Optical spectral singularities as threshold resonances

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Spectral singularities are among generic mathematical features of complex scattering potentials. Physically they correspond to scattering states that behave like zero-width resonances. For a simple optical system, we show that a spectral singularity appears whenever the gain coefficient coincides with its threshold value and other parameters of the system are selected properly. We explore a concrete realization of spectral singularities for a typical semiconductor gain medium and propose a method of constructing a tunable laser that operates at threshold gain.

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Consider an infinite planar slab gain medium that is aligned along the x axis and the electromagnetic wave given by $\vec{E}(z,t) = E\, e^{i(\mathfrak{K}z-\omega t)}\hat{e}_x$, where E is a constant, $\mathfrak{K}$ is the propagation constant, and $\hat{e}_x$ stands for the unit vector pointing along the positive x axis. It is easy to show that while traveling through the gain medium the wave is amplified by a factor of e^{gL}, where L is the width of the gain medium and g is the gain coefficient. The latter is related to the imaginary part κ of the the complex refractive index of the medium,

$$\mathfrak{n} = \eta + i\kappa, \tag{1}$$

and the wavelength $\lambda := 2\pi c/\omega$ according to [1]:

$$g = -\frac{4\pi\kappa}{\lambda}. \tag{2}$$

Often, one places the gain medium between two mirrors to produce a (Fabry-Perot) resonator. This extends the length of the path of the wave through the gain medium and yields a much larger amplification of the wave for the resonance frequencies of the resonator.

In [2,3], we have outlined an alternative amplification effect that does not involve mirrors. For the system we consider, it requires adjusting L and g so that the system supports a spectral singularity. This is a generic mathematical feature of complex scattering potentials [4] that obstructs the completeness of the eigenfunctions of the corresponding non-Hermitian Hamiltonian operator. Physically, a spectral singularity is the energy of a scattering state that behaves exactly like a zero-width resonance [2,5–7]. In this Brief Report, we first reveal the relationship between spectral singularities and the well-known laser threshold condition [1], and then explore the possibility of tuning the wavelength of the spectral singularity by adjusting the pump intensity. This turns the system into a tunable laser that operates at the threshold gain.

It is easy to show that

$$\vec{E}(z,t) = E\, e^{-i\omega t}\psi(z)\hat{e}_x, \quad \vec{B}(z,t) = -i\omega^{-1}E\, e^{-i\omega t}\psi'(z)\hat{e}_y$$

is a solution of Maxwell's equations for the above system provided that $\hat{e}_y$ stands for the unit vector along the positive y axis; ψ is a continuously differentiable solution of the time-independent Schrödinger equation:

$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z), \tag{3}$$

where $k := \omega/c = 2\pi/\lambda$ and v is the complex barrier potential:

$$v(z) := \begin{cases} k^2\hat{\mathfrak{z}} & \text{for} \quad |z| < \frac{L}{2}, \\ 0 & \text{for} \quad |z| \geqslant \frac{L}{2}, \end{cases} \qquad \hat{\mathfrak{z}} := 1 - \mathfrak{n}^2. \tag{4}$$

Solving the Schrödinger equation (3) we find

$$\psi(z) = \begin{cases} A_-e^{ikz} + B_-e^{-ikz} & \text{for} \quad z \leqslant -L/2, \\ A_0e^{i\mathfrak{n}kz} + B_0e^{-i\mathfrak{n}kz} & \text{for} \quad |z| < L/2, \\ A_+e^{ikz} + B_+e^{-ikz} & \text{for} \quad z \geqslant L/2, \end{cases} \tag{5}$$

where $k \in \mathbb{R}^+$, A_- and B_- are free complex coefficients, and A_0, B_0, A_+, and B_+ are complex coefficients related to A_- and B_-. For example, $\binom{A_+}{B_+} = \mathbf{M}\binom{A_-}{B_-}$, where

$$\mathbf{M} := \frac{1}{4\mathfrak{n}}\begin{pmatrix} e^{-iLk}f(\mathfrak{n}, -\frac{Lk}{2}) & 2i(\mathfrak{n}^2-1)\sin(\mathfrak{n}Lk) \\ -2i(\mathfrak{n}^2-1)\sin(\mathfrak{n}Lk) & e^{iLk}f(\mathfrak{n}, \frac{Lk}{2}) \end{pmatrix}$$

is the transfer matrix, and for all $z_1, z_2 \in \mathbb{C}$,

$$f(z_1,z_2) := e^{-2iz_1z_2}(1+z_1)^2 - e^{2iz_1z_2}(1-z_1)^2. \tag{6}$$

Because v is an even function of z, the left and right reflection and transmission amplitudes coincide. They are respectively given by $R = -M_{21}/M_{22} = M_{12}/M_{22}$ and $T = 1/M_{22}$ [2].

The spectral singularities are the k^2 values for which $M_{22} = 0$ [2], i.e., the real k values satisfying

$$f\left(\mathfrak{n}, \frac{Lk}{2}\right) = 0. \tag{7}$$

Because f is a complex-valued function, Eq. (7) is equivalent to a pair of coupled real transcendental equations for three unknown real variables $\mathrm{Re}(\hat{\mathfrak{z}})$, $\mathrm{Im}(\hat{\mathfrak{z}})$, and αk. In Ref. [3], we outline a method of decoupling these equations. Here we give a more direct solution that reveals some previously unknown aspects of the problem.

First, we use (6) to express (7) in the form

$$e^{-2i\mathfrak{n}Lk} - \left(\frac{1-\mathfrak{n}}{1+\mathfrak{n}}\right)^2 = 0. \tag{8}$$

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Noting that the boundaries of the gain region have a reflectivity of

$$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2, \tag{9}$$

we can write (8) as $e^{-2iLk\mathfrak{n}} = \mathcal{R}$. Next, we substitute (1) in this equation and use (2) to express its left-hand side in terms of g. This gives

$$e^{-2gL} e^{-4iLk\eta} = \mathcal{R}^2. \tag{10}$$

Taking the modulus of both sides of this equation yields

$$e^{-2gL} = |\mathcal{R}|^2. \tag{11}$$

This is precisely the laser threshold condition [1]. In other words, a spectral singularity and the associated zero-width resonance appear at the threshold gain:

$$g = g_{\rm th} := \frac{1}{2L}\ln\frac{1}{|\mathcal{R}|^2}. \tag{12}$$

We wish to stress that the threshold condition (11) is only a necessary condition for having a spectral singularity. It is by no means sufficient. A necessary and sufficient condition is Eq. (8) that we can express as

$$kL = -\frac{1}{2i\mathfrak{n}}\ln\mathcal{R}. \tag{13}$$

A key observation that reveals the discrete nature of spectral singularities is that, as a complex-valued function, $\ln\mathcal{R}$ has infinitely many values; in view of (1) and (9),

$$\ln\mathcal{R} = \ln|\mathcal{R}| + 2i\left[\tan^{-1}\left(\frac{2\kappa}{\eta^2+\kappa^2-1}\right) - \pi m\right], \tag{14}$$

where m is an arbitrary integer. Using (1) and (9), we also find

$$|\mathcal{R}| = \frac{(\eta-1)^2+\kappa^2}{(\eta+1)^2+\kappa^2}. \tag{15}$$

Because $\eta > 0$, this equation implies $|\mathcal{R}| < 1$. Furthermore, in view of (12), we have

$$g = g_{\rm th} = \frac{1}{L}\ln\left[\frac{(\eta+1)^2+\kappa^2}{(\eta-1)^2+\kappa^2}\right]. \tag{16}$$

Next, we return to Eq. (13). Because kL is real, the imaginary part of the right-hand side of this equation must vanish. This gives

$$\eta\ln|\mathcal{R}| + 2\kappa\left[\tan^{-1}\left(\frac{2\kappa}{\eta^2+\kappa^2-1}\right) - \pi m\right] = 0. \tag{17}$$

Furthermore, we can express (13) as

$$kL = \frac{\ln|\mathcal{R}|}{2\kappa}. \tag{18}$$

Because $kL > 0$ and $|\mathcal{R}| < 1$, (18) implies $\kappa < 0$. According to (2), this corresponds to the situation that the medium has a positive gain coefficient. This is a remarkable manifestation of the conservation of energy, because whenever we arrange the parameters of the system so that a spectral singularity is generated, the system begins emitting radiation. This can happen only for a gain medium, i.e., when $g > 0$. In [3], we could only demonstrate this graphically. Here we have derived it rigorously.

Equation (17) determines the location of spectral singularities in the η-κ plane. In view of the inequalities $\eta > 0$, $\kappa < 0$, $|\mathcal{R}| < 1$, and the fact that $\tan^{-1}$ is an odd function taking values in $(-\frac{\pi}{2},\frac{\pi}{2})$, we can satisfy (17) only for $m \geqslant 0$. It is also instructive to note that solving for $\ln|\mathcal{R}|$ in (17), substituting the result in (18), and using $k = 2\pi\nu/c$, we find the following expression for the frequency ν of the spectral singularity:

$$\nu = \frac{\nu_\phi m}{\eta} - \frac{\nu_\phi}{\pi\eta}\tan^{-1}\left(\frac{2\kappa}{\eta^2+\kappa^2-1}\right), \tag{19}$$

where $\nu_\phi := \frac{c}{2L}$. The first term on the right-hand side of (19) is the usual resonance frequency of a resonator of length L with perfectly reflecting boundaries. It is important to note that because κ and η are frequency-dependent quantities, there will be certain frequencies for which they satisfy (17). A spectral singularity will arise if and only if at least one of these frequencies coincides with one of the values fulfilling (19). It turns out that if we fix all the physical parameters of the system this can happen only for a single critical frequency, i.e., a particular mode number m.

In order to explore this phenomenon suppose that the gain medium is obtained by doping a host medium of refraction index n_0 and that it is modeled by a two-level atomic system with lower and upper level population densities N_l and N_u, resonance frequency ω_0, and damping coefficient γ. Then its permittivity ($\varepsilon := \varepsilon_0\mathfrak{n}^2$) is given by

$$\varepsilon = \varepsilon_0\left[n_0^2 - \frac{\omega_p^2}{\omega^2-\omega_0^2+i\gamma\,\omega}\right], \tag{20}$$

where ε_0 is the permittivity of the vacuum, $\omega_p^2 := (N_l - N_u)e^2/(m\varepsilon_0)$, and e and m are the electron's charge and mass, respectively [1,8]. In view of (1), (2), and (20), we can express ω_p^2 in terms of the gain coefficient g_0 at the resonance frequency ω_0. Introducing $\lambda_0 := 2\pi c/\omega_0$ and

$$\kappa_0 := -\frac{\lambda_0 g_0}{4\pi}, \tag{21}$$

and using

$$\varepsilon/\varepsilon_0 = \mathfrak{n}^2 = (\eta + i\kappa)^2, \tag{22}$$

and (2), we have

$$\frac{\omega_p^2}{\omega_0^2} = \frac{2\gamma\kappa_0\sqrt{n_0^2+\kappa_0^2}}{\omega_0} = -\frac{cg_0\gamma}{\omega_0^2}\sqrt{n_0^2+\left(\frac{cg_0}{2\omega_0}\right)^2}. \tag{23}$$

For example for a semiconductor gain medium [1] with

$$n_0 = 3.4, \quad \lambda_0 = 1500\,\text{nm}, \qquad \frac{\gamma}{\omega_0} = 0.02, \quad g_0 = 40\,\text{cm}^{-1}, \tag{24}$$

we have $\kappa_0 = -4.7747\times10^{-4}$ and $\frac{\omega_p^2}{\omega_0^2} = -6.4935\times10^{-5}$. Substituting these values in (20) and using (22), we can express $\delta\eta := \eta - n_0$ and κ as functions of ω and use them to plot a parametric curve C representing the dispersion relation (20). See Fig. 1. Next, we return to Eq. (17) and consider it as an

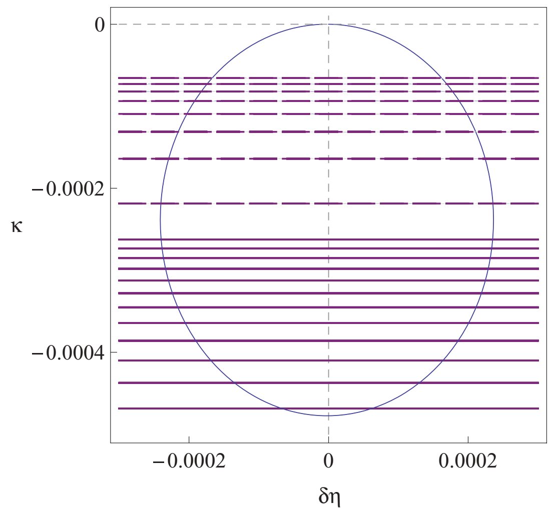

FIG. 1. (Color online) Plots of the curves of spectral singularities C_m (full and dashed thick purple curves) and the dispersion relation C (thin blue curve) for a gain medium with specifications (24). The spectral label m of the C_m that are displayed ranges over 1400,1500,1600, ... ,2500 for the thick full (purple) curves from bottom to top, and 3000,4000,5000, ... ,10 000 for the thick dashed (purple) curves. As one increases m these curves approach the $\delta\eta$ axis from below. The thin dashed (gray) lines are the coordinate axes.

implicit function for each value of m. Figure 1 also shows the graph of these functions for several values of m. These are the curves of spectral singularities that we label by C_m. As seen from Fig. 1 for sufficiently large values of m (i.e., $m \geqslant 1374$), C_m intersects C at two points. One of these (p_{m+}) has a positive $\delta\eta$ coordinate and corresponds to a frequency ω_{m+} that is larger than ω_0. The other (p_{m+}) has a negative $\delta\eta$ and corresponds to a frequency ω_{m-} that is smaller than ω_0. We can compute the coordinates $(\delta\eta_{m\pm},\kappa_{m\pm})$ of $p_{m\pm}$, find $\omega_{m\pm}$ and $k_{m\pm} := \omega_{m\pm}/c$, and use (18) to determine L. This gives rise to a set of values $L_{m\pm}$ for the length of the gain medium that would allow for the existence of a spectral singularity. For the gain medium (24), setting $m = 1374$ we find a pair of spectral singularities with extremely close wavelengths to λ_0, namely $\lambda = 1499.9$ and 1500.2 nm. These correspond to $L = 303.074$ and 303.133 μm, respectively.

Next, we consider the more realistic situation that L is fixed and the gain coefficient g_0 is adjustable. This is simply done by changing the pump intensity. It is not difficult to see that spectral singularities appear for specific discrete values of g_0 and λ. This may be viewed as means for producing a tunable laser that would function at the very threshold gain. To implement this idea we insert (23) in (20) and use (22) and $\lambda = 2\pi c/\omega$ to express $\delta\eta$ and κ as functions of g_0 and λ. Substituting the resulting expressions in (17) and (18) and using $k = 2\pi/\lambda$ we obtain a pair of equations for g_0 and λ for each value of the mode number m. The solution of these equations yield the desired values of g_0 and λ for which a spectral singularity appears.

As we see from Fig. 1, the relevant range of values of $\delta\eta$ and κ is several orders of magnitude smaller than n_0. This shows that we can obtain reliable approximate forms of (17) and (18) by neglecting quadratic and higher order terms in $\delta\eta$ and κ. Applying this approximation to (22) and using (20), we have

$$\delta\eta \approx -\kappa_0 f_1, \quad \kappa \approx \kappa_0 f_2, \tag{25}$$

where

$$f_1 := \frac{1-\hat{\omega}^2}{(1-\hat{\omega}^2)^2+\hat{\gamma}^2\hat{\omega}^2}, \quad f_2 := \frac{\hat{\gamma}^2\hat{\omega}}{(1-\hat{\omega}^2)^2+\hat{\gamma}^2\hat{\omega}^2}, \tag{26}$$

$\hat{\omega} := \omega/\omega_0 = \lambda_0/\lambda$, and $\hat{\gamma} := \gamma/\omega_0$. In view of (25), the above-mentioned approximation scheme is equivalent to neglecting terms of order two and higher in κ_0. Inserting (25) in (18) and using $k = 2\pi/\lambda$, we find

$$\kappa_0 \approx -\frac{\rho_0}{2}\left(\frac{f_1}{n_0^2-1}+\frac{\pi L\hat{\omega} f_2}{\lambda_0}\right)^{-1}, \tag{27}$$

where $\rho_0 := \ln[(n_0+1)/(n_0-1)]$. In light of (21), (27) is equivalent to

$$g_0 \approx 2\pi\rho_0\left(\frac{\lambda_0 f_1}{n_0^2-1}+\pi L\hat{\omega} f_2\right)^{-1}. \tag{28}$$

Next, we substitute (25) in (18), neglect the quadratic and higher order terms in κ_0, and use (27) and (26) to simplify the result. This yields, after some lucky cancelations,

$$\left(\frac{2\pi L n_0}{\lambda_0}-\frac{\rho_0}{\hat{\gamma}^2}\right)\hat{\omega}^2 - \pi m\,\hat{\omega} + \frac{\rho_0}{\hat{\gamma}^2} \approx 0.$$

Only for $m \geqslant \mu := \frac{2\rho_0}{\pi\hat{\gamma}}\sqrt{\frac{2\pi L n_0}{\rho_0\lambda_0}-\frac{1}{\hat{\gamma}^2}}$ does this equation have real solutions. These give the frequency (wavelength) of the spectral singularities. Substituting them in (28), we find the corresponding g_0 values. The approximate values of λ and g_0 obtained in this way allow for a more effective numerical solution of the exact Eqs. (17) and (18).

As a concrete example, we take $L = 300\,\mu$m for the gain medium described by (24) and obtain the wavelength of the spectral singularities that are produced as we change g_0 in the range 0–115 cm^{-1}. These turn out to correspond to the mode numbers 1355–1365 which are consistent with the result, $m \geqslant \mu \approx 1320$, of our approximate calculations. The corresponding λ and g_0 values are given in Table I. The spectral singularity with lowest g_0 value (40.4 cm^{-1}) has a wavelength that is extremely close to $\lambda_0 = 1500$ nm. It corresponds to $m = 1360$. As we increase g_0 from 40.4 to 115 cm^{-1}, there appear 10 more spectral singularities with wavelengths ranging between 1482 and 1520 nm. These should in principle be detectable, if we gradually increase the pump intensity. For example a periodic change of g_0 in the range

TABLE I. Wavelength λ of the spectral singularities for the gain coefficient g_0 ranging over 0–115 cm^{-1} and $L = 300\,\mu$m.

m	λ (nm)	g_0 (cm^{-1})	m	λ (nm)	g_0 (cm^{-1})
1355	1519.832	110.2	1361	1495.662	43.8
1356	1515.380	82.5	1362	1494.582	45.7
1357	1512.058	66.3	1363	1489.203	61.5
1358	1507.651	50.9	1364	1484.927	81.7
1359	1504.363	43.8	1365	1481.736	101.1
1360	1500.000	40.4			

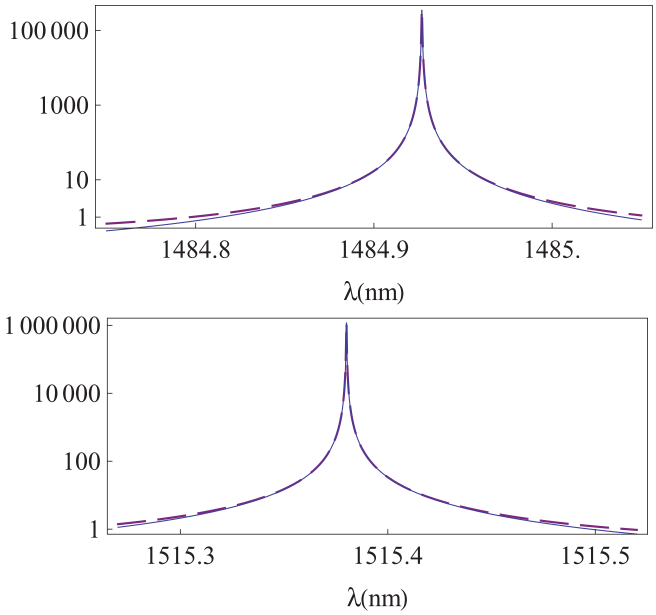

FIG. 2. (Color online) Logarithmic plots of the reflection (thick dashed curves) and transmission (thin full curves) coefficients as functions of λ for $L = 300\,\mu$m and $g = 81.7$/cm (top graph) and $g = 82.5$/cm (bottom graph).

70–90 cm^{-1} should produce periodic emissions of radiation at the wavelengths $\lambda = 1484.927$ nm (for $m = 1364$ and $g_0 = 81.6821\text{cm}^{-1}$) and $\lambda = 1515.380$ nm (for $m = 1356$ and $g_0 = 82.4918\,\text{cm}^{-1}$) with no emitted wave of comparable amplitude at the resonance wavelength 1500 nm. This is a remarkable feature of the spectral singularity related resonance effect. Using the above values of λ and g_0 to compute the reflection and transmission coefficients $|R|^2$ and $|T|^2$, which give the amplification factor for the emitted electromagnetic energy density, we obtain $|R|^2 \approx |T|^2 \approx 1.1 \times 10^6$ and 3.1×10^5, respectively. In other words we obtain an amplification of the background electromagnetic energy density at these wavelengths by a factor of $|R|^2 + |T|^2 \approx 2.2 \times 10^6$ and 6.2×10^5, respectively. It turns out that these numbers are extremely sensitive to the value of λ but not g_0. Using the less accurate values 81.7 and 82.5 cm^{-1} for g_0 and the same values for λ, we find $|R|^2 \approx |T|^2 \approx 1.0 \times 10^6$ and 2.8×10^5. But, as we can see from Fig. 2, changing the above values of λ by 0.01 nm reduces $|R|^2$ and $|T|^2$ by three to four orders of magnitude. This shows that the detection of spectral singularities would require a spectrometer with a band width of 0.01 nm or smaller.

To summarize, we have explored spectral singularities of a simple optical system and shown that the equation (10) that determines spectral singularities reduces to the laser threshold condition $g = g_{\text{th}}$, if we take the modulus of both sides of this equation. Equating the phase of both sides of this equation gives rise to an independent condition for the existence of the spectral singularities that involves an integer (mode) number m. It turns out that this equation and the threshold condition can be satisfied only for particular values of the physical parameters of the system and this corresponds to a single value of m and a corresponding critical wavelength. We have also explored the idea of tuning this wavelength by adjusting the pump intensity for a typical semiconductor gain medium. A remarkable feature of the spectral singularity related resonance effect is that if we increase the pump intensity so that the gain exceeds the threshold value, this effect disappears. This marks a clear distinction between the zero-width resonances associated with spectral singularities and the usual resonances that we encounter in optical resonators.

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