Geometric phases in the presence of a composite environment

Paula I. Villar* and Fernando C. Lombardo†

Departamento de Física Juan José Giambiagi, FCEyN UBA, Facultad de Ciencias Exactas y Naturales, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

(Received 3 December 2010; published 24 May 2011)

We compute the geometric phase for a spin-1/2 particle under the presence of a composite environment, composed of an external bath (modeled by an infinite set of harmonic oscillators) and another spin-1/2 particle. We consider both cases: an initial entanglement between the spin-1/2 particles and an initial product state in order to see if the initial entanglement has an enhancement effect on the geometric phase of one of the spins. We follow the nonunitary evolution of the reduced density matrix and evaluate the geometric phase for a single two-level system. We also show that the initial entanglement enhances the sturdiness of the geometric phase under the presence of an external composite environment.

DOI: 10.1103/PhysRevA.83.052121 PACS number(s): 03.65.Vf, 03.65.Ud, 03.67.Pp

I. INTRODUCTION

It is widely known that a system can retain the information of its motion when it undergoes a cyclic evolution, in the form of a geometric phase (GP). This was first put forward by Pancharatnam in optics [1] and later studied explicitly by Berry in a general quantal system [2]. Since then, great progress has been achieved in this field as it became clear that geometric phases had important consequences for quantum systems. The original notion of Berry phase has been extended to the case of nonadiabatic evolutions [3]. As an important evolvement, the application of the geometric phase has been proposed in many fields, such as the geometric quantum computation. Due to its global properties, the GP is propitious to construct fault tolerant quantum gates. The study of the GP was soon extended to open quantum systems. In this context, many authors have analyzed the correction to the GP under the influence of an external environment using different approaches [4–11]. The interest on geometric phases has lately reached composite systems such as bipartite systems. In a previous article [12], we have computed the GP for the bipartite system under the influence of a nonunitary evolution, induced by a bosonic or fermionic environment. There, we have shown that entanglement plays an important role in the robustness of the GP for open systems. In [13], it was shown a particular situation in which an initial separable state remains separable so that the GP of the system is always equal to the sum of the geometric phases of its subsystems under an unitary evolution.

The presence of an environment can destroy all the traces of the quantumness of a system. All real world quantum systems interact with their surrounding environment to a greater or lesser extent. As the quantum system is in interaction with an environment defined as any degrees of freedom coupled to the system which can entangle its states, a degradation of pure states into mixtures takes place. No matter how weak the coupling that prevents the system from being isolated, the evolution of an open quantum system is eventually plagued by nonunitary features like decoherence and dissipation. Decoherence, in particular, is a quantum effect whereby the system loses its ability to exhibit coherent behavior. Nowadays, decoherence stands as a serious obstacle in quantum information processing.

In this context, there are many efforts to find a GP that "survives" the presence of the environment. What is more, the GP for a state under nonunitary evolution has not yet been directly measured. Recently, using a tomographic approach, the GP was measured for a qubit coupled to a critical environment using a nuclear magnetic resonance (NMR) quantum simulator [14].

In this paper, we study the geometric phase (GP) of a qubit (spin 1) in a noisy "composite" environment. That means, we start with a two-qubit state (spin 1 and spin 2) coupled to an external reservoir and then, we focus on only one qubit (by tracing out the other spin 2) in order to study the geometric phase (GP) of the resulting spin 1 (our subsystem). In the end, the model is different to other preexisting models, such as, for example, two qubits in a noisy environment and/or a solely spin in a noisy environment. The reason for choosing this model is twofold. On one side, it is the natural sequel to other previous studies of the GP in open quantum systems and has interesting features that differentiate this model from the others studied previously. On the other side, by taking into account new results about the existence of an environmentally induced GP for quantum states [15], we present a new insight on the effect that the degree of entanglement has on the geometric phase of a particle of spin 1/2 and, a possible way to measure it using NMR quantum simulators. Following this line, we shall start by describing the model and computing the geometric phase for a general environment in Sec. II. In Sec. III, we shall focus on a bosonic environment writing explicitly the decoherence factors involved. We shall consider different situations for the initial state of the bipartite: either entangled (Sec. III A) or a product state (Sec. III B). Further, in Sec. III C we shall study the differences that arise in the former computations if one of the spin-1/2 particles is not coupled to the external environment. Finally, we conclude our results in Sec. IV.

II. DECOHERENCE INDUCED ON ONE TWO-LEVEL SYSTEM DUE TO A COMPOSITE ENVIRONMENT

We shall consider a bipartite system (as in [12]), that is to say, two interacting two-level systems, both coupled

*paula@df.uba.ar
†lombardo@df.uba.ar

 ©2011 American Physical Society

to an external reservoir. We shall study a model where the Hamiltonians that describe the complete evolution are given by $H = H_S + H_I + H_B$ where the system's and interaction Hamiltonians are, respectively, defined as

$$H_S = \frac{\hbar\Omega_1}{2}\sigma_z^1 + \frac{\hbar\Omega_2}{2}\sigma_z^2 + \chi\sigma_z^1 \otimes \sigma_z^2, \tag{1}$$

$$H_I = \sigma_z^1 \otimes \sum_{n=1}^{N} \lambda_{1n} q_n + \sigma_z^2 \otimes \sum_{n=1}^{N} \lambda_{2n} q_n. \tag{2}$$

The environment's Hamiltonian H_B will be left without definition until we are faced with a particular example.

One approach to know the nonunitary features induced by the environment in the bipartite subsystem is by tracing out the degrees of freedom of the environment as suggested in Ref. [16]. For a general arbitrary two-qubit initial state,

$$|\Psi(0)\rangle = \alpha|00\rangle + \beta|01\rangle + \zeta|10\rangle + \delta|11\rangle, \tag{3}$$

the reduced density matrix takes the form,

$$\rho_{\mathrm{r}}(t) = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i(2\chi+\Omega_2)t} F_{12} & \alpha\zeta^* e^{-i(2\chi+\Omega_1)t} F_{13} & \alpha\delta^* e^{-i(\Omega_1+\Omega_2)t} F_{14} \\ \beta\alpha^* e^{i(2\chi+\Omega_2)t} F_{12}^* & |\beta|^2 & \beta\zeta^* e^{-i(\Omega_1-\Omega_2)t} F_{23} & \beta\delta^* e^{-i(\Omega_1-2\chi)t} F_{24} \\ \zeta\alpha^* e^{i(2\chi+\Omega_1)t} F_{13}^* & \zeta\beta^* e^{i(\Omega_1-\Omega_2)t} F_{23}^* & |\zeta|^2 & \zeta\delta^* e^{-i(\Omega_2-2\chi)t} F_{34} \\ \delta\alpha^* e^{i(\Omega_1+\Omega_2)t} F_{14}^* & \delta\beta^* e^{i(\Omega_1-2\chi)t} F_{24}^* & \delta\zeta^* e^{i(\Omega_2-2\chi)t} F_{34}^* & |\delta|^2 \end{pmatrix}. \tag{4}$$

By F_i, with $i = 12,13,14,23,24,34$, we are implying the decoherence and dissipation factors that appear due to the presence of the environment in the reduced density matrix. These factors are obtained from noise and dissipation kernels similarly as in the familiar spin-boson treatment. The master equation for the reduced density matrix of a bipartite system coupled to a bosonic environment Eq. (4) and the calculation of these terms can be read in the Appendix A of [12]. However, we are not interested in the explicit form of these coefficients right now, since we want to focus on the geometric phase acquired by only one spin of the bipartite subsystem. For this, we need to know the dynamics of this only spin ruled by its own Hamiltonian and the presence of the composite environment: the reservoir plus the other spin-1/2 particle of the subsystem. It is important to note that both spin-1/2 particles interact by the coupling constant χ defined in H_S and each of them interact as well with the external environment by the coupling constants λ_{1n} and λ_{2n}, respectively. We could further define the coefficients of the initial bipartite state so as to be able to interpret the results in terms of the known Bloch ball [we have not done so before because of the difficulty of writing Eq. (4)]. Therefore, we can assume that

$$\begin{aligned} \alpha &= \sqrt{\lambda_0}\cos(\theta_0/2), \\ \beta &= -\sqrt{\lambda_1}\sin(\theta_0/2), \\ \zeta &= \sqrt{\lambda_0}\sin(\theta_0/2), \\ \delta &= \sqrt{\lambda_1}\cos(\theta_0/2). \end{aligned} \tag{5}$$

The normalization of this state implies $\lambda_1 = 1 - \lambda_0$. The concurrence of this state is $\mathcal{C} = 2\sqrt{\lambda_0\lambda_1}$. Hence, we see that when $\lambda_0 = \lambda_1 = 1/2$ we have a maximally entangled state (MES). The angle θ_0 has here a geometric interpretation: coordinates $X = (1 - 2\lambda_0)\sin(\theta_0)$, $Y = 0$, and $Z = (2\lambda_0 - 1)\cos(\theta_0)$ of the state vector in the Bloch ball are given, using the results obtained in [17]. The radius of the Bloch ball is $R = |2\lambda_0 - 1|$ and all states lying on its surface have the same degree of purity (and come from two-qubit states having the same amount of entanglement). The angle θ_0 is the one that makes the density matrix vector with the z axis.

In order to know the dynamics of only one spin we need to trace out the degrees of freedom corresponding to the spin 2. After doing so, in this familiar notation, the reduced density matrix $\tilde{\rho}_{\mathrm{r}}(t)$ for the dynamics of spin 1 is

$$\tilde{\rho}_{\mathrm{r}}(t) = \begin{pmatrix} (\lambda_0 - 1/2)\cos\theta_0 + 1/2 & 1/2\sin\theta_0 e^{-i\Omega_1 t}\Gamma(t) \\ 1/2\sin\theta_0 e^{i\Omega_1 t}\Gamma(t)^* & (1/2 - \lambda_0)\cos\theta_0 + 1/2 \end{pmatrix}. \tag{6}$$

It is important to note that the $\Gamma(t)$ factor includes a real decaying decoherence factor as well as an imaginary dissipation term, both induced by the external parties to our subsystem spin 1. Therefore, we can write this coefficient as $\Gamma(t) = r(t)\exp[i\vartheta(t)]$, where $r(t)$ is the modulus of the complex number and $\vartheta(t)$ its argument.

The GP for a state under nonunitary evolution has been defined in [7] as

$$\Phi_g = \arg\left\{ \sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(\tau)}\langle\Psi_k(0)|\Psi_k(\tau)\rangle e^{-\int_0^\tau dt\langle\Psi_k|\frac{\partial}{\partial t}|\Psi_k\rangle} \right\}, \tag{7}$$

where $\varepsilon_k(t)$ are the eigenvalues and $|\Psi_k\rangle$ the eigenstates of the reduced density matrix $\tilde{\rho}_{\mathrm{r}}$, and τ is a time ($\tau > 0$) at which we study our system. Since we are working under the weak coupling limit, it is useful to consider a quasicyclic path $\mathcal{P} : t\epsilon[0,\tau]$, with $\tau = 2\pi/\Omega_1$ (Ω_1 is the system's characteristic frequency). As we can see from the GP definition, in order to compute the phase gained by the system, it is crucial to know the system's dynamics at all times. This is why this is known as the kinematic approach in contrast to other approaches for GP in open quantum systems [4–6]. The central result of Eq. (7) is to extract from the global phase gain acquired during the evolution, by a proper choice of the "parallel transport condition," the purification independent part which can be termed a geometric phase because it is gauge invariant and reduces to the known results in the limit of an unitary evolution.

The eigenvectors and eigenvalues of Eq. (6) are easily computed as

$$\varepsilon_\pm = \pm\frac{1}{2}\sqrt{\cos^2\theta_0 + r^2\sin^2\theta_0 + 4\lambda_0(\lambda_0-1)\cos^2\theta_0} + \frac{1}{2},$$

$$|\upsilon_\pm\rangle = \frac{e^{-i\Omega_1 t + i\vartheta(t)} r(t)\sin\theta_0}{\sqrt{r^2\sin^2\theta_0 + [(2\varepsilon_\pm - 1) + (1-2\lambda_0)\cos\theta_0]^2}}|0\rangle + \frac{[(2\varepsilon_\pm - 1) + (1-2\lambda_0)]}{\sqrt{r^2\sin^2\theta_0 + [(2\varepsilon_\pm - 1) + (1-2\lambda_0)\cos\theta_0]^2}}|1\rangle.$$

Unfamiliarly to all examples done before, this time, for an arbitrary initial state, we see that $\varepsilon_+(0) \neq 1$ and $\varepsilon_-(0) \neq 0$. Then, there is contribution to the GP coming from both eigenvalues. The only real factors in Eq. (7) are $\alpha = \sqrt{\varepsilon_+(0)\varepsilon_+(\tau)}$ and $\beta = \sqrt{\varepsilon_-(0)\varepsilon_-(\tau)}$. After some algebra, we obtain the geometric phase gained by spin 1 under this nonunitary evolution in a time τ,

$$\Phi_g = \arg(\alpha r_+(\tau)e^{i\varphi_+}e^{i\varphi} + \beta r_-(\tau)e^{i\varphi_-}e^{i\tilde{\varphi}}), \tag{8}$$

where we have defined

$$r_+(\tau)e^{i\varphi_+} = \langle\upsilon_+(0)|\upsilon_+(\tau)\rangle, \tag{9}$$

$$r_-(\tau)e^{i\varphi_-} = \langle\upsilon_-(0)|\upsilon_-(\tau)\rangle, \tag{10}$$

$$\varphi = \int_0^\tau \left(\Omega_1 + \frac{d\vartheta}{dt}\right)\cos^2\theta_+(t)\,dt, \tag{11}$$

$$\tilde{\varphi} = \int_0^\tau \left(\Omega_1 + \frac{d\vartheta}{dt}\right)\cos^2\theta_-(t)\,dt. \tag{12}$$

We have also used

$$\cos\theta_\pm(t) = \frac{r(t)\sin\theta_0}{\sqrt{r^2\sin^2\theta_0 + [(2\varepsilon_\pm - 1) + (1-2\lambda_0)\cos\theta_0]^2}},$$

so as to be able to write the eigenvector in an easier way:

$$|\upsilon_\pm\rangle = e^{-i\Omega_1 t + i\vartheta(t)}\cos\theta_\pm(t)|0\rangle + \sin\theta_\pm(t)|1\rangle. \tag{13}$$

III. BOSONIC ENVIRONMENT

In this section, we investigate the effect of a bosonic environment coupled to the composite system of two spin-1/2 particles, whose Hamiltonian is defined as

$$H_B = \sum_{n=1}^{N} \hbar\omega_n a_n^\dagger a_n. \tag{14}$$

One assumption we shall make is that the spectral density of the bosonic environment $J(\omega)$ is a reasonably smooth function of ω, and that is of the form ω^n up to some cutoff frequency Λ that may be large compared to Ω_1 and Ω_2. The spectral density function can be written as $J(\omega) = \gamma_0/4\omega^n\Lambda^{n-1}e^{-\omega/\Lambda}$, where dimensionless $\gamma_{01} \sim \lambda_1^2$ and $\gamma_{02} \sim \lambda_2^2$ [18]. We can consider an ohmic spectral density for such an environment, particularly one that goes as $J(\omega) \sim \omega$. In that case, the $\Gamma(t)$ factor that appears due to the tracing out of the degrees of freedom of the environment and the degrees of freedom of the spin 2, is

$$\Gamma(t) = e^{-2\gamma_0\ln(1+\Lambda^2t^2)}[(2\lambda_0 - 1)\cos\Omega_R t - i\sin\Omega_R t], \tag{15}$$

with $\Omega_R = (2\chi - \gamma_0\Lambda)$. We have so far considered $\gamma_{01} = \gamma_{02} = \gamma_0$ for the sake of simplicity.

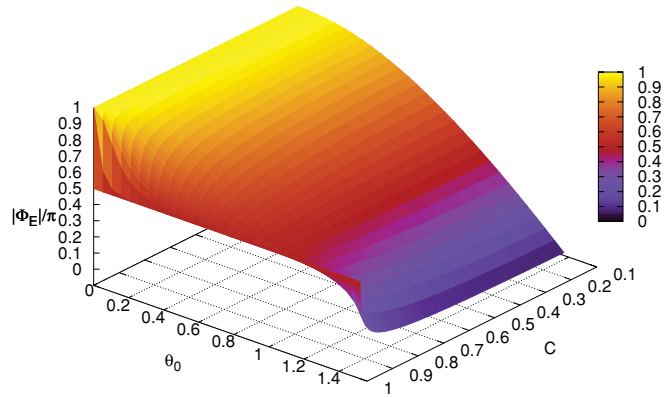

FIG. 1. (Color online) Geometric phase Φ_E as a function of the concurrence $\mathcal{C}$ and the angle θ_0 for the uncoupled case $\chi = 0$ and $\gamma_0 = 0$. The phase is measured in units of π.

A. Numerical results for an initial entangled state

We shall start by considering the isolated spin, initially set up in a bipartite state, but with no external interaction, that is to say $\gamma_0 = 0$ and $\chi = 0$. Hence, in this uncoupled situation of the bipartite initial state and after tracing out spin number 2, there is a decoherence factor $\Gamma(t) = (2\lambda_0 - 1)$ for spin 1. This means that the initial degree of entanglement will affect the dynamics through the decoherence factor. It is also important to note, that when $\lambda_0 = 1/2$, we have a maximal entangled state (MES), and the decoherence factor is $\Gamma = 0$, killing all coherences in Eq. (6) as expected. In Fig. 1 we can see the behavior of the geometric phase (that we call Φ_E in the case of an initially entangled state) in the case spin 1 is completely isolated but initially belonged to an entangled bipartite state. It is easy to see that it does not correspond to the geometric phase of a unitary single spin [i.e., $\Phi^U = \pi(1 - \cos\theta_0)$] due to the initial entanglement. However, it can serve as a reference for studying either the influence of the environment or the entanglement of the bipartite state. One important feature to note is the monotonic behavior of the geometric phase in this case.

If the degree of entanglement is $\lambda_0 = 1/2$, we have an MES and the concurrence is $\mathcal{C} = 1$, for all values of θ_0. In Fig. 2 we present the geometric phase for two particular initial entangled states of different concurrences for a better understanding of the behavior. In this figure, we present a solid red curve for maximum entanglement and a dashed purple one for a very low initial entanglement. We can note that for a spin initially set up in a MES the GP is $\pi/2$. On the other side, we can see that for an "almost" initial separable state ($\mathcal{C} \sim 0$), the value of the geometric phase for only one spin depends strongly on the value of θ_0 (purple dashed line in Fig. 2). In any case, an initially separable (product) state shall be further discussed in a following section.

Let us recall that for a bipartite state, we can no longer use the Bloch sphere to seek a geometric representation of that state. In [19] it has been shown that a geometric representation of a bipartite state can be obtained by using a Bloch ball and a SO3 sphere. Therefore, the total phase gained by a state is a combination of not only the dynamical and geometrical phase, but also the topological phase. Similarly to one-qubit states, MESs also gain a total phase of π (or $n\pi$) under a cyclic

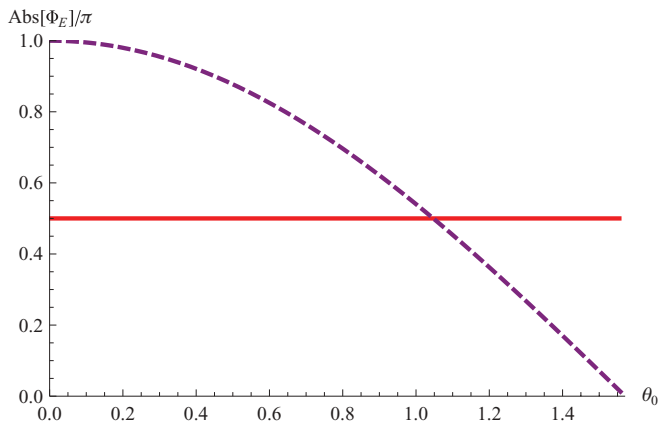

FIG. 2. (Color online) Geometric phase of one spin initially set up in an entangled bipartite state for the isolated case $\gamma_0 = 0 = \chi$ as a function of θ_0. Red solid curve for $\mathcal{C} = 1$ and purple dashed curve for $\mathcal{C} = 0.06$. The phase is measured in units of π.

evolution. However, this phase is of topological origin. It is already known that for a qubit the total phase gained is π (or $n\pi$) and it is due to a combination of the dynamical phase and the GP. Now, we are obtaining like the "partial" phase of that entangled state, that is, the GP of one spin-1/2 particle after tracing out the "external" degree of freedom of the other spin, both initially set up in an isolated bipartite state. As an MES cannot be represented in the Bloch sphere (i.e., it is reduced to the central point of it), the eigenvalues of the reduced density matrix for $\lambda_0 = 1/2$ are degenerate. By using the definition of the GP for the degenerate case also proposed in [7], which in this case reduces mainly to the same formulation multiplied by the degeneracy of the eigenvalues, it is easy to obtain the value of the GP for this case, namely $\pi/2$. Another case that can be easily analyzed is when $\theta_0 = 0$. Hence, the initial state reduces to $|\Psi(0)\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{1-\lambda_0}|11\rangle$, and it is easy to see that the $\cos\theta_\pm(t) = \pi$. Then, doing some algebra, we obtain that the $\Phi_E = \pi$ for all $\lambda_0 \neq 1/2$. On the other hand, if we have $\theta_0 = \pi$, which is equivalent to having the initial state $|\Psi(0)\rangle = -\sqrt{1-\lambda_0}|01\rangle + \sqrt{\lambda_0}|10\rangle$, the GP for a spin is zero for all $\lambda_0 \neq 1/2$. These two initial states mentioned are the Werner states studied in [12]. Thus, we see that the GP of a spin initially entangled in a bipartite state, yields a similar geometric phase to the one obtained for the whole bipartite isolated system.

As we are focusing on the geometric phase of one spin, we can find the path traversed in the Bloch sphere by studying the several components of the reduced density matrix $\tilde{\rho}_r$. The three-dimensional coordinates in the Bloch sphere are $x = \tilde{\rho}_{r_{12}} + \tilde{\rho}_{r_{21}}$, $y = i(\tilde{\rho}_{r_{12}} - \tilde{\rho}_{r_{21}})$, and $z = \tilde{\rho}_{r_{11}} - \tilde{\rho}_{r_{22}}$. By the use of Eq. (6), it is easy to generate the trajectories of Fig. 3. Thus, we can see a point for an MES, and bigger circles corresponding to the trajectories of the states with smaller degrees of initial entanglement (i.e., smaller values of $\mathcal{C}$, when $\chi = 0$ and $\gamma_0 = 0$).

If we consider that there is no bosonic environment ($\gamma_0 = 0$) but the two spin-1/2 particles are coupled through the χ constant, then the decoherence factor is

$$\Gamma_\chi(t) = (2\lambda_0 - 1)\cos(2\chi t) - i\sin(2\chi t),$$

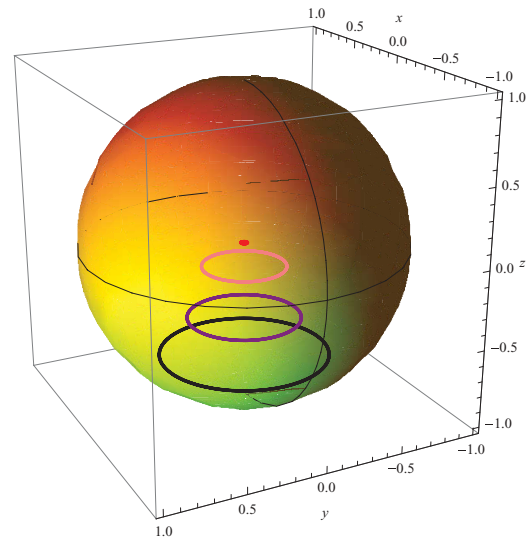

FIG. 3. (Color online) The evolution of the system can be illustrated by the path traversed in the Bloch sphere in the case that $\gamma_0 = 0$ and $\chi = 0$. Red curve for $\mathcal{C} = 1$, pink curve for $\mathcal{C} = 0.95$, purple curve for $\mathcal{C} = 0.8$, and black curve $\mathcal{C} = 0.43$. Time is measured in units of Ω_1.

which explicitly shows that the coupling between both particles not only affects the dynamics through the time-dependent decoherence complex factor but the geometric phase as well. In all cases, except for $\lambda_0 = 1/2$, the degree of entanglement and coupling constant χ act as a source of noise and dissipation (through its real and imaginary parts) for the system particle. In Fig. 4 we show the behavior of the geometric phase of the spin coupled to the other spin. One important feature that can be noted in Fig. 4 is that the monotonic behavior is broken. The main reason for this is the contribution of both eigenenergies φ and $\tilde{\varphi}$ in Eqs. (11) and (12) and the presence of a dissipation factor. Another interesting feature is that for a maximally entangled state (i.e., $\lambda_0 = 1/2$, $\mathcal{C} = 1$) the geometric phase is similar to that of the isolated case (40% bigger). The presence of the valley is due to the intrinsic dynamics between the spin introduced by the coupling constant χ.

As a further step in our study, we include the interaction with the external reservoir. In Fig. 5 we can see the behavior of the GP as a function of the degree of entanglement λ_0 and the initial angle θ_0 for $\gamma_0 \neq 0$ and $\chi \neq 0$. The presence

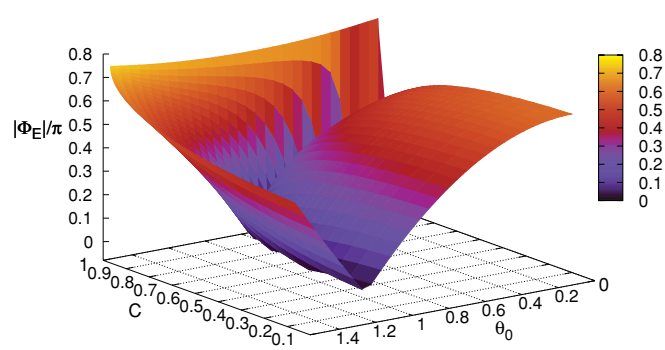

FIG. 4. (Color online) Geometric phase Φ_E as a function of the initial degree the entanglement through $\mathcal{C}$ and the angle θ_0 for $\gamma_0 = 0$ and $\chi = 0.1\Omega_1$. The phase is normalized by π.

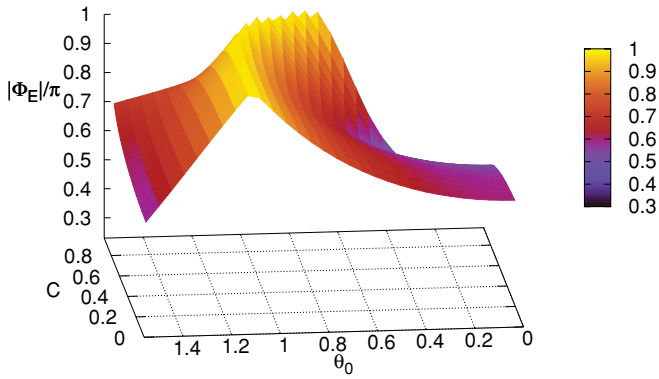

FIG. 5. (Color online) Geometric phase Φ_E as a function of $\mathcal{C}$ and θ_0. The value of the geometric phase is measured in units of π. Parameters used are as follows: $\chi = 0.1\Omega_1$, $\gamma_0 = 0.02$, and $\Lambda = 20\Omega_1$. The choice of the parameters used is for graphical reasons in order to compare with those plots of the preceding section.

of a stronger environment will contribute to the symmetry break and nonmonotonic behavior, as can be seen in our case for a weak environment in Fig. 5. Once again, we can note that the "open" GP gets a higher value for maximal initial concurrence and some particular values of θ_0. It is easy to note the nonmonotonic behavior of the geometric phase, which exhibits a local maximum and minimum. This characteristic feature of an energy exchanging process between system and environment (composite in this case) coincides with that shown in Ref. [20] for a single qubit under a dissipative interaction. As can be seen, the stability of phase can be significantly improved via a proper choice of the initial state determined by θ_0 which may be crucial for the effectiveness of quantum computation. Another feature that can be noted in Fig. 5 is that, as indicated in [21], in some cases there is "saturation" value which is a characteristic value for a given configuration of parameters that do not change much in time. For example, an initial angle $\theta_0 = \pi/3 \sim 1$, gives a maximum value of GP for a quasicyclic period of time. In order to study the importance of the initial state, we shall analyze the behavior of the geometric phase for different initial states in the weak coupling limit.

The geometric nature is the main feature of this phase in unitary closed systems. However, in the open quantum systems framework, this characteristic is not that evident. There have been many studies for one-qubit systems but none of them has been conclusive about this topic. A nice way to see that the change of the GP is associated with the path traversed by the state of system is to plot the path traced by the state in the Bloch sphere. Therefore, Figs. 6 and 7 are presented to study the geometric nature of the GP in open quantum systems. Both figures can be related by noting that states with a higher value of initial concurrence, have a smaller rate of change of the GP in time (Fig. 6) and a smaller change in the path traversed (Fig. 7), at least for small values of θ_0. This is consistent with the geometric nature of the GP, which might be rephrased as it depends on the path traversed by the state of the system and not on the dynamics. The latter feature can be seen in the Bloch sphere, as initially bigger values of λ_0 imply an initially smaller radius of the spiral trajectory and a smaller change in time of the GP undergoing an external influence. For example, if we compare the dotted-dashed black curve with

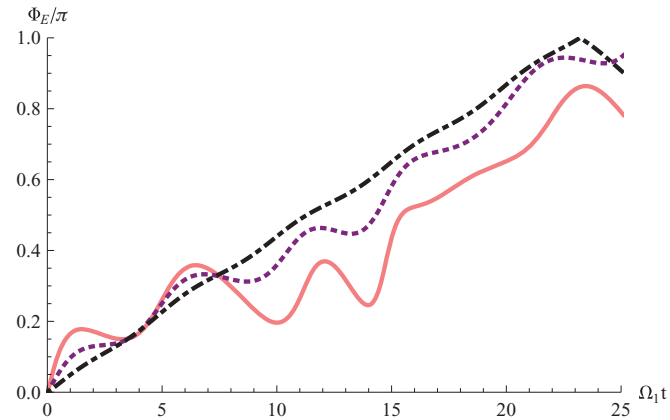

FIG. 6. (Color online) The geometric phase for the entangled initial state Φ_E as a function of time. The dashed pink line represents the initial state with $\lambda_0 = 0.2$; the dotted-dashed purple line $\lambda_0 = 0.1$, and the black dotted-dashed line is for $\lambda_0 = 0.01$, all curves for $\theta_0 = \pi/5$. Parameters used are as follows: $\Lambda = 20$, $\gamma_0 = 0.02$, and $\chi = 0.1$. The geometric phase is normalized with π.

the dashed pink one in Fig. 6, we can see that the latter has a smaller rate of change in time than the former one. Likewise, in Fig. 7 we can note that initially the black spiral radius is bigger and hence the change of the path and in the geometric phase is notable (during the first cycle, the black one is 25%; meanwhile the pink trajectory has changed only 9% and the purple one changes 16%). Therefore, we note that not only is it important to have a proper choice of the initial value of θ_0, but the value of the initial concurrence plays a crucial role in order to have a robust GP undergoing an external influence.

In this model we have several environmental parameters such as γ_0, Λ, and χ. We can investigate the effect of the strength of the external bath, assuming a weak coupling limit, as we show the behavior of the phase as a function of the

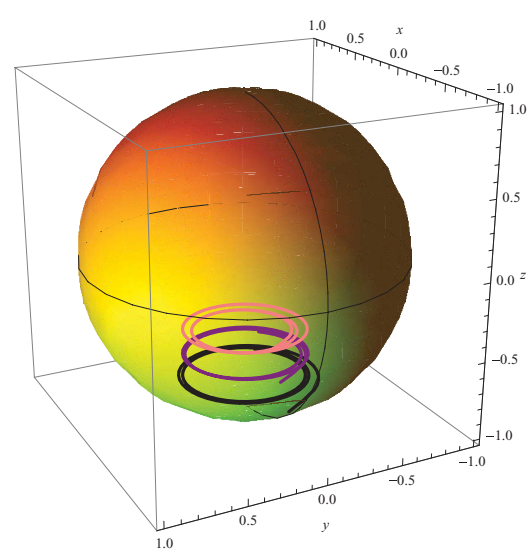

FIG. 7. (Color online) The evolution of the system can be illustrated by the path traversed in the Bloch sphere. Pink curve for $\mathcal{C} = 0.91$, purple curve for $\mathcal{C} = 0.71$, black curve for $\mathcal{C} = 0.43$. Parameters used are as follows: $\Lambda = 20\Omega_1$, $\gamma_0 = 0.02$, $\chi = 0.1\Omega_1$, and $\theta_0 = \pi/5$.

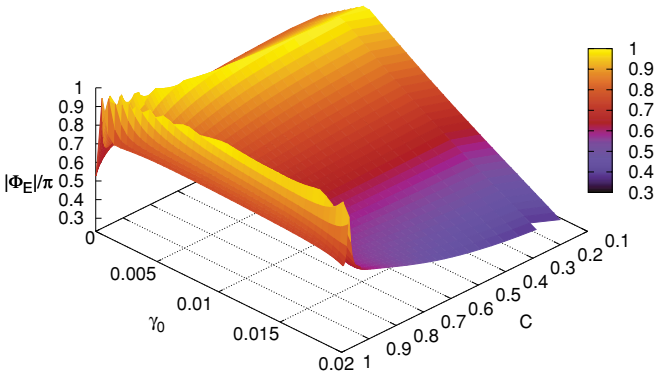

FIG. 8. (Color online) The geometric phase, measured in units of π, as a function of the coupling constant to the bosonic bath γ_0, and the concurrence $\mathcal{C}$. Parameters used are as follows: $\Lambda = 20\Omega_1$, $\chi = 0$, $\theta_0 = \pi/3$, and $\tau = 2\pi/\Omega_1$.

initial concurrence and the coupling constant to the external bath. In Fig. 8, we can see how this bosonic environment affects the different initial states of the system. It is easy to note that initially stronger correlations seem to contribute to the robustness of the GP in the presence of external couplings χ and γ_0. The value of χ adds oscillations to the dynamics of the system. Once again, we can note that the stability of the phase can be significantly improved via a proper choice of the initial state determined in this case by the value of λ_0 and θ_0. As for this figure, we can enhance our statement that initially stronger correlated states are prone to being less influenced by the presence of the environment and hence, the phase change is smaller for those states. In addition, this figure serves for a future comparison among the equivalent behavior of the phase if the initial state is a product state as we will show in the next section.

It is noteworthy to mention the possibility of implementing this model in a NMR-quantum simulator, similar to the one used in Ref. [14]. In this case, we have measured the GP corrections over a qubit induced by another qubit emulating a critical bath. We have obtained this correction by measuring the nonunitary evolution of the reduced density matrix of a spin 1/2 coupled to an environment. The experiments were done using a NMR quantum simulator, where we have emulated qualitatively the influence of a critical environment using just a simple one-qubit model. By adding stochastic fields and further spins, we believe we can quantum simulate more realistic environments and couplings to the system. With the same idea, it is possible to experimentally test the present model in which the entanglement parameter λ_0 plays an important role. In fact, we believe that this configuration will raise a new possibility of measuring the GP corrections implementing a tomography of the reduced density matrix. However, this experimental test is out of the scope of the present paper.

B. Numerical results for an initial product state

In this section, we shall see what happens with the geometric phase (set as Φ_p) of the spin-1/2 particle if the initial state is not entangled, namely $|\Psi(0)\rangle = |\Psi_1(0)\rangle \times |\Psi_2(0)\rangle$, where

$$\begin{aligned} |\Psi_1(0)\rangle &= \sqrt{1-p}|0_1\rangle + \sqrt{p}|1_1\rangle, \\ |\Psi_2(0)\rangle &= \sqrt{1-q}|0_2\rangle + \sqrt{q}|1_2\rangle. \end{aligned} \tag{16}$$

In this case, it is easy to associate the new parameters with the coefficients α,β,ζ, and δ in order to re-obtain the corresponding new dynamical equations:

$$\begin{aligned} \alpha &= \sqrt{1-p}\sqrt{1-q}, \\ \beta &= \sqrt{1-p}\sqrt{q}, \\ \zeta &= \sqrt{p}\sqrt{1-q}, \\ \delta &= \sqrt{p}\sqrt{q}. \end{aligned} \tag{17}$$

An important consideration is to write the initial state of the bipartite system with equal parameters as in the entangled case, in order to be able to compare both cases. Therefore, we can associate p to $\cos^2(\theta_0/2)$ and write

$$\begin{aligned} |\Psi_1(0)\rangle &= \cos(\theta_0/2)|0_1\rangle + \sin(\theta_0/2)|1_1\rangle, \\ |\Psi_2(0)\rangle &= \sqrt{1-q}|0_2\rangle + \sqrt{q}|1_2\rangle. \end{aligned} \tag{18}$$

In this way, we can describe the initial state of the system as a function of an angular and radial coordinates. As before, we can start by considering the case in which the bipartite system is isolated, that is to say $\gamma_0 = \chi = 0$. In such a case, we note that the geometric phase obtained is similar to that of a spin under unitary evolution $\Phi_g^p = 2\pi(1-p)$, or equivalently $\Phi_g^U = 2\pi \sin^2(\theta_0/2)$, with mod$(2\pi)$, as shown in Fig. 9.

The next step in complexity is to consider that both spin-1/2 particles are coupled to each other through the χ constant and/or to the external bath through the coupling constant γ_0. In such cases, the decoherence factor for the spin 1 (after tracing out spin 2) is

$$\Gamma_p(t) = e^{-2\gamma_0 \ln(1+\Lambda^2 t^2)}[\cos(\Omega_R t) + i(2q-1)\sin(\Omega_R t)]. \tag{19}$$

In this case, there are also two eigenvalues, but only one contributes since $\varepsilon_-(t=0) = 0$. So, the only contribution comes from the eigenvalue ε_+. As the decoherence factor is complex once again, namely $\Gamma_p(t) = r(t)e^{i\vartheta_p(t)}$, the computation of the geometric phase is different from the unitary one, namely,

$$\Phi_p = \arg\{\sqrt{\varepsilon_+^p(\tau)}\langle \upsilon_+^p(0)|\upsilon_+^p(\tau)\rangle e^{i\varphi_p(t)}\}, \tag{20}$$

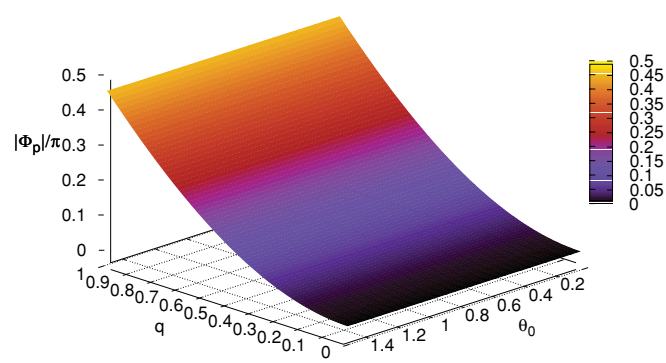

FIG. 9. (Color online) Geometric phase for the product initial state Φ_p as a function of q (λ_0) and θ_0, for $\Lambda = 20\Omega_1$, $\chi = 0$, and $\gamma_0 = 0$.

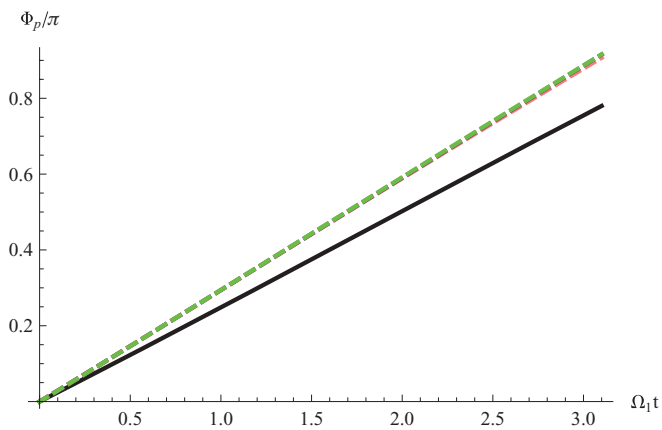

FIG. 10. (Color online) Geometric phase as a function of time for different initial states. Dashed lines (superposed) represent different values of q and $\theta_0 = \pi/5$ and the solid line is for $q = 0.05$ and $\theta_0 = \pi/3$. Parameters used are as follows: $\chi = 0.1\Omega_1$, $\Lambda = 20\Omega_1$, $\gamma_0 = 0.02$. The choice of the parameters used is for graphical reasons in order to compare with those plots of the preceding section.

and

$$|\upsilon_+^p\rangle = \cos\theta_+^p(t)e^{-i\Omega_1 t}e^{i\vartheta_p(t)}|0\rangle + \sin\theta_+^p(t), \tag{21}$$

with

$$\begin{aligned}\varphi_p(t) &= \int_0^\tau \left(\Omega_1 - \frac{\partial\vartheta}{dt}\right)\cos^2\theta_+^p(t)\,dt,\\ \cos\theta_+^p(t) &= \frac{\sqrt{p(1-p)}r_p(t)}{\sqrt{p(1-p)r_p(t)^2 + p^2}},\\ \sin\theta_+^p(t) &= \frac{\sqrt{p}}{\sqrt{p(1-p)r_p(t)^2 + p^2}}.\end{aligned} \tag{22}$$

In order to analyze the importance of the initial state in the robustness of the GP in open quantum systems, we shall make the same analysis than in the preceding section. In Fig. 10 we present the GP as a function of the time for different initial states in the limit of weak coupling with the environment. It is easy to see that the real crucial parameter is the initial value of θ_0 which sets up the initial state of $|\Psi_1(0)\rangle$. The same behavior can be seen in Fig. 11 for the path traversed in the Bloch sphere. Let us recall that in the preceding section there was a hierarchy imposed by the value of the initial concurrence.

Finally, in Fig. 12 we present the GP for different initial states (the same ones considered before) as a function of the coupling constant γ_0, for $\chi = 0$ and $\chi \neq 0$. We can see that the presence of the internal dynamics introduced by χ reduces considerably the value of the GP in the cases considered.

C. The second spin is not coupled to the environment

There is another situation that we can study which may be interesting. It is the case where the spin 2 is not coupled to the external environment; that means that the spin 1 is coupled to an external environment through the λ_{1n} constant and to spin 2 through the χ constant. This leads to a redefinition of the decoherence factors that appear in the reduced density matrix Eq. (4). For example, $F_{12} = F_{34} = 1$ and $F_{13} = F_{14} = F_{23} = F_{24} = \Gamma_1$ where $\Gamma_1 = e^{-2\gamma_0\ln(1+\Lambda^2t^2)}$ is the decoherence

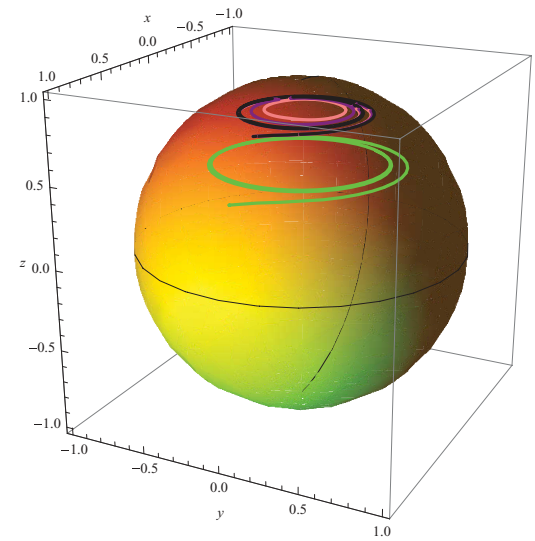

FIG. 11. (Color online) Trajectories in the Bloch sphere for the same initial states of Fig. 7. Parameters used are as follows: $\chi = 0.1\Omega_1$, $\Lambda = 20\Omega_1$, $\gamma_0 = 0.02$.

factor obtained after tracing out the degrees of the environment, namely,

$$\Gamma^C(t) = e^{-2\gamma_0\ln(1+\Lambda^2t^2)}[(2\lambda_0 - 1)\cos(2\chi t) - i\sin(2\chi t)].$$

After tracing out the second spin, we get the reduced density matrix for the dynamics of spin 1, for the case where the initial state is initially entangled.

$$\tilde{\rho}_{\mathrm{r}}(t) = \begin{pmatrix} (\lambda_0 - 1/2)\cos(\theta_0) + 1/2 & 1/2\sin(\theta_0)e^{-i\Omega_1 t}\Gamma^C(t) \\ 1/2\sin(\theta_0)e^{i\Omega_1 t}\Gamma^{C*}(t) & (1/2 - \lambda_0)\cos(\theta_0) + 1/2 \end{pmatrix}. \tag{23}$$

This expression is very similar to that obtained for the case when the spin 2 is coupled to the environment in Sec. III A. However, if we look at the $\Gamma^C(t)$ factor of Eq. (15) we see

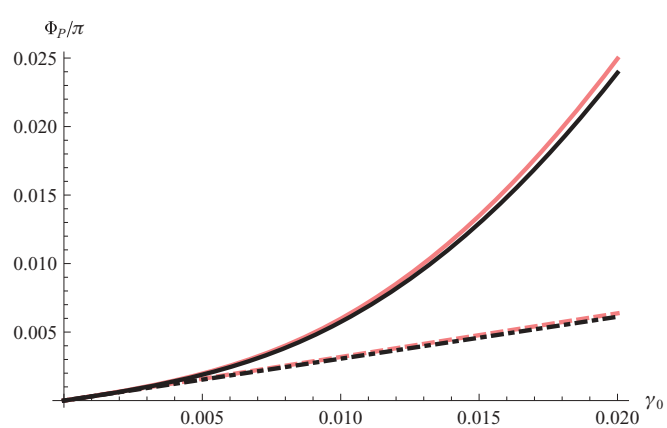

FIG. 12. (Color online) The geometric phase, measured in units of π, for different initial states as a function of the dimensionless coupling constant to the bosonic bath γ_0. Pink lines for $q = 0.4$, $\theta = \pi/3$; black lines for $q = 0.01$, $\theta = \pi/3$. Dotted and dashed lines imply that $\chi = 0.1\Omega_1$ while solid lines imply $\chi = 0$. Parameters used are as follows: $\Lambda = 20\Omega_1$ and $\tau = 2\pi/\Omega_1$.

that in the former case there was dissipation introduced by the coupling of the spin 2 and the environment, while in this case there is not. This is a somewhat reasonable result since if the spin 2 is effectively coupled to the environment, when we ignore those degrees of freedom (by tracing them out), we expect to have a loss of energy. On the other side, just by tracing out the degrees of freedom of one spin, we do not expect to have a significant loss of energy. It is important to note that we have chosen to trace out the spin 2 but it is irrelevant whether we trace spin 1 or spin 2. The results are similar if we study the reduced density matrix for spin 2 by tracing out the degrees of freedom of spin 1.

IV. CONCLUSIONS

In this article, we have continued with the analysis of the geometric phase in the framework of a quantum open system, as a natural sequel to some previous works done in the field. In particular, we have analyzed the geometric phase of a spin $1/2$ coupled to a noisy composite environment. Initially we have started with a bipartite state coupled to an external bath and traced out the degrees of freedom of one of the spins in order to focus on the GP of only one spin. The richness of the model lies in the fact that the initial bipartite state could be entangled or not, by introducing the initial entanglement as another free parameter of the model. We consider it as a major goal to understand the role that the degree of entanglement plays in the dynamics of the coupled system, as well as in the correction to the geometric phase.

We have calculated the decoherence factor for different initial states of our system of interest and, consequently, we have analyzed the geometric phase corrections from the nonunitary evolution. We have mainly considered two major groups: initial entangled bipartite states and initial product bipartite states, in order to study the role of the entanglement in the changes suffered by the geometric phase. In each case, we have computed the different decoherence factors and numerically obtained the open geometric phase for different physical situations. With the help of the study of the geometric nature of the GP, we have shown that the entanglement enhances the sturdiness of the geometric phase under the presence of an external environment for small values of θ_0. States with smaller values of initial entanglement seem to be less robust than others. This is related to the fact that the radius of the path traversed is bigger and its curvature is hence more affected by changes originated by the presence of the environment. Another important feature shown is that MES states still remain "special" or privileged as we have proved in Ref. [12] due to the topological nature of the phase of these states. As shown, in order to implement a measurement of the noise-induced corrections to the GP, the choice of the initial state is crucial. However, we have shown so far that not only is a proper choice of the initial value of θ_0 important, but the value of the initial concurrence also plays a crucial role in order to have a robust GP undergoing an external influence. In addition, it is important to state that by considering the case in which the temperature T is finite, the main properties of the geometric phase remain similar. However, it can be expected that the values of the maximum diminish with increasing temperature.

Furthermore, by comparison between the entangled state and the product initial bipartite state, we have seen that having an initial entangled bipartite state yields a bigger geometric phase than in the case of an initial product bipartite state. In the particular case in which the central spin (system of interest) is completely isolated, it is easy to see that the GP of the initially entangled state does not correspond to the geometric phase of a unitarily evolving single spin, that is, $\Phi_g^U = \pi(1 - \cos\theta_0)$ (due to the initial entanglement). However, for an initial product state, we have found that the GP, for the uncoupled bipartite system, is similar to that of a spin $1/2$ under a unitary evolution. This suggests that the degree of entanglement works as an effective coupling between the two particles in the system, even though one of them is traced out.

Finally, we have noted that the stability of geometric phase with respect to decoherence and dissipation is crucial for effectiveness of holonomic quantum computation. It is evident that the stability of phase can be significantly improved via a proper choice of the initial state determined by the parameters of the initial state (whether λ_0 for the entangled state and θ_0 for the product state). Thus, we consider the better choice an initial entangled bipartite state as opposed to a product initial state and with the feasible higher value of concurrence.

All in all, we are reporting about a possible scenario where the phase measuring in an open system is feasible. We claim that the best choice in order to measure the open system GP must take into account the degree of entanglement of the initial state. Our model admits the possibility of preparing a MES, which is the best option to ensure a measure. This conclusion is a byproduct of having this type of model, in which one can prepare a two-particle initial state, but using just one qubit to perform the experiment.

ACKNOWLEDGMENTS

F.C.L. was supported by UBA, CONICET, and ANPCyT, Argentina. P.I.V. acknowledges financial support from the UNESCO-L'OREAL Women in Science Programme.

[1] S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 247 (1956).
[2] M. V. Berry, Proc. R. Soc. Lond. A **392**, 45 (1984).
[3] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987); J. Anandan and Y. Aharonov, Phys. Rev. D **38**, 1863 (1988).
[4] R. S. Whitney and Y. Gefen, Phys. Rev. Lett. **90**, 190402 (2003); R. S. Whitney, Y. Makhlin, A. Shnirman, and Y. Gefen, *ibid.* **94**, 070407 (2005).
[5] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **90**, 160402 (2003); **92**, 020402 (2004).
[6] G. De Chiara, A. Lozinski, and G. M. Palma, Eur. Phys. J. D **41**, 179 (2007); G. De Chiara and G. M. Palma, Phys. Rev. Lett. **91**, 090404 (2003).
[7] D. M. Tong, E. Sjoqvist, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. **93**, 080405 (2004).

[8] F. C. Lombardo and P. I. Villar, Phys. Rev. A **74**, 042311 (2006).
[9] F. C. Lombardo and P. I. Villar, Int. J. Quantum Inf. **6**, 707713 (2008).
[10] P. I. Villar, Phys. Lett. A **373**, 206 (2009).
[11] K. P. Marzlin, S. Ghose, and B. C. Sanders, Phys. Rev. Lett. **93**, 260402 (2004).
[12] F. C. Lombardo and P. I. Villar, Phys. Rev. A **81**, 022115 (2010).
[13] C. W. Niu, G. F. Xu, L. Liu, L. Kang, and D. M. Tong, Phys. Rev. A **81**, 012116 (2010).
[14] F. M. Cucchietti, J.-F. Zhang, F. C. Lombardo, P. I. Villar, and R. Laflamme, Phys. Rev. Lett. **105**, 240406 (2010).
[15] E. Sjqvist, Phys. Lett. A **374**, 1431 (2010).
[16] J. P. Paz and W. H. Zurek, in *Environment Induced Superselection and the Transition from Quantum to Classical* in *Coherent Matter Waves, Les Houches Session LXXII*, edited by R. Kaiser, C. Westbrook, and F. David (EDP Sciences, Springer-Verlag, Berlin, 2001), pp. 533–614; W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[17] P. Milman, Phys. Rev. A **73**, 062118 (2006).
[18] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).
[19] P. Milman and R. Mosseri, Phys. Rev. Lett. **90**, 230403 (2003); P. Milman, Phys. Rev. A **73**, 062118 (2006).
[20] J. Dajka, J. Luczka, and P. Hanggi, e-print arXiv:1004.3432v1 [quant-ph] (2010).
[21] S. Yin and D. M. Tong, Phys. Rev. A **79**, 044303 (2009).