Entanglement and symmetry in permutation-symmetric states

Damian J. H. Markham*

CNRS, LTCI, Telecom ParisTech, 37/39 rue Dareau, 75014 Paris, France

(Received 2 July 2010; published 25 April 2011)

We investigate the relationship between multipartite entanglement and symmetry, focusing on permutation symmetric states. We give a highly intuitive geometric interpretation to entanglement via the Majorana representation, where these states correspond to points on a unit sphere. We use this to show how various entanglement properties are determined by the symmetry properties of the states. The geometric measure of entanglement is thus phrased entirely as a geometric optimization and a condition for the equivalence of entanglement measures written in terms of point symmetries. Finally, we see that different symmetries of the states correspond to different types of entanglement with respect to interconvertibility under stochastic local operations and classical communication.

DOI: 10.1103/PhysRevA.83.042332 PACS number(s): 03.67.Mn, 03.75.Gg

I. INTRODUCTION

Entanglement and symmetry are two main concepts at the heart of quantum mechanics. For a while now there have been enticing hints of a general connection between the two. On an intuitive level, we may understand that changing global symmetry (or topological properties) should be a global operation, so one that effects the entanglement of the system at hand. The relationship is of great interest, particularly because of the relation between symmetry and phase transitions. There is by now a vast array of instances where entanglement shows some relationship to symmetry breaking, for example, in quantum phase transitions where phase transition coincides with change in entanglement properties [1–3]. Indeed, it has been suggested that entanglement may be able to observe phase transitions where conventional order parameters fail. However, a concrete relationship remains unclear; for example, it is known that change in some symmetry properties need not effect the entanglement, and vice versa. For a recent review of these issues see [4].

At the same time, symmetry properties of states have been used to simplify the study of their entanglement, for example, in the calculation of entanglement [5,6] and questions of separability [7]. A particular feature of multipartite entanglement is that it is possible to have different "types" of entanglement, whereby we mean different classes under stochastic local operations and classical communication (SLOCC) [8]. This property has been largely overlooked in the in the study of phase transitions and the use of symmetries in entanglement theory. Two states are SLOCC inequivalent (belong to different classes) if they cannot be converted to one another via local operations and classical communications, even probabilistically, which signifies them as potentially different resources in the context of quantum information processing (QIP). Alongside this comes a plethora of entanglement measures, with a variety of different operational interpretations, and which may be suited to quantifying one type of entanglement more than another. The question naturally arises, can symmetry help us to explore this vast landscape, and can a relationship between symmetry and the types of entanglement be made.

In this work we focus on permutation symmetric states. These states are useful in a variety of QIP tasks, occur naturally as ground states, for example, in some Hubbard models, and certain of these states have recently been implemented experimentally [9,10]. Various entanglement properties of these states have also been studied, such as the clarification of separability conditions [7], calculation of the geometric measure of entanglement [11,12], and the identification of SLOCC classes [13]. In all these cases, however, permutation symmetry is essentially used as a tool for simplification in calculations. We would like to see if further symmetry properties can be useful, if a deeper connection between symmetry and entanglement properties can be found, and if there may be some insight into the role of entanglement in many-body physics. To this extent we observe that symmetries with respect to local operations (rather than permutation) determine several entanglement properties, with intriguing mirrors in spinor Bose-Einstein condensates (BECs).

In particular, by using the Majorana representation [14], we see how symmetry allows us to calculate the geometric measure of entanglement [15] and identify the most entangled state.[1] We then show that the existence of certain symmetries guarantees equivalence of three different entanglement measures—the geometric measure of entanglement, the logarithmic robustness of entanglement [16], and the relative entropy of entanglement [17]. Finally, we see how the different symmetries reflect different types of entanglement (in terms of SLOCC classes), indicating an intriguing relationship between symmetries and types of entanglement. We close with some remarks on occasions where these same symmetries coincide with different phases for spinor BEC and how these states may be generated experimentally.

II. ENTANGLEMENT IN THE MAJORANA REPRESENTATION

We first present the Majorana representation [14]. This way of seeing states has been used recently to simplify the

*markham@enst.fr

[1] Preliminary versions of these results were presented in the QIT 16 workshop in Japan, D. Markham, Proceedings of QIT 16, Japan (2007).

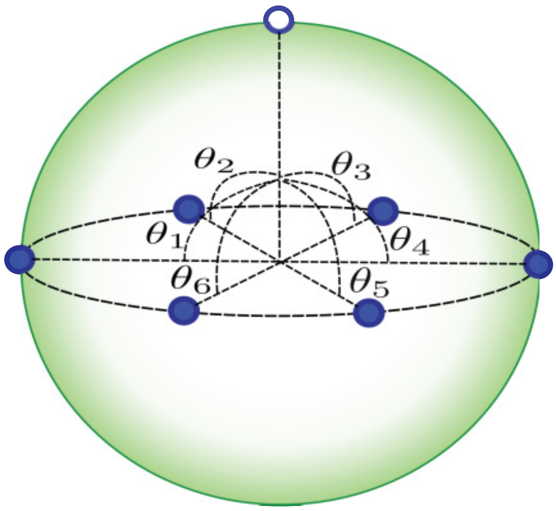

FIG. 1. (Color online) The Majorana representation of the n-party GHZ state $|\mathrm{GHZ}_n\rangle := (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2})$, having n MPs equally spaced around the equator, here for $n = 6$ in solid points. The hollow point at the north pole is the point of one of the closest product states, maximizing $\prod_i |\langle\phi|\eta_i\rangle|^2 = \prod_i [\cos(\theta_i/2)]^2$.

classification of symmetric states into SLOCC classes [13,18]. For n qubits, all permutation symmetric states can be written in the form [14]

$$|\psi\rangle = \frac{e^{i\alpha}}{\sqrt{K}} \sum_{\mathrm{PERM}} |\eta_1\rangle|\eta_2\rangle...|\eta_n\rangle, \qquad (1)$$

where the sum is over all permutations and K is a normalization constant. The Majorana representation consists of n points corresponding to the n states from this decomposition $|\eta_i\rangle = \cos(\theta_i/2)|0\rangle + e^{i\phi_i}\sin(\theta_i/2)|1\rangle$ via the standard Bloch sphere, i.e., a point on the unit sphere at position θ_i,ϕ_i. We call these the Majorana points (MPs), and they define the state up to global phase $e^{i\alpha}$ (see Fig. 1). For more details see Appendix A.

To see how entanglement can be visualized in the Majorana representation, we first notice that the product of local unitaries on a symmetric state $U \otimes U \otimes ...U|\psi\rangle$ is just a rotation of the Majorana sphere, since each point gets rotated by the same U. In fact, it can be shown that if two symmetric states $|\psi\rangle,|\phi\rangle$ are related by local unitaries $U_1 \otimes U_2 \otimes ...U_n|\psi\rangle = |\phi\rangle$, there is always some U such that they can be connected by the same unitary $U \otimes U \otimes ...U|\psi\rangle = |\phi\rangle$. This fact is shown for the more general case of local invertible operations in [13,18], and the same proof works for unitaries. Furthermore, this shows that the symmetry of the state under local unitaries $U^{\otimes n}$ is reflected by the symmetry of the MPs (see also Appendix A). This will be a main tool throughout the paper.

We can make the connection to entanglement more explicit using the geometric measure of entanglement [15],

$$E_g(|\psi\rangle) = \min_{|\Phi\rangle\in\mathrm{PROD}} -\log_2(|\langle\Phi|\psi\rangle|^2), \qquad (2)$$

where PROD is the set of product states. It has recently been shown that for permutation symmetric states, we can always take a symmetric product state $|\Phi\rangle = |\phi\rangle|\phi\rangle...|\phi\rangle$ in this optimization [11], for which the Majorana representation consists of n points at the position of $|\phi\rangle$. For the general n partite symmetric state (1) we then have

$$E_g(|\psi\rangle) = -\log_2[\Lambda(\psi)], \quad \Lambda(\psi) = \max_{|\phi\rangle} |\langle\phi|^{\otimes n}|\psi\rangle|^2$$
$$= \frac{1}{K} n!^2 \max_{|\phi\rangle} |\langle\phi|\eta_1\rangle|^2|\langle\phi|\eta_2\rangle|^2...|\langle\phi|\eta_n\rangle|^2.$$

Hence the optimization problem of finding the closest product state has the geometric interpretation of maximizing the product of angles $|\langle\phi|\eta_i\rangle|^2$. The example of $|\mathrm{GHZ}_6\rangle$ is illustrated in Fig. 1.

This geometric phrasing of the problem allows us to use geometric properties, for example, symmetry of the MP distribution to calculate entanglement, and to search for the most entangled states in this class. In a sense we can say that the most entangled states will be those which spread the points out the most (though this does not necessarily coincide with other definitions of "spread" such as Tammes' problem). This direction is pursued in detail in follow-up work [19] and has been independently studied in [20].

We now proceed to see how the Majorana representation can allow us to identify symmetries indicating states for which several distancelike entanglement measures coincide, and show that these states represent different SLOCC classes of entangled states.

III. EQUIVALENCE OF ENTANGLEMENT MEASURES

In Ref. [12] the relationship between the geometric measure of entanglement and two other distancelike entanglement measures, the relative entropy of entanglement E_R [17] and the logarithmic robustness of entanglement [16], are studied. In particular, it was shown that if there exists a local unitary group for which the state in question $\psi = |\psi\rangle\langle\psi|$ is itself an invariant subspace of the group, then we have $E_G(|\psi\rangle) = E_R(|\psi\rangle) = E_{Rob}(|\psi\rangle)$.

Equivalence of measures is desirable for several reasons, primarily because the different measures have different interpretations. For example, E_{Rob} signifies the ability of the state to withstand noise [16], and the relative entropy being an entropic quantity, E_R naturally has several information theoretic interpretations [17]. Since E_G is easier to calculate, it is easier to verify these operational properties. Further significance of the equivalence is discussed in [12], in particular, its significance for local accessibility of information and the construction of optimal entanglement witnesses.

Using the equivalence between symmetry of points and of states, we are able to phrase the problem solely in terms of the Majorana representation (see Appendix B for more details).

Lemma 1. If a permutation symmetric state $|\psi\rangle$ has MPs such that they are invariant under some subgroup $X \subset \mathrm{SO}(3)$, and for any small change of the points this invariance disappears, it satisfies

$$E_G(|\psi\rangle) = E_R(|\psi\rangle) = E_{Rob}(|\psi\rangle).$$

We call such subgroups $X \subset \mathrm{SO}(3)$ the symmetry groups and say such states are totally invariant. Intriguingly, this is exactly the condition for finding inert states in the context of spinor condensates [21], which is discussed more in the concluding remarks.

The Majorana representation then allows us to identify symmetries to show equality of the entanglement measures for many new sets of states. The complete set of all the possible subgroups of SO(3) are the continuous groups, orthogonal O(2), and special orthogonal SO(2), and discrete groups cyclic C_m, dihedral D_m, tetrahedral T, octehedral O, and isocahedral Y. One can then systematically go through all of these groups to find these special states, as done in [21] in the context of inert states. For the subgroup of arbitrary rotations about a fixed axis SO(2), we see that states with MPs only at either pole of the rotation axis satisfy our condition. If the rotations are around the Z axis, these are the states,

$$|S(n,k)\rangle := \frac{1}{\sqrt{\binom{n}{k}}}\left(\sum_{\text{PERM}} |\underbrace{00...0}_{n-k}\underbrace{11..1}_{k}\rangle\right),$$

also known as Dicke states, and we can see here pictorially the proof of equivalence for these states reported in [12]. Note that for even n and $k = n/2$, these states also satisfy our condition for the group O(2) (arbitrary rotations around the Z axis, and a flip on some axis in the $X - Y$ plane). In such cases we associate the state with the smallest subgroup. The cyclic group C_n has no truly invariant states, since if all points are moved together up and down the axis of rotation the symmetry is not lost. The dihedral group D_m (consisting of rotations through $2\pi/m$ and a flip on the axis of rotation) has m totally invariant states for each value m (see Fig. 2). T, O, and Y only have truly invariant states for certain n. For the tetrahedral group T, truly invariant states are made up of tetrahedrons, their antipode tetrahedrons, and octagons with at most two MPs on any tetrahedron point and three MPs at any octahedron point, so that there are only truly invariant states for $n \leqslant 34$. For the octahedral group O truly invariant states have MPs at the points of the cube and the octahedron with at most three

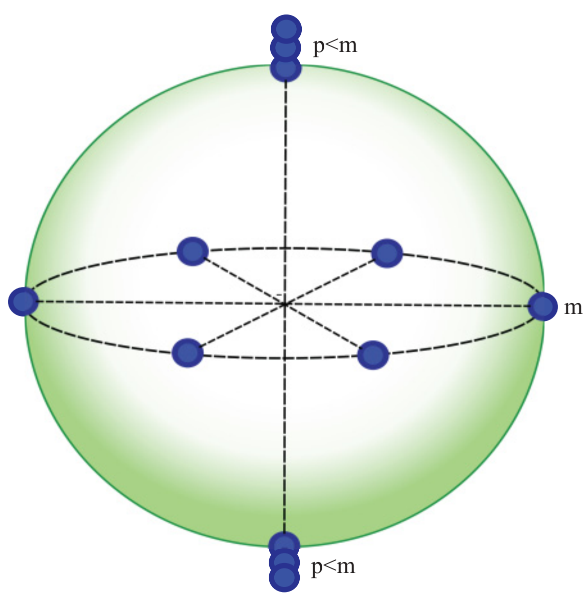

FIG. 2. (Color online) States of $n = m + 2p$ qubits which are totally invariant for the dihedral symmetry groups D_m, $|D_m(n,p)\rangle = 1/\sqrt{2}[|S(n,p)\rangle + |S(n,n-p)\rangle]$. The state has p MPs at each pole and $m = n - 2p$ MPs distributed evenly around the equator. For all n,p, these states satisfy $E = E_{Rob} = E_R = E_G$.

and two MPs at each, respectively, so that they only exist for $n \leqslant 34$. All truly invariant states of the isocahedral group Y are made up of combinations of isocahedrons (with 12 vertices) and dodecahedrons (with 20 vertices), with at most three and two MPs at each, respectively; hence they exist only for $n \leqslant 88$. For four qubits there are four entangled states, satisfying the condition, $|T\rangle = 1/\sqrt{3}|S(4,0)\rangle + \sqrt{2/3}|S(4,3)\rangle$, $|\text{GHZ}_4\rangle$, $|S(4,2)\rangle$ and $|W_4\rangle = |S(4,1)\rangle$, as shown in Fig. 3.

IV. SYMMETRY AND SLOCC ENTANGLEMENT CLASSES

We now look at how these different symmetries also correspond to different SLOCC entanglement classes. First of all, it is shown in [13,18] that if two states have different degeneracies of MPs (that is, the number of MPs which are on top of each other is different), they are SLOCC inequivalent. From this it is clear that

Lemma 2. For any number of qubits greater than two the totally invariant states with respect to the groups O(2), SO(2), and D_m are of different entanglement types.

This is true since they have different degeneracies. This fact also means that in addition, the totally invariant states for the dihedral group $|D_m(n,p)\rangle$ are SLOCC inequivalent for all $p\rangle 0$ (see Fig. 2).

For the remaining symmetry groups T, O, and Y there are only a finite number of possible totally invariant states. We can then use a combination of the degeneracy and other methods to attempt to show the same for these all subgroups. Consider the four-qubit case in Fig. 3. From the above we can see that $|S(4,2)\rangle$ (with two sets of two degenerate MPs) is in a different class to $|W_4\rangle$ (with a three-degenerate point), and they are both in different classes to $|T\rangle,|\text{GHZ}_4\rangle$ (which have all four MPs separate). To see that the $|T\rangle,|\text{GHZ}_4\rangle$ are different, we use the fact [8] that under SLOCC the minimal number of terms r for any expansion of the state in terms of only product states (the log of which is the Schmidt measure [22]) remains unchanged. It is straightforward to see that taking some minimum decomposition, from definition (2) we have $E_G \leqslant \log_2(r)$. We know that $r(|\text{GHZ}_4\rangle) = 2$ [8], and in [19] it is shown that $E_G(|T\rangle) = \log_2(3)$; hence, $r(|T\rangle) \geqslant 3 > r(|\text{GHZ}_4\rangle)$ and so they are in different SLOCC classes also. For larger n one can in principle go through all cases individually (since their number is finite) and check using similar methods. Such an exhaustive search was beyond the scope of this manuscript; however, the same techniques as above can be used to show the SLOCC inequivalence for *all* the totally invariant states up to seven qubits, and it seems plausible

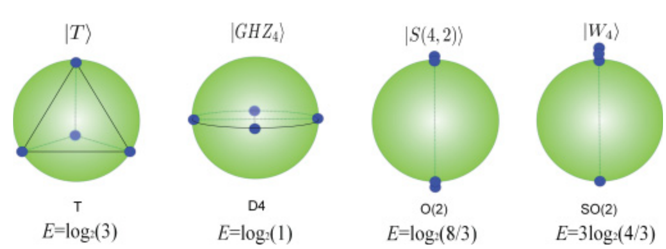

FIG. 3. (Color online) Different symmetries for four qubits giving states such that $E = E_{Rob} = E_R = E_G$. The symmetry group and the entanglement are written below the sphere. Each state is in a different SLOCC class.

that indeed all different symmetries do imply different classes of entanglement.

V. DISCUSSIONS

In this work we have presented a geometrical representation of the entanglement of permutation symmetric states in the form of the Majorana representation. This has allowed us to phrase the geometric measure of entanglement in a simple way and to look at how the further symmetry properties of states effect their entanglement properties, in particular, showing the equivalence of three different distancelike measures. This equivalence simplifies calculation and allows for wider operational understanding as the operational interpretations coincide. Finally, we show that for these states the different symmetries correspond to different types of entanglement. This presents a very interesting relationship between symmetries and types of entanglement. Though we are not able to confirm that this is a general connection, it is very interesting, seems possible, and is worth deeper investigation.

Intriguingly, within the context of spinor condensates, similar symmetry arguments have been used to identify and characterize different phases of matter [23–25]. In this case the Majorana representation is used not to describe n symmetric qubits but rather a single-spin $S = n/2$ system (through the well-known isometry between the two) [14]. Because of this caution is required when talking about entanglement in this context, but it is not impossible for the two pictures to coincide, for example, the total spin can be a result of combined spin half-systems in exactly the permutation symmetric space we look at, which really would be entangled as we discuss here. In this sense we would see that phase transitions through symmetry are incidental with phase transitions through entanglement, raising the prospect of entanglement type as an indicator of different phases. Indeed, in [23] a phase diagram is presented for a spin two-spinor condensate where each phase is identified exactly with different symmetry types presented in Fig. 3. Where these connections are most explicit and general is in the case of inert states—often good candidates for ground states in spinor BECs—where the conditions of equivalence of E_G, E_{Rob}, and E_R coincide exactly in terms of the MPs [21], pointing to deeper possible connections.

The states discussed here can also be experimentally prepared in a variety of ways and media. For example, in optics the six-party $|S(6,3)\rangle$ (Dicke) state and several five- and four-party states have recently been generated and their entanglement properties verified [9,10]. Furthermore, a general scheme has been developed recently which can generate any symmetric state [26] (including all those here) which works for any Λ-system photon emitters, such as trapped ions or neutral atoms or quantum dots, and so may be long lived and within reach of current experimentation.

Note added: Recently several related works have emerged [20,27–30].

ACKNOWLEDGMENTS

We are very grateful to Shashank Virmani, Martin Aulbach, Mio Murao, and Vlatko Vedral for insightful comments and discussions.

APPENDIX A: MAJORANA REPRESENTATION

The permutation symmetric subspace of n qubits is spanned by the Dicke states

$$|S(n,k)\rangle := \frac{1}{\sqrt{\binom{n}{k}}}\left(\sum_{\text{PERM}} |\underbrace{00\ldots0}_{n-k}\underbrace{11..1}_{k}\rangle\right), \tag{A1}$$

which can be understood as the symmetric states with k excitations. Thus any permutation symmetric state can be written as

$$|\psi\rangle = \sum a_k |S(n,k)\rangle. \tag{A2}$$

Alternatively, all symmetric states can be written in the Majorana representation [14],

$$|\psi\rangle = \frac{e^{i\alpha}}{\sqrt{K}} \sum_{\text{PERM}} |\eta_1\rangle|\eta_2\rangle\ldots|\eta_n\rangle, \tag{A3}$$

where the sum is over all permutations and K is a normalization constant.

To find the Majorana representation (A3), we consider the overlap with product state $|\phi\rangle^{\otimes n}, |\phi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|1\rangle$. It is clear by comparison to Eq. (A3) that $|\phi\rangle$ orthoganol to the MP $|\eta_i\rangle$ will give zero overlap. This is exactly how we find the MPs. For simplicity we take a multiple of the overlap, sometimes called the characteristic polynomial, Majorana polynomial, amplitude function, or coherent state decomposition,

$$f(\psi) := \cos^{-n}\left(\frac{\theta}{2}\right)\langle\phi|^{\otimes n}|\psi\rangle = \sum_{k=0}^{n}\sqrt{\binom{n}{k}}a_k\alpha^k, \tag{A4}$$

which is a complex polynomial in $\alpha := e^{-i\varphi}\tan(\frac{\theta}{2})$. By the fundamental theorem of algebra, this has unique zeros up to multiplication by some complex. Hence the zeros $\alpha_j = e^{-i\varphi_j}\tan(\frac{\theta_j}{2})$ define the state $|\psi\rangle$ up to a global phase. The corresponding MPs are at position $\theta_j' = \theta_j + \pi$, $\varphi_j' = \varphi_j + \pi$.

Note that we can understand the state $|\phi\rangle^{\otimes n}$ as a kind of generalized coherent state [31,32], defined by the action of a group on some chosen fiducial state (so that certain properties apply such as overcompleteness). For our case the group is SU(2) as represented by $U^{\otimes n}$, with U a rotation through θ,φ and the fiducial state $|0\rangle^{\otimes n}$, that is

$$|\phi\rangle^{\otimes n} = U^{\otimes n}|0\rangle^{\otimes n}. \tag{A5}$$

When viewing the symmetric subspace as one spin $S = n/2$ system, these are equivalent to spin coherent states [33,34]. In this sense the Majorana representation is a kind of condensed coherent state representation of states [since it is only concerned with the zeros of the coherent state decomposition (A4)].

APPENDIX B: EQUIVALENCE OF ENTANGLEMENT MEASURES

We now come to the proof of the equivalence between entanglement measures and the symmetry of the MP distributions. The entanglement measures in question are the geometric measure of entanglement [15], the relative entropy of entanglement E_R [17], and the logarithmic robustness of entanglement [16] is studied. Equality between the measures is guaranteed for a state $|\psi\rangle$, if it is possible to find a separable state of the form [12]

$$\omega_{sep} = \Lambda(\psi)|\psi\rangle\langle\psi| + [1 - \Lambda(\psi)]\Delta, \tag{B1}$$

where Δ can be any density matrix and $\Lambda(\psi)$ is the maximum overlap with a product state as defined in (3). The state (B1) can be understood as the "closest" separable state with respect to the robustness of entanglement, which is deemed equal to the geometric measure of entanglement by its form [12].

The trick used in [12] is to take techniques from group averaging to find such a state (see also [5]). For a group G, any particular representation $U(g)$, $g \in G$ can always be expanded as a product sum over irreducible representations (irreps, which we enumerate by k), and the irreps give a decomposition of the total Hilbert space,

$$U(g) = \bigoplus_{k=1}^{K} \mathbf{1}_{A_k} \otimes U_{B_k}(g) \tag{B2}$$

$$H = \bigoplus_{k=1}^{K} H_{A_k} \otimes H_{B_k}, \tag{B3}$$

where $U_{B_k}(g)$ is the representation of $g \in G$ for irrep k acting on Hilbert space H_{B_k}. The role of H_{A_k} is just to give a compact form to express multiplicity—the multiplicity of irrep k is given by $\dim(H_{A_k}) = \mathrm{Tr}(\mathbf{1}_{A_k})$. Note that the tensor product in the above has nothing to do with the separation of parties defining entanglement. By direct application of Shur's lemma, averaging over the group gives [12]

$$\omega = \int U(g)\rho U(g)^\dagger dg$$
$$= \sum_{k=1} \frac{1}{\dim(H_{B_k})} Tr_{B_k}\{P_{A_k\otimes B_k}\rho P_{A_k\otimes B_k}\} \otimes \mathbf{1}_{B_k}. \tag{B4}$$

If we now average over a local unitary group on a product state $|\Phi\rangle$, which achieves the maximum overlap $\Lambda(\psi)$, we will get a separable state, which is our candidate for (B1). If, furthermore, the state $\psi = |\psi\rangle\langle\psi|$ is an invariant subspace associated to a one-dimensional irrep (say $k = 1$) with multiplicity equal to one we have

$$\omega_{sep} = \int U(g)\Phi U(g)^\dagger dg = \Lambda(\psi)|\psi\rangle\langle\psi|$$
$$+ \sum_{k=2} \frac{Tr_{B_k}\{P_{A_k\otimes B_k}|\Phi\rangle\langle\Phi|P_{A_k\otimes B_k}\}}{\dim(H_{B_k})} \otimes \mathbf{1}_{B_k}, \tag{B5}$$

which is indeed of form (B1), implying equality of the entanglement measures.

In terms of states, $\psi = |\psi\rangle\langle\psi|$ corresponds to a one-dimensional irrep if it is invariant under group action. A one dimensional irrep is a phase which acts over a space of dimension equal to the multiplicity. Any state [one-dimensional (1D) matrix] in this space is unchanged and so can itself be considered a 1D irrep. Since it is possible to continuously change states through this space, it means that a state which is a 1D irrep, and therefore invariant, can be continuously changed to another state which is also a 1D irrep, and hence also invariant. If, on the other hand, a small shift breaks the invariance, the state has multiplicity of only one, as we desire.

The groups we consider in this work are naturally enough subgroups of SU(2), as represented by the local unitaries $U^{\otimes n}$. Again, we see from definition (1), such operations are simply rotations [in SO(3)] of the Majorana sphere itself. Since we are only interested in the state matrix ψ (where global phases do not matter), the invariance the MPs implies a state is a 1D irrep. If no small change in the positions of the points is also invariant, this implies there is no multiplicity within the symmetric subspace. Although this is not immediate enough to show the group-averaged state is of the form (B1), it can be proven as follows. The only remaining possibility for multiplicity is if it has part outside the symmetric subspace. In fact, a projection onto it (say for irrep k) must be of the form $P_{A_k\otimes B_k} = |\psi\rangle\langle\psi| + |\psi^\perp\rangle\langle\psi^\perp|$, where $|\psi^\perp\rangle$ has no components in the symmetric subspace. This is true since its representation is $U^{\otimes n}$ and so any 1D irrep cannot stretch over the symmetric subspace and another subspace but must be distinctly in one or the other. If we put this into (B4) (with again $\rho = \Phi$), we indeed get the form (B5).

Thus the condition for equality of measures stated in the main text is correct and complete. For example, for the subgroup of arbitrary rotations about a fixed axis SO(2), we see that states with MPs only at either pole of the rotation axis satisfy our condition. If the rotations are around the Z axis, these are the Dicke states, and we can see here pictorially the proof of equivalence for these states reported in [12].

[1] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature **416**, 608 (2002).

[2] T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).

[3] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).

[4] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).

[5] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A **64**, 062307 (2001).

[6] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Phys. Rev. A **67**, 022112 (2003).

[7] G. Toth and O. Gühne, Phys. Rev. Lett. **102**, 170503 (2009).

[8] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[9] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **103**, 020503 (2009).

[10] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Toth, and H. Weinfurter, Phys. Rev. Lett. **103**, 020504 (2009).

[11] R. Hübener, M. Kleinmann, T.-C. Wei, C. Gonzalez-Guillen, and O. Gühne, Phys. Rev. A **80**, 032324 (2009).
[12] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Phys. Rev. A **77**, 012104 (2008).
[13] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. **103**, 070503 (2009).
[14] E. Majorana, Nuovo Cimento **9**, 43 (1932).
[15] A. Shimony, Ann. N.Y. Acad. Sci. **755**, 675 (1995).
[16] D. Cavalcanti, Phys. Rev. A **73**, 044302 (2006); G. Vidal and R. Tarrach, *ibid.* **59**, 141 (1999); A. W. Harrow and M. A. Nielsen, *ibid.* **68**, 012308 (2003); M. Steiner, *ibid.* **67**, 054305 (2003).
[17] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[18] P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, Phys. Rev. A **81**, 052315 (2010).
[19] M. Aulbach, D. Markham, and M. Murao, New J. Phys. **12**, 073025 (2010).
[20] J. Martin, O. Giraud, P. A. Braun, D. Braun, and T. Bastin, Phys. Rev. A **81**, 062347 (2010).
[21] H. Mäkelä and K.-A. Suominen, Phys. Rev. Lett. **99**, 190408 (2007).
[22] J. Eisert and H.-J. Briegel, Phys. Rev. A **64**, 022306 (2001).
[23] R. Barnett, A. Turner, and E. Demler, Phys. Rev. Lett. **97**, 180412 (2006).
[24] R. Barnett, A. Turner, and E. Demler, Phys. Rev. A **76**, 013605 (2007).
[25] R. Barnett, S. Mukerjee, and J. E. Moore, Phys. Rev. Lett. **100**, 240405 (2008).
[26] T. Bastin, C. Thiel, J. von Zanthier, L. Lamata, E. Solano, and G. S. Agarwal, Phys. Rev. Lett. **102**, 053601 (2009); N. Kiesel, W. Wieczorek, S. Krins, T. Bastin, H. Weinfurter, and E. Solano, Phys. Rev. A **81**, 032316 (2010).
[27] T. Bastin, P. Mathonet, and E. Solano, e-print arXiv:1011.1243 [quant-ph].
[28] C. D. Cenci, D. W. Lyons, and S. N. Walck, e-print arXiv:1011.5229 [quant-ph].
[29] P. Ribeiro and R. Mosseri, e-print arXiv:1101.2828 [quant-ph].
[30] M. Aulbach, e-print arXiv:1103.0271 [quant-ph].
[31] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Heidelberg, 1986).
[32] J. R. Klauder and B. Skagerstam, *Coherent States* (World Scientific, Singapore, 1985).
[33] J. M. Radcliffe, J. Phys. A **4**, 313 (1971).
[34] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).