Reduce, reuse, recycle for robust cluster-state generation

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(Received 27 July 2010; published 21 April 2011)

Efficient generation of cluster states is crucial for engineering large-scale measurement-based quantum computers. Hybrid matter-optical systems offer a robust, scalable path to this goal. Such systems have an ancilla which acts as a bus connecting the qubits. We show that by generating the cluster in smaller sections of interlocking bricks, reusing one ancilla per brick, the cluster can be produced with maximal efficiency, requiring fewer than half the operations compared with no bus reuse. By reducing the time required to prepare sections of the cluster, bus reuse more than doubles the size of the computational workspace that can be used before decoherence effects dominate. A row of buses in parallel provides fully scalable cluster-state generation requiring only 20 controlled-PHASE gates per bus use.

DOI: 10.1103/PhysRevA.83.042327   PACS number(s): 03.67.Lx, 42.50.Ex, 03.67.Mn, 32.80.−t

I. INTRODUCTION

Hybrid schemes for quantum information processing are among the most promising for scalable quantum computers. Such systems combine both matter and optical elements, where the computational gates between qubits of one type can be mediated by a shared bus of the other type [1–3]. A computational model for such hybrid systems has recently been characterized as *ancilla-based computation* [4], in contrast to the usual quantum computation models that use direct qubit-qubit gates. Ancilla-driven schemes are important for chip-based quantum computing architectures, where a flying ancilla mediates between fixed qubits [5–7].

Hybrid architectures form a natural substrate for measurement-based quantum computing (MBQC) [8], one type of which (the topological model based on the surface code [9]) has the best error threshold for quantum computing [10]. In MBQC a highly entangled *cluster state* is generated, and then computation is performed by sequential qubit measurements. The quantum processing task is to generate the cluster state, after which it becomes a matter of measurement and classical processing to feed forward the measurement outcomes. The first proposal for cluster-state construction was a one-shot scheme, where the entire cluster was created by a small number of global operations [8]. Since the cluster qubits are measured sequentially in scalable physical realizations, the cluster is prepared dynamically, a few rows at a time [11,12]. This avoids the need for long coherence times for entangled qubits [13,14], a critical requirement for scalable schemes. Photonic schemes for constructing cluster states probabilistically [14] exploit the linear optics quantum computing scheme of Knill, Laflamme, and Milburn [15]. The disadvantage of this approach is the large number of repeated operations required to successfully build the cluster. To reduce this overhead, heralded controlled-PHASE (CPHASE) operations occurring between two qubits were proposed by Browne *et al.* [16]. Duan *et al.* [17] showed that this probabilistic generation does indeed allow the cluster to grow, and Gross *et al.* [18] determined the optimal growth strategies for regimes with low and high probabilities of success per operation. Louis *et al.* [19] showed that using a three-qubit entangling gate instead of a two-qubit entangling gate increased the success probability from $1/2$ to $3/4$. The advantages of deterministic gates were explored by exploiting ancilla-based schemes [5,19–22]. Wang *et al.* [23] proposed a method to transfer an atomic cluster state to photonic qubits, inverting the usual role of the qubit and ancilla between the matter and optical systems.

As the cluster state is the fundamental quantum resource of a measurement-based computation, it becomes extremely important to make it as error free as possible. Errors in constructing the cluster can propagate rapidly through a computation because of the highly entangled nature of the state, leading to failure of the computation. Topological surface encodings on cluster states provide a robust fault tolerance for quantum computation, provided each component in the system has an error below a certain threshold [7,24,25]. The construction of the cluster itself is one such component, and schemes to reduce cluster error can enable systems that would otherwise be unusable to reach the threshold for use with error correction. Hybrid systems are susceptible to specific types of error that other systems are not, because of the use of the mediating ancilla. In cases where the ancilla is not destroyed after each gate, there is the additional possibility of errors propagating through ancilla reuse. We show there is a trade-off between increasing efficiency by using the same bus for multiple gates and increasing errors because of this.

In this paper we present the optimal scheme for dynamic two-dimensional (2D) cluster-state generation in hybrid systems where the mediating system (bus) can be used for more than one gate operation without being reset. We divide the cluster state into interlocking bricks, each of which is built with a single bus. We give the optimal method for constructing the bricks, reducing the number of system-bus entanglements.

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We then show how to determine the brick size based on the error threshold of the system being used. We find that, even when the probability of error in the system is high, this scheme can still deliver significant efficiency savings through bus reuse, enabling a larger cluster to be generated. The paper is organized as follows. In Sec. II we give an overview of the qubus system, the particular ancilla-based scheme we will focus on. Section III explains how to reduce the number of operations required when reusing the qubus for multiple gates. In Sec. IV we introduce our error model for reusing the qubus, and in Sec. V we apply bus reuse to generating a 2D cluster state. In Sec. VI we calculate the optimal bus reuse scheme in the ideal case, and in Sec. VII we combine this with our error model to give the optimal bus reuse scheme with dephasing. Section VIII discusses how to apply our results to dynamic generation schemes, and in Sec. IX we calculate the optimal brick size in terms of the system parameters. Section X summarizes our conclusions.

II. QUBUS SYSTEM

To provide a concrete setting for our calculations, we will focus on the qubus system, which consists of matter qubits and a photonic field as the mediating ancilla [1,26,27]. The cluster state we are generating is a regular square lattice of qubits with nearest neighbors entangled. Each qubit is initialized in the state $(|0\rangle + |1\rangle)/\sqrt{2}$, then CPHASE gates are applied between neighboring qubits. Using the qubus, CPHASE gates are performed using a conditional evolution

$$U_e = \exp(-i H_{\text{int}}\tau/\hbar), \tag{1}$$

where τ is the fixed time for one such operation and

$$H_{\text{int}} = \hbar\chi\sigma_z(a^\dagger e^{i\theta} + a e^{-i\theta}), \tag{2}$$

where χ is nonlinearity strength, a ($a^\dagger$) are the field annihilation (creation) operators, and $\theta = 0$ $(\pi/2)$ describes coupling of the qubit to the position (momentum) quadrature of the field. This interaction results in deterministic displacements along discrete paths in phase space, of amplitude $\beta = \chi\tau$. The application of $U_e(\pm x_j)$ applies a displacement of β in the positive (negative) direction in position space for the jth qubit, and $U_e(\pm p_k)$ applies a displacement of β along the positive (negative) axis in momentum space for the kth qubit. The sequence

$$U_e(x_2)U_e(-p_1)U_e(-x_2)U_e(p_1) \tag{3}$$

performs a geometric phase gate between qubits 1 and 2, with the phase change proportional to the area traced out [1,28]. When $\beta^2 = \pi/8$, this provides the CPHASE gate required for cluster-state construction. The qubus thus acts as a discrete-level system with two partitions, equivalent to two coupled qudit ancillas. There are two options for using such an ancilla-based system to construct a cluster state: Either the ancilla is discarded after every gate, or it is recycled for use with further gates.

III. REUSING THE BUS

If each CPHASE gate is performed by a different bus, then each qubit in the cluster (apart from the perimeter) needs to be operated on by four different buses to generate the four entanglements it is part of. For a cluster of $m \times n$ qubits we therefore need

$$N = 8mn - 4(m+n) \tag{4}$$

bus operations to complete it—one entangling and one disentangling operation per qubit per gate. However, if we are able to reuse the bus, then we can use fewer operations. Consider the following sequence of unitaries for three qubits:

$$U_e(x_3)U_e(-p_2)U_e(-x_3)U_e(-x_1)U_e(p_2)U_e(x_1). \tag{5}$$

Reading from the right, a CPHASE is performed between qubits 1 and 2, and then qubit 1 is disentangled from the bus. Qubit 2 is kept on the bus, and qubit 3 is entangled with the position quadrature. Finally, qubits 2 and 3 are disentangled from the bus (in that order). The result is CPHASE gates between both (1,2) and (2,3) using six bus operations rather than the eight needed if qubit 2 were disentangled after the first interaction. Such sequential operations are possible in all ancilla-based systems which can reuse the ancilla.

IV. DEPHASING ERRORS

Reusing the ancilla reduces the total number of operations required, speeding up the process and hence reducing the length of time decoherence acts on the cluster qubits. For N bus operations taking a total time $N\tau$ to perform, the probability of a phase-flip error due to qubit dephasing is $[1 - \exp(-N\gamma\tau)]/2$, where γ is the dephasing rate for one qubit. Fewer bus operations, therefore, mean less dephasing. However, we have to take into account error accumulating on the ancilla. For the qubus, the errors come from photon loss. The probability of a phase-flip error due to photon loss on the bus is $[1 - \exp(-4C\eta\beta^2)]/2$, where C is the number of CPHASE gates constructed per bus and η is the loss parameter for the bus. Combining these gives the total probability of dephasing:

$$\varepsilon = \frac{1}{2}[1 - \exp(-N\gamma\tau - 4C\eta\beta^2)]. \tag{6}$$

We can therefore trade off the two dephasings by reusing the bus, which reduces N but increases C. If we minimize N for a given ε, this then enables a maximum number of CPHASE gates to be completed before the dephasing reaches a critical value.

V. CLUSTER-STATE GENERATION

We now apply bus reuse to more efficient cluster-state construction. Extending the bus reuse sequence in Eq. (5) to further qubits allows one ancilla to generate a line of entangled qubits with just two operations per qubit, one entangling followed by one disentangling. A set of such lines of length L arranged to form an $L \times L$ grid generates a 2D cluster, as proposed by Louis *et al.* [19,29]. The minimum number of operations required to build a cluster from 1D entangled lines of qubits can be obtained by a simple combinatorial argument. Consider the cluster as a 2D lattice graph, with qubits as vertices and entanglement links as edges. We count how many edges of the graph can be generated using a line of qubits when each qubit is only visited once; this corresponds to being connected to the bus once only, thus minimizing bus use. For the cluster of $m \times n$ qubits, the total number of vertices is mn. The maximum number of edges that can be generated is therefore $mn - 1$. Each entangling action requires two bus operations per qubit (one to connect to the bus and one to disconnect). We can therefore generate $mn - 1$ edges with $2mn$ bus operations. The total number of edges in the cluster is $m(n-1) + n(m-1)$, so we are left with $(n-1)(m-1)$ edges to fill in. The path we have generated can connect a maximum of two edges to each vertex in the lattice. All vertices except the corners require more than two edges. Therefore all qubits except the four corners will require reactivation in order to fill in the extra edges, requiring $2mn - 8$ additional operations. We therefore have a minimum number of bus operations $4mn - 8$ to generate the cluster state using an ancilla system where at most two qubits are coupled to the ancilla at any time. The method of Louis *et al.* [19] achieves the minimum up to a constant. If the total number of operations one bus can perform is limited by the errors accumulating, each change to a new bus requires, in general, an extra disentangling of the old bus and re-entangling of the new bus: a total of two extra operations per extra bus.

VI. OPTIMAL BUS REUSE

Generation of a line of entangled qubits is the simplest use of sequential operations, with a maximum of one qubit on each quadrature of the bus at any time. However, it does not use the full power of the qubus to reduce the number of bus operations per gate. The displacement operators on the qubus allow a qubit on one quadrature to become entangled with all qubits on the other. If we start by connecting one qubit to, say, the position quadrature, then all its neighbors can be simultaneously coupled to the momentum quadrature. However, if we then try to connect any other qubit to the position quadrature there will be cross-entanglements generated that are not part of the required cluster state (where qubits are entangled only with nearest neighbors). Only two of the momentum-quadrature qubits can remain on the bus and not generate unwanted entanglement. These qubits must neighbor both of the position-quadrature qubits; this then forms a closed box in the lattice.

In contrast with the previous scenario, we now need to consider a path across the lattice that is two qubits wide, rather than a single-qubit line (Fig. 1). By inspection, the maximum number of edges that can be generated on such a path that visits all mn qubits in the cluster only once is $3mn/2 - 2$, for even mn. The number of cluster edges remaining after $2mn$ sequential bus operations is therefore $\frac{1}{2}mn - (m+n) + 2$. We could either finish the sequential operations and then generate these edges separately (requiring two qubits per edge to be reconnected with the bus), or we could construct these edges as we go along. In the latter case, before a qubit is disconnected from the bus, we generate the extra edges required for that qubit. Then only one qubit per additional edge needs to be connected to the bus again. With two bus operations per connection, this requires a further $mn - 2(m+n) + 4$ uses of the bus. This gives a lower bound of

$$N_{\min} = 3mn - 2(m+n) + 4 \tag{7}$$

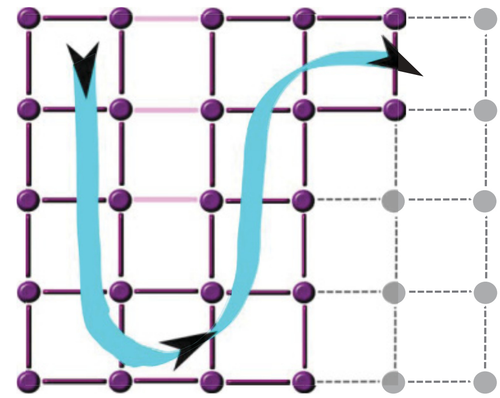

FIG. 1. (Color online) A path of width two generating a 5×6 cluster. Dark edges represent entanglement between qubits that have connected to the bus once, light edges require at least one qubit to connect to the bus twice, and dotted edges indicate CPHASE gates not yet performed.

operations to construct the cluster. We show elsewhere [30] that using a spiral pattern for the path of width two achieves the bound $N_{\min}$. A spiral path does not allow dynamic generation, so in practice we will use a zigzag path (Fig. 1). The U-shaped turns require up to two extra operations per turn, so we will want to minimize their number to minimize the actual cost. The zigzag path in Fig. 1 is the minimum turn arrangement for dynamically generating rectangular clusters.

Equation (7) tells us the most efficient a scheme can be when the bus acts as an ancilla partitioned into two. It is clear that, in general, an ancilla can have more than two partitions, although multipartition ancillas are more naturally suited to multiqubit gates. With a path of width a, the number of operations using a single bus would only improve to order $2mn + 2(mn/a)$. Thus, we can see that going to large partition sizes significantly increases the ancilla complexity for a rapidly reducing payoff in terms of bus efficiency.

VII. REUSE WITH DEPHASING

We now consider the case where building the entire cluster with one bus would take us beyond the threshold value of the error as given by Eq. (6). In such a situation we would need to use multiple buses, each one creating a smaller part of the cluster. Since there are always at least two qubits entangled with the bus for the path of width two, changing buses requires two qubit disconnects and reconnects, a total of four extra operations per extra bus. These sections of the cluster generated by one bus we call bricks (Fig. 2). If these bricks have length b [see Fig. 2(a)], then an $m \times n$ cluster will contain $mn/2b$ bricks. The number of extra operations is thus $4(mn/2b - 1)$, giving a minimum number of operations to create the cluster using multiple buses of

$$N_{\min}(b) = (3 + 2/b)mn - 2(m+n). \tag{8}$$

Figure 2(b) shows how the bricks fit together in the cluster, with shared qubits being reactivated by different buses during the construction. The bus thus entangles a total of $3b + 2$ qubits to construct a brick that adds $2b$ qubits to the cluster. Where whole bricks fit neatly into the cluster (as shown), we achieve

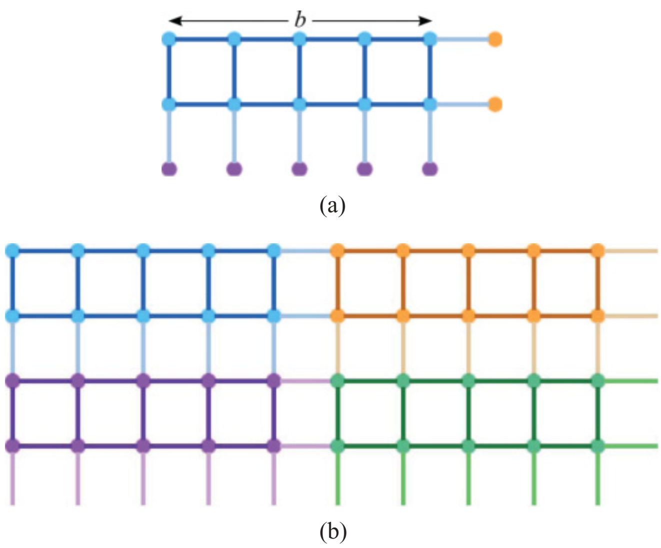

FIG. 2. (Color online) (a) A brick of size $b = 5$, consisting of a core of $2b = 10$ qubits and $(b + 2) = 7$ connections to qubits in neighboring bricks. (b) Four bricks joined together, showing how qubits are shared between bricks and thus entangled by more than one bus (at different times) during cluster construction.

the bound in Eq. (8). Each brick needs $6b + 4$ bus operations to produce it (two operations per qubit), so by multiplying by the number of bricks $mn/2b$, and subtracting the $2(m + n)$ operations not required for the sides not connected to further bricks, the total number of operations $N_{\min}(b)$ is obtained.

VIII. DYNAMIC GENERATION

For dynamic generation of our cluster, we need to produce a strip a few qubits wide with the measurements that perform the computation applied just behind the construction process. When a whole number of bricks fit across the cluster, it can be dynamically generated without any loss of efficiency. When bus-changing operations do not happen conveniently at the edge of the cluster, we will need to turn a corner within a brick. These shaped turns will cost at most two extra operations per turn [30].

Our brick method also facilitates optimally efficient dynamic generation where multiple buses are used in parallel to produce a fully scalable cluster-state scheme. We orient our bricks along the growth direction (see Fig. 3), producing parallel connected paths of width two. To avoid the buses entangling to the same qubit at the same time, alternate buses must be started six operations apart. Since we want a wide enough strip to allow room for the measurements to follow behind the cluster construction, we can also use twice as many buses (one per qubit row). This is less efficient in operations per bus but generates a wider strip in the same time frame [30]. The optimal choice will depend on the decoherence rates and the cost of extra buses for the particular system.

IX. OPTIMAL BRICK SIZE

The system's error threshold ε will determine the size of our bricks. A brick has $3b + 2$ qubits, each operated on twice

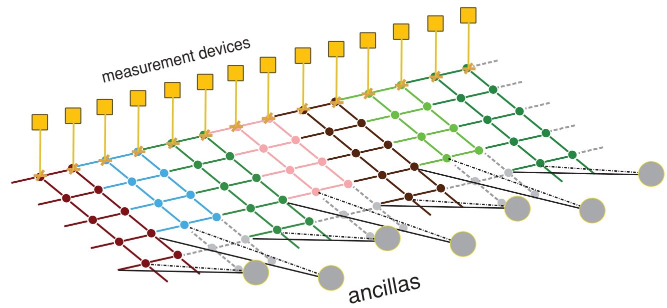

FIG. 3. (Color online) Dynamic generation using multiple ancillas.

for a total of $6b + 4$ bus operations, and $4b$ edges (CPHASE gates). Using Eq. (6) we require

$$\tfrac{1}{2}\{1 - \exp[-(6b + 4)\gamma\tau - 16b\eta\beta^2]\} \leqslant \varepsilon. \tag{9}$$

For a given set of experimental parameters γ, τ, and η, and desired dephasing limit ε, this determines b.

Let us now compare our scheme to the capabilities of one without bus reuse. If we use one bus per CPHASE gate to generate a brick, Eq. (6) gives

$$\tfrac{1}{2}[1 - \exp(-16b\gamma\tau - 4\eta\beta^2)] \leqslant \varepsilon. \tag{10}$$

Comparing Eqs. (9) and (10), we find our scheme produces less qubit dephasing than using one bus per CPHASE gate provided $\eta\beta^2 \lesssim \gamma\tau/2$. For example, if $\gamma\tau = 5 \times 10^{-4}$ and $\eta = 10^{-4}$, then for an error threshold of $\varepsilon = 10^{-2}$, the bus-per-gate method could generate only 8 CPHASE gates between 8 qubits ($b = 2$) before reaching the threshold, while our brick method would be able to connect at least 17 qubits with 20 CPHASE gates ($b = 5$) before the same dephasing occurred. For the case using multiple buses in parallel, this would give a coherent strip of cluster four qubits wide, just enough to apply the measurements behind the construction, as shown in Fig. 3.

X. CONCLUSIONS

We have described the optimally efficient method for generating cluster states in ancilla-based computation, based on dividing the cluster into interlocking bricks, each of which is constructed with a single, reused ancilla. We have shown how, in the specific case of the qubus system, the reduction in ancilla operations can offset the increased noise due to bus reuse, allowing approximately twice the number of qubits to be connected into a cluster state compared to single-bus use. Compared with $8mn - 4(m + n)$ bus operations with no bus reuse, for large clusters, the interlocking-brick scheme uses fewer than half for $b > 2$, $O(3mn)$ compared to $O(8mn)$. Even for $b = 1$, the reduction is to $5mn - 2(m + n)$, equivalent to the method in [19] when limited to five qubits per bus. This will therefore be the method of choice for any deterministic ancilla-based cluster generation that allows bus reuse (see [18] for optimal probabilistic schemes). This form of bus reuse can provide savings in many other contexts, including the quantum Fourier transform [31].

While the exact error model will vary with the underlying physical system, our analysis can be generalized to all ancilla-based cluster generation schemes. Our results are directly applicable to bus-based experimental production of cluster states, enabling the same resources to produce dynamically generated cluster states of twice the size compared to single-gate bus use. For multibus dynamic schemes, this means fully scalable operation can be achieved with half the coherence time compared to single-gate buses. In practical terms, this needs as few as 20 gates per bus, independent of cluster size.

ACKNOWLEDGMENTS

We thank Aram Harrow, Ashley Stephens, and Simon Devitt for helpful discussions. D.H. was supported by the EU Integrated Project Qubit Applications and the Bristol Centre for Nanoscience and Quantum Information. K.L.B. was supported by the United Kingdom Engineering and Physical Sciences Research Council and Hewlett-Packard. V.M.K. was supported by the United Kingdom Royal Society. W.J.M. acknowledges partial support from the EU project HIP and MEXT in Japan.

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