, b_1\hat{x}_1]|, \tag{41}$$

which is equivalent to (due to Lemma 1)

$$\inf_{|\Psi\rangle}\left\{H\left(\hat{x}_1\cos\theta + \hat{p}_1\sin\theta + \frac{a_3}{a}\hat{x}_2\right) + H(\hat{x}_1)\right\} = 1 + \ln\pi + \ln|[\hat{x}_1\cos\theta + \hat{p}_1\sin\theta, \hat{x}_1]| = 1 + \ln\pi + \ln|\sin\theta|, \tag{42}$$

where $a_1 = a\cos\theta$, $a_2 = a\sin\theta$. Now we see that Eq. (42) is precisely Eq. (37).

We have completed our discussion in H_2. Generally, Theorem in $H_n (n > 2)$ can be proved similarly with only the following minor revision (no essential new ideas included). We should introduce $R \in \mathrm{SO}(n)$ (n-dimensional rotation) to replace the role of R_θ (two-dimensional rotation) in Lemma 3(b), simply because the global rotation becomes an n-dimensional rotation in H_n. The n-dimensional version of Eq. (24) can be verified by direct computation and making use of $R \in \mathrm{SO}(n)$, or by simply using the fact that R is a symplectic transformation.

We have completed the proof of the main theorem. Due to the concavity of the differential Shannon entropy, we observe that Eq. (7) also holds for mixed states, which are probabilistic mixtures of pure states.

III. DISCUSSIONS

Proposition. Equation (7) implies the variance-based uncertainty principle Eq. (5). It also implies Serafini's multidimensional uncertainty principle—Eq. (8) in [6].

Proof. We first provide an equivalent description of Eq. (5). Let d, d' be two $2n$-dimensional real vectors: $d = (a_1, a_1', a_2, \ldots, a_n, a_n')^T$ and $d' = (b_1, b_1', b_2, \ldots, b_n, b_n')^T$. Obviously,

$$\gamma + i\Omega \geqslant 0 \Longleftrightarrow (d + id')^\dagger(\gamma + i\Omega)(d + id') \geqslant 0 \;\forall\, d, d'. \tag{43}$$

If we define operators $\hat{A}, \hat{B}$ as Eq. (8), it is equivalent to

$$\Delta\hat{A} + \Delta\hat{B} \geqslant |d'^T\Omega d| = \left|\sum_{i=1}^{n}(a_i b_i' - b_i a_i')\right| = |[\hat{A}, \hat{B}]|, \tag{44}$$

because $d^T\gamma d = 2\Delta\hat{A}$ and $d'^T\gamma d' = 2\Delta\hat{B}$. We thus see that Eq. (5) is simply a direct consequence of the Heisenberg uncertainty principle. $H(\hat{A})$ and $\Delta\hat{A}$ are, respectively, the differential Shannon entropy and the variance of the same distribution. From [20], we have $\Delta\hat{A} \geqslant \frac{1}{2\pi}\exp[2H(\hat{A}) - 1]$, $\Delta\hat{B} \geqslant \frac{1}{2\pi}\exp[2H(\hat{B}) - 1]$. By basic inequalities, we obtain

$$\begin{aligned}\Delta\hat{A} + \Delta\hat{B} &\geqslant 2\sqrt{\Delta\hat{A}\Delta\hat{B}} \\ &\geqslant 2\sqrt{\frac{1}{4\pi^2}\exp[2H(\hat{A}) + 2H(\hat{B}) - 2]} \\ &\geqslant \frac{1}{\pi}\exp(1 + \ln\pi + \ln|[\hat{A}, \hat{B}]| - 1) = |[\hat{A}, \hat{B}]|.\end{aligned} \tag{45}$$

Finally, Eq. (7) implies Serafini's uncertainty principle for multimode states (Eq. (8) in [6]) because Eq. (8) in [6] is a necessary condition of Eq. (5) [6].

IV. CONCLUSION AND OUTLOOK

I have derived the commutator-based entropic uncertainty relation Eq. (7), which holds for more general Hermitian operators on multidimensional position and momentum spaces, twofold generalizing the previous entropic uncertainty relation Eq. (6). The lower bound in Eq. (7) is optimal, and Eq. (7) implies the multidimensional variance-based uncertainty principle Eq. (5). Every time we use Eq. (5) in applications, we might think of using Eq. (7) instead to produce better results.

One might try to seek for a simplified proof of the main theorem by using the Stone–von Neumann theorem [21]. However, the present proof at least has the advantage of being elementary.

A fundamental and interesting problem is to study how far we can generalize Eq. (7). We restrict $\hat{A}, \hat{B}$ to be of the form Eq. (8) in the present work, but does Eq. (7) hold for general Hermitian operators? At least we should modify Eq. (7) in the case that $[\hat{A}, \hat{B}]$ is not a number operator. I propose the following open conjecture, which (if holds) implies the Heisenberg uncertainty principle Eq. (1).

Conjecture. For arbitrary Hermitian operators $\hat{A}, \hat{B}$ on multidimensional position and momentum spaces, we have

$$H(\hat{A}, |\Psi\rangle) + H(\hat{B}, |\Psi\rangle) \geqslant 1 + \ln\pi + \ln|\langle\Psi|[\hat{A}, \hat{B}]|\Psi\rangle|. \tag{46}$$

If the conjecture is false, then can we add some loose restrictions on $\hat{A}, \hat{B}$ [not as strong as the restriction that $\hat{A}, \hat{B}$ should take the form of Eq. (8)] so that Eq. (46) holds?

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