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Quantum mechanics of hyperbolic orbits in the Kepler problem

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The problem of deriving macroscopic properties from the Hamiltonian of the hydrogen atom is resumed by extending previous results in the literature, which predicted elliptic orbits, into the region of hyperbolic orbits. As a main tool, coherent states of the harmonic oscillator are used which are continued to imaginary frequencies. The Kustaanheimo-Stiefel (KS) map is applied to transform the original configuration space into the product space of four harmonic oscillators with a constraint. The relation derived between real time and oscillator (pseudo) time includes quantum corrections. In the limit $\hbar \to 0$, the time-dependent mean values of position and velocity describe the classical motion on a hyperbola and a circular hodograph, respectively. Moreover, the connection between pseudotime and real time comes out in analogy to Kepler's equation for elliptic orbits. The mean-square-root deviations of position and velocity components behave similarly in time to the corresponding ones of a spreading Gaussian wave packet in free space. To check the approximate treatment of the constraint, its contribution to the mean energy is determined with the result that it is negligible except for energy values close to the parabolic orbit with eccentricity equal to 1. It is inevitable to introduce a suitable scalar product in $\mathbb{R}^4$ which makes both the transformed Hamiltonian and the velocity operators Hermitian. An elementary necessary criterion is given for the energy interval where the constraint can be approximated by averaging.

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I. INTRODUCTION

The prediction of Schrödinger "that in quite a similar way as for the harmonic oscillator, with certainty, one will construct wave groups which turn around high-quantized Kepler ellipses" [1] took about half a century to be realized. A major achievement is due to Nauenberg [2], who constructed coherent states that have minimal quantum fluctuations in the noncommuting components of the Runge-Lenz vector; he applied the theory, in particular, to a Rydberg state with principal quantum number $n = 40$. Three years before, Gerry [3] presented a different approach which, in the limit $\hbar \to 0$, led to the classical equation of motion. He used coherent states of the harmonic oscillator which minimize the uncertainty product of position and momentum; the original three-dimensional configuration space was connected with the product space of four harmonic oscillators by means of the Kustaanheimo-Stiefel (KS) transformation [4]. The KS map requires one to identify the subspace of $\mathbb{R}^4$ where the transformation is one-to-one. The latter amounts to introducing a projection operator (constraint) which, when strictly implemented, unfortunately, prevents one from working with simple states formed by the product of coherent oscillator states. In [3], the constraint was taken into account in the mean, which allowed expectation values to be determined analytically with the classical equation of motion emerging in the form

$$\mu\langle\ddot{\boldsymbol{r}}\rangle = -\frac{\kappa}{|\langle\boldsymbol{r}\rangle|^3}\langle\boldsymbol{r}\rangle,$$

where μ and κ denote the reduced mass and the coupling constant of the two-body problem, respectively.

As compared to previous investigations [2,3], which pertain to bound classical orbits with negative energy, in the present work we consider positive values of energy which correspond to hyperbolic orbits. As compared to [3], we address the following additional tasks: (i) calculation of the mean-square-root deviations from the classical orbit, i.e., considering quantum corrections to mean position and velocity; (ii) determination of the mean constraint contribution to the energy, H_X; and (iii) modification of the connection between (pseudo) time in oscillator space and real time. In order to achieve analytical results, we keep the wave function simple by taking into account the constraint in the mean only, as was done in [3]. However, we check the consistency of this assumption, point (ii), by examining the expectation value of H_X in comparison with the mean energy of the transformed Hamiltonian H_u. The latter comes out with the reciprocal distance operator, $1/r$, in front of the four-dimensional Laplacian, in addition to the potential energy and the constraint part H_X, which would be zero in the correct subspace of the oscillator space $\mathbb{R}^4$. We infer a suitable volume element in $\mathbb{R}^4$ from the original one in $\mathbb{R}^3$. Within the modified metric, both the transformed Hamiltonian H_u and the velocity operator are Hermitian. Expectation values are calculated in $\mathbb{R}^4$ with the aid of the modified scalar product.

In part of the literature, there appears to be some confusion in transferring pseudotime σ of the KS harmonic oscillator to real time t. Fictitious, or pseudo, time was introduced in the path integral treatment of the hydrogen atom in [5] (see also [6]) by parametrizing time as $t \to \sigma$ along a given path with $\sigma(t) = \int^t dt'/r_{\text{path}}(t')$. Occasionally, this gave rise to the ill-defined transformation rule $dt/d\sigma = r(t)$ which makes t an operator rather than a scalar parameter. Indeed, the function $t \to \sigma$ is different for different paths, in general. In Sec. III, we define the map by comparing the expectation values of commutators of the position operator and two different Hamiltonians. In the limit $\hbar \to 0$, we recover the function $t \to \sigma$, as proposed in [3].

Coherent states for the harmonic oscillator were first proposed in [1]. They have the structure

$$\psi(x,\sigma) = C(\sigma)\exp[a(\sigma)x - \Gamma x^2],$$

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where x and σ denote the space variable and time parameter, respectively. After the KS transformation, the harmonic oscillator Hamiltonian emerges in the form

$$H_{\rm os} = -\frac{\hbar^2}{8\mu}\frac{\partial^2}{\partial u^2} - Eu^2, \quad -E = \frac{\omega^2}{2\mu}.$$

Here, E denotes the orbit energy, and u^2 has the dimension of a length rather than the square of it. As a consequence, the fictitious time σ has dimension s/m. For bound orbits, E is negative and leads to the real oscillator frequency ω. Positive values of the orbit energy, $E > 0$, on the other hand, produce imaginary values of the frequency. The usual construction of coherent states, e.g., by means of creation and annihilation operators [7], will then no longer work. However, as is shown in Sec. III, one succeeds by assuming the localization parameter Γ to be time dependent and complex. With respect to the configuration variables, the new coherent states have the same structure as those for elliptic orbits, and, thus, the efforts to calculate expectation values are about the same.

In the present theory, the mean-square-root deviations (msrd) from the classical orbit vanish in the limit $\hbar \to 0$; i.e., they are of quantum nature. Hyperbolic orbits give rise to position msrd which increase linearly with time in the asymptotic limit, in much the same way as it is found for the spreading of a Gaussian wave packet in free space. One may be tempted to speculate whether in realistic situations quantum fluctuations of position could become comparable with the geometrical dimension of a space vehicle, such as to modify cross sections with respect to radiation pressure or with respect to collision with rest particles in space, for example. We found that such effects are illusionary in macroscopic situations, at least in the energy range where our method should give reliable results. Our method does not cover a small energy interval around a parabolic orbit where the classical orbit energy amounts to $E = 0$ with eccentricity $e = 1$. Here, the effect of the constraint becomes relevant beyond the mean value approximation adopted. There may be a chance to examine the parabolic energy interval, or part of it, by methods which do not rely on the KS constraint, such as the method of [2], which is restricted to negative values of orbit energy, or the construction of coherent states based on irreducible representations of Lie groups [8,9].

This work contributes to verifing that the Schrödinger equation of the hydrogen atom can be extended to the macroscopic Kepler problem which, with respect to a dimensionless quantum parameter of this article, scales by about 80 decimal powers as compared to the ground state of the H atom.

II. KS TRANSFORMATION

The Kustaanheimo-Stiefel (KS) transformation was introduced in [4], in order to remove the collision singularity of the two-body problem of celestial mechanics. It was applied to the quantum mechanics of the H atom, e.g., in [3,10–12]. In what follows, we supplement these applications by deriving a suitable scalar product which ensures hermiticity of the transformed Hamiltonian and the velocity operator. Furthermore, by considering the transformation of the Schrödinger equation, including the wave function, we get an elementary understanding of the origin of the constraint, which, e.g., in [10] and [11] is group theoretically founded: as a superselection rule under the transformation of a one-parametric subgroup of O(4).

By the convention of [3], the KS transformation from the space $\mathbf{u} \in \mathbf{R}^4$ to the original space $\mathbf{x} \in \mathbb{R}^3$ reads

$$x_1 = 2(u_1u_3 - u_2u_4), \quad x_2 = 2(u_1u_4 + u_2u_3), \quad x_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2. \tag{1}$$

The transformation implies the following properties, see [4],

$$r \equiv u_1^2 + u_2^2 + u_3^2 + u_4^2 = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{2}$$

and any rotation $\mathbf{u}' = \mathrm{T}(\Phi)\mathbf{u}$, with $0 \leqslant \Phi < 2\pi$, lets x_1, x_2, and x_3 invariant:

$$\mathrm{T}(\Phi) = \begin{pmatrix} \cos(\Phi), & \sin(\Phi), & 0, & 0 \\ -\sin(\Phi), & \cos(\Phi), & 0, & 0 \\ 0, & 0, & \cos(\Phi), & -\sin(\Phi) \\ 0, & 0, & \sin(\Phi), & \cos(\Phi) \end{pmatrix}. \tag{3}$$

By interpreting Φ as the forth variable in x space, $x_4 = \Phi$, and using the polar coordinate representation $x_1 = r\sin(\theta)\cos(\varphi)$, $x_2 = r\sin(\theta)\sin(\varphi)$, and $x_3 = r\cos(\theta)$, one obtains the 1-1 map $(u_1,u_2,u_3,u_4) \leftrightarrow (r,\theta,\varphi,\Phi)$ with $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi$, and $\Phi < 2\pi$, where

$$u_1 = \sqrt{r}\cos(\theta/2)\cos(\varphi - \Phi), \quad u_2 = \sqrt{r}\cos(\theta/2)\sin(\varphi - \Phi), \quad u_3 = \sqrt{r}\sin(\theta/2)\cos(\Phi), \quad u_4 = \sqrt{r}\sin(\theta/2)\sin(\Phi). \tag{4}$$

The corresponding functional determinant reads

$$du_1du_2du_3du_4 = \frac{\partial(u_1,u_2,u_3,u_4)}{\partial(r,\theta,\varphi,\Phi)}\,drd\theta d\varphi d\Phi = \frac{1}{8}r\sin(\theta)drd\theta d\varphi d\Phi = \frac{1}{8r}dx_1dx_2dx_3d\Phi, \tag{5}$$

which implies the connection between volume elements

$$\frac{1}{2\pi}\int_0^{2\pi} d\Phi \int dx_1dx_2dx_3\, f(x_1,x_2,x_3) \equiv \int dx_1dx_2dx_3 f(x_1,x_2,x_3) = \frac{4}{\pi}\int r(u)\,du_1du_2du_3du_4 f(x_1(u),x_2(u),x_3(u)). \tag{6}$$

This suggests to introduce the metric factor $r(u)$ into the four-dimensional scalar product with the factor $4/\pi$ consumed in the normalization of the wave function. Indeed, the transformed Hamiltonian H_u and the velocity operator, derived below, will be Hermitian in such a metric.

In terms of the components x_1, x_2, x_3, and Φ, the inverse transformation reads

$$u_1 = \frac{x_1\cos(\Phi) + x_2\sin(\Phi)}{\sqrt{2(r - x_3)}}, \quad u_2 = \frac{x_2\cos(\Phi) - x_1\sin(\Phi)}{\sqrt{2(r - x_3)}}, \quad u_3 = \frac{1}{2}\sqrt{2(r - x_3)}\,\cos(\Phi), \quad u_4 = \frac{1}{2}\sqrt{2(r - x_3)}\,\sin(\Phi), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{7}$$

As a check, when (4) is used and inserted on the right-hand side of (1), then the polar representation of x_1, x_2, x_3 is reproduced, independently of Φ. From (7), one infers immediately

$$\frac{\partial u_1}{\partial \Phi} = u_2, \quad \frac{\partial u_2}{\partial \Phi} = -u_1, \quad \frac{\partial u_3}{\partial \Phi} = -u_4, \quad \frac{\partial u_4}{\partial \Phi} = u_3, \tag{8}$$

which, in consistency with (1), implies the properties

$$\frac{\partial}{\partial \Phi} x_k(\mathbf{u}) = 0, \quad k = 1,2,3; \quad \frac{\partial}{\partial \Phi} r(\mathbf{u}) = 0. \tag{9}$$

To transform the Laplacian Δ_x of the one-particle Hamiltonian into u space, we restrict ourselves, at first, to functions which depend via (1) on the u components, briefly called xu space:

$$F(\mathbf{u}) := f(x_1(u), x_2(u), x_3(u)), \quad f \in C^2. \tag{10}$$

Then, the following relation applies:

$$\Delta_u F(\mathbf{u}) \equiv \left[\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}\right] F(\mathbf{u}) = 4r\Delta_x f(\mathbf{x}). \tag{11}$$

Such a property is an immediate consequence of (1) which implies

$$\Delta_u[x_i(\mathbf{u})] = 0, \quad \text{and} \quad \sum_{j=1}^{4} \frac{\partial x_i(\mathbf{u})}{\partial u_j} \frac{\partial x_k(\mathbf{u})}{\partial u_j} = 4r\delta_{ik}, \quad i,k = 1,2,3, \tag{12}$$

where δ is the Kronecker symbol.

We write the stationary Schrödinger equation in configuration space $\mathbf{x} \in \mathbb{R}^3$ as

$$H_x \psi(\mathbf{x}) = E\psi(\mathbf{x}), \quad \text{with} \quad H_x = -\frac{\hbar^2}{2\mu}\Delta_x - \frac{\kappa}{r}. \tag{13}$$

The coupling constant is specified as $\kappa = GmM$ and $\kappa = q_e^2/(4\pi\epsilon_0)$ in the case of two gravitational masses, m and M, and the hydrogen problem, respectively; μ denotes the reduced mass. Transforming (13) into xu space, using (12), we obtain

$$H_{xu}\Psi(\mathbf{x}(\mathbf{u})) = E\Psi(\mathbf{x}(\mathbf{u})), \quad \text{with} \quad H_{xu} = -\frac{\hbar^2}{2\mu}\frac{1}{4r}\Delta_u - \frac{\kappa}{r}. \tag{14}$$

After multiplying (14) with $r \equiv \mathbf{u}^2$, we get

$$H_{\text{os}}\Psi(\mathbf{u}) = \kappa\Psi(\mathbf{u}), \quad H_{\text{os}} = -\frac{\hbar^2}{8\mu}\Delta_u - E\mathbf{u}^2. \tag{15}$$

Thus, one has transformed the original Hamiltonian into the sum of four harmonic oscillators. However, the oscillators are coupled in xu space with $\Psi(\mathbf{u}) = \psi(\mathbf{x}(\mathbf{u}))$, which implies that $\Psi(\mathbf{u})$ does not separate into a product of four independent eigenfunctions of the harmonic oscillator, in general.

In the following, we turn to the standard method of the transformation $H_x \to H_u$ with $\psi(\mathbf{u})$ being unrestricted. To this end a fourth differential δx_4 is introduced in addition to the complete differentials dx_1, dx_2, and dx_3, see, e.g., [12]:

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ \delta x_4 \end{pmatrix} \equiv A \begin{pmatrix} du_1 \\ du_2 \\ du_3 \\ du_4 \end{pmatrix}, \quad A = \begin{pmatrix} 2u_3, & -2u_4, & 2u_1, & -2u_2 \\ 2u_4, & 2u_3, & 2u_2, & 2u_1 \\ 2u_1, & 2u_2, & -2u_3, & -2u_4 \\ u_2, & -u_1, & -u_4, & u_3 \end{pmatrix}. \tag{16}$$

The mutual orthogonality of the row vectors of the matrix A leads to the following transformed Laplacian and Hamiltonian H_u [the prefactor of X in [3] is corrected here from $1/(4r)$ to $1/(4r^2)$]:

$$\Delta_x \to \frac{1}{4r}\Delta_u - \frac{1}{4r^2}X^2; \quad X = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}; \tag{17}$$

$$H_u = -\frac{\hbar^2}{2\mu}\left\{\frac{1}{4r}\Delta_u - \frac{1}{4r^2}X^2\right\} - \frac{\kappa}{r}; \quad r = u_1^2 + u_2^2 + u_3^2 + u_4^2. \tag{18}$$

The KS transformation (1) implies for all functions $f \in C^1$ the following property of the constraint operator X:

$$Xf(x_1(\mathbf{u}), x_2(\mathbf{u}), \quad x_3(\mathbf{u})) \equiv 0. \tag{19}$$

This follows from

$$X\, x_i(\mathbf{u}) \equiv 0, \quad i = 1,2,3, \tag{20}$$

which is an immediate consequence of transformation (1). Property (19) represents the constraint operator in a form which is alternative to representations used in [3,10–12].

In order to guarantee hermiticity of H_{os} and H_u and related observables, respectively, we adopt the two scalar products

$$1 = \langle\psi|\psi\rangle_{\text{os}} \equiv C_{\text{os}}^2 \int du_1du_2du_3du_4\psi_0^*\psi_0, \quad 1 = \langle\psi|\psi\rangle_u \equiv C_u^2 \int du_1du_2du_3du_4 r(u)\psi_0^*\psi_0. \tag{21}$$

III. COHERENT STATES FOR IMAGINARY OSCILLATOR FREQUENCY

We are looking for coherent states of the harmonic oscillator in the case of imaginary values of the oscillator frequency which correspond to positive values of the orbit energy E. To this end, we extend the usual coherent states with real frequency, see, e.g., [7], by assuming the localization parameter Γ to be time dependent and complex: $\Gamma(\sigma) = \Gamma_R(\sigma) + i\,\Gamma_I(\sigma)$. To solve the Schrödinger equation,

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial\sigma} = H_{\text{os}}\psi, \quad H_{\text{os}} = -\frac{\hbar^2}{8\mu}\frac{\partial^2}{\partial u^2} - E\,u^2, \quad E > 0, \tag{22}$$

we make the following ansatz:

$$\psi(u,\sigma) = C_0\exp[\beta(\sigma) + a(\sigma)u - \Gamma(\sigma)u^2/2], \quad u \in \mathbb{R}, \quad \sigma \geqslant 0. \tag{23}$$

Equation (22) will be fulfilled if the parameters obey the following differential equations:

$$\Gamma'(\sigma) = -2\frac{i}{\hbar}\left[E + \frac{\hbar^2}{8\mu}\Gamma^2\right]; \tag{24}$$

$$a'(\sigma) = -2\frac{i}{\hbar}\frac{\hbar^2}{8\mu}\Gamma(\sigma)a(\sigma); \tag{25}$$

$$\beta'(\sigma) = -\frac{i}{\hbar}\frac{\hbar^2}{8\mu}[-a^2(\sigma) + \Gamma(\sigma)]. \tag{26}$$

We introduce the two constants

$$\Omega^2 = \frac{E}{2\mu} > 0, \quad \Gamma_\Omega = \frac{4\mu\Omega}{\hbar} > 0, \tag{27}$$

and the scaled functions α and γ,

$$\alpha(\sigma) = a(\sigma)/\sqrt{\Gamma_\Omega}, \quad \gamma(\sigma) = \Gamma(\sigma)/\Gamma_\Omega, \tag{28}$$

to write

$$\gamma'(\sigma) = -i\Omega[1+\gamma^2(\sigma)]; \quad \alpha'(\sigma) = -i\Omega\gamma(\sigma)\alpha(\sigma);$$
$$\beta'(\sigma) = i\frac{\Omega}{2}[\alpha^2(\sigma) - \gamma(\sigma)]. \tag{29}$$

Integration is straightforward going consecutively from the first equation in (29) to the third one. With the abbreviations $\alpha_0 = \alpha(0)$, $\gamma_0 = \gamma(0)$, and $\Upsilon = 2\,\Omega\,\sigma$, the solutions read

$$\gamma(\sigma) = -i\tanh(\Upsilon/2 + iC_1), \quad \Upsilon = 2\,\Omega\sigma; \tag{30}$$

$$\alpha(\sigma) = C_2\frac{1}{\cosh(\Upsilon/2 + iC_1)}; \tag{31}$$

$$\beta(\sigma) = C_3 - \frac{1}{2}\ln[\cosh(\Upsilon/2 + iC_1)] + \frac{i}{2}C_2^2\tanh(\Upsilon/2 + iC_1); \tag{32}$$

$$C_1 = \arctan(\gamma_0), \quad C_2 = \frac{\alpha_0}{\sqrt{1+\gamma_0^2}}, \quad C_3 = -\frac{1}{2\gamma_0}\frac{\alpha_0^2}{1+\gamma_0^2}. \tag{33}$$

The (time-independent) normalization constant C_0 turns out as

$$C_0^2 = \sqrt{\frac{\gamma_0}{\pi\Gamma_\Omega\left(1+\gamma_0^2\right)}}. \tag{34}$$

We partly work out the real and imaginary parts and replace α by a:

$$\gamma(\sigma) = \frac{1}{\cosh^2(\Upsilon/2) + \gamma_0^2\,\sinh^2(\Upsilon/2)} \times \left[\gamma_0 - \frac{i}{2}\left(1+\gamma_0^2\right)\,\sinh(2\Upsilon)\right]; \tag{35}$$

$$a(\sigma) = \frac{a(0)}{\cosh^2(\Upsilon/2) + \gamma_0^2\sinh^2(\Upsilon/2)} \times [\cosh(\Upsilon/2) - i\gamma_0\sinh(\Upsilon/2)]\,; \tag{36}$$

$$\exp[\beta(\sigma)] = \frac{1}{\sqrt{\cosh(\Upsilon/2 + iC_1)}} \times \exp\left[C_3 - \frac{\alpha_0^2}{2\left(1+\gamma_0^2\right)}\gamma(\sigma)\right]. \tag{37}$$

If the initial parameter $\gamma_0 > 0$, then the real part of $\gamma(\sigma)$, obviously, is larger than zero for all time values σ, and, thus, the state (23) is normalizable. Clearly, if ψ obeys the time-dependent Schrödinger equation (22), then its norm remains constant for all time values:

$$\frac{d}{d\sigma}\langle\psi|\psi\rangle_{\text{os}} = \langle\dot\psi|\psi\rangle_{\text{os}} + \langle\psi|\dot\psi\rangle_{\text{os}} = \frac{i}{\hbar}[\langle H_{\text{os}}\psi|\psi\rangle_{\text{os}} - \langle\psi|H_{\text{os}}\psi\rangle_{\text{os}}] = 0. \tag{38}$$

An explicit normalization check of the solutions (30) to (34) is sketched in Appendix A.

We extend the coherent state (23) to four dimensions

$$\psi = C\psi_0, \quad \psi_0 = \exp[\mathbf{a}\cdot\mathbf{u} - \Gamma u^2/2], \quad \mathbf{u} \in \mathbb{R}^4, \tag{39}$$

where

$$\mathbf{a}(\sigma) = \frac{\mathbf{a}(0)}{\sqrt{1+\gamma_0^2}}\,\frac{1}{\cosh(\Upsilon/2 + iC_1)}, \quad \Gamma(\sigma) = \Gamma_\Omega\,\gamma(\sigma); \tag{40}$$

$$C(\sigma) = \frac{(\Gamma_R)^2}{\pi^2}\,\exp\left[-\frac{A^2}{4\Gamma_R}\right], \quad A^2 = \sum_{k=1}^{4}(a_k + a_k^*)^2. \tag{41}$$

Further below, we set the disposable localization parameter $\gamma_0 = 1$, for simplicity.

It should be noticed that, also for negative values of the energy E, coherent states of type (23) can be found which have a time-dependent complex localization parameter. However, the corresponding uncertainty product is minimal only for a constant value $\Gamma = \Gamma_R$:

$$\Delta u\Delta p_u = \frac{\hbar}{2}\sqrt{\left(1+\Gamma_I^2/\Gamma_R^2\right)}, \quad E < 0. \tag{42}$$

This product is minimal, if $\Gamma_I = 0$, which is possible for all time values only if $\Gamma_R = \Gamma = 4\mu\,|\Omega|/\hbar$.

The coherent states for elliptic orbits with negative E, which corresponds to (39), can be readily written down by means of the formula for the one-dimensional harmonic oscillator (see, e.g., [7]):

$$\psi(\mathbf{u},\sigma) = \frac{\Gamma}{\pi}\exp[-i\omega\sigma/2 - a^2(\sigma)/(4\Gamma)] \times \exp[\mathbf{a}(\sigma)\cdot\mathbf{u} - \Gamma u^2/2]; \tag{43}$$
$$\mathbf{a}(\sigma) = \mathbf{a}(0)\,\exp[-i\omega\sigma];$$
$$\Gamma = 4\mu\omega/\hbar; \quad \omega^2 = (-E)/(2\mu); \quad E < 0.$$

Wave function (43) is equivalent to the second quantization form given in [3].

IV. PSEUDOTIME AND REAL TIME

The time parameter in the Schrödinger equation (23), σ, and the pseudofrequency Ω have dimensions s/m and m/s, respectively. Real time t, on the other hand, is connected with the Hamiltonian H_u, the transformed version of the Hamiltonian of the Kepler problem. To transfer pseudotime to real time, different schemes can be found in the literature. In [3], one finds the recipe $t = \langle r\rangle\,\sigma$, which gives the desired classical mean values, but somehow comes out as deus ex machina. In the following, we choose the velocity operator as observable.

We define time propagation of the position operator by means of the commutator with the Hamiltonian. Thus, pseudotime σ and real time t are defined through

$$w_i \equiv \frac{dx_i}{d\sigma} = \frac{i}{\hbar}[H_{\rm os},x_i(\mathbf{u})],$$
$$v_i \equiv \frac{dx_i}{dt} = \frac{i}{\hbar}[H_u,x_i(\mathbf{u})], \quad i = 1,2,3. \tag{44}$$

Evaluating the commutators, one obtains the velocity operators in the form

$$w_1 = -i\frac{\hbar}{2\mu}\left[u_3\frac{\partial}{\partial u_1} - u_4\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_3} - u_2\frac{\partial}{\partial u_4}\right];$$
$$w_2 = -i\frac{\hbar}{2\mu}\left[u_4\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_3} + u_1\frac{\partial}{\partial u_4}\right]; \tag{45}$$
$$w_3 = -i\frac{\hbar}{2\mu}\left[u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_4}\right];$$

$$v_i = \frac{1}{r}w_i, \quad r = u_1^2 + u_2^2 + u_3^2 + u_4^2, \tag{46}$$

where, for the latter equation, we applied (20) which implies that the commutators $[x_i,X]$ vanish. As is observed, the velocity operator $\mathbf{v}$ does not depend on the constraint operator X.

With the aid of (21), we easily find the normalization constants of the coherent state (39) for the two metrics

$$C_{\rm os}^2 = \frac{\Gamma_R^2}{\pi^2}\exp[-A^2/(4\Gamma_R)] \quad \text{and}$$
$$C_u^2 = \frac{4\Gamma_R^4}{\pi^2}\frac{\exp[-A^2/(4\Gamma_R)]}{A^2 + 8\Gamma_R}. \tag{47}$$

Clearly, the velocity operators w_i and v_i are Hermitian in the corresponding metrics. The expectation values $\langle w_i\rangle_{\rm os}$ and $\langle v_i\rangle_u$ amount to the same integrals which differ only by the normalization constants

$$\langle v_i\rangle_u = \frac{1}{S}\langle w_i\rangle_{\rm os}; \quad S = \left(\frac{C_{\rm os}}{C_u}\right)^2 = \frac{A^2 + 8\Gamma_R}{4\Gamma_R^2}. \tag{48}$$

Through A^2 as defined in (41) and (40), and through $\Gamma_R(\sigma) \equiv \Gamma_\Omega\,\gamma_R(\sigma)$, the scalar function S becomes σ dependent and has the dimension of a length. Thus, we obtain the following connection between σ and t:

$$dt = S(\sigma)\,d\sigma. \tag{49}$$

V. IMPLEMENTING INITIAL CONDITIONS

We have in mind an orbit in the $[x_1,x_2]$ plane with $\langle x_3\rangle = 0$. The eight real parameters μ_k, ν_k, defined through $a_k(0) = \mu_k + i\,\nu_k$, $k = 1,\dots 4$, will be determined by identifying, at zero time, the mean values of position and velocity with the initial point $\mathbf{r}_0 = \{r_0,0,0\}$, $\mathbf{v}_0 = \{0,v_0,0\}$. Anticipating hyperbolic orbits, the above point is a vertex and implies that our coordinate axes coincide with the principal ones of a hyperbola branch. The mean values are practically sharp in the case of a macroscopic two-body system, see Sec. VIII. As an additional condition, we require, as was done in [3], that the mean value of the constraint operator X vanishes at time $\sigma = t = 0$. Altogether, this poses seven conditions for the eight real parameters. As a matter of fact, the complex parameters of the state (39) will be fixed up to a KS phase Φ defined in (3). As it turns out, the mean orbit will stay in the $[x_1,x_2]$ plane for all time values, and also, if $\langle X\rangle = 0$ at zero time, then this will transfer to all time values. Furthermore, as is proved in Appendix B, Sec. III, the mean angular momentum is constant in time. A dimensionless, time-dependent, function $g = 4\Gamma_R/A^2$ will emerge, see (56) below, which characterizes the quantum corrections to the classical orbit.

The calculation of mean values is based on parameter differentiation and integration. Factors u_i, r, and $1/r$ are replaced in the integrand as

$$u_i \to \frac{\partial}{\partial a_i}G(A^2,\Gamma_R), \quad r \to \left(-\frac{\partial}{\partial\Gamma_R}G(A^2,\Gamma_R)\right),$$
$$\frac{1}{r} \to \int_{\Gamma_R}^{\infty} ds\,G(A^2,s), \tag{50}$$

where

$$G(A^2,\Gamma_R) = \int du_1du_2du_3du_4\psi_0^*\psi_0 \equiv 1/C_{\rm os}^2. \tag{51}$$

For example,

$$\langle x_1\rangle_u = 2C_u^2\int du_1du_2du_3du_4r(u)(u_2u_3 - u_2u_4)\psi_0^*\psi_0$$
$$= 2C_u^2\left(-\frac{\partial}{\partial\Gamma_R}\right)\left(\frac{\partial}{\partial a_1}\frac{\partial}{\partial a_3} - \frac{\partial}{\partial a_2}\frac{\partial}{\partial a_4}\right)G(A^2,\Gamma_R). \tag{52}$$

We used MATHEMATICA [13] to support the partially involved algebraic manipulations. In Appendix B we give more details.

We find the following for the mean values of position and velocity:

$$\langle x_1\rangle_u = \xi[(a_1 + a_1^*)(a_3 + a_3^*) - (a_2 + a_2^*)(a_4 + a_4^*)],$$
$$\langle x_2\rangle_u = \xi[(a_1 + a_1^*)(a_4 + a_4^*) + (a_2 + a_2^*)(a_3 + a_3^*)], \tag{53}$$
$$\langle x_3\rangle_u = \tfrac{1}{2}\xi[(a_1 + a_1^*)^2 + (a_2 + a_2^*)^2 - (a_3 + a_3^*)^2 - (a_4 + a_4^*)^2],$$

$$\langle v_1\rangle_u = -i\eta\{[a_1a_3 - a_1^*a_3^* - a_2a_4 + a_2^*a_4^*]\Gamma_R - i[(a_1 + a_1^*)(a_3 + a_3^*) - (a_2 + a_2^*)(a_4 + a_4^*)]\Gamma_I\},$$
$$\langle v_2\rangle_u = -i\eta\{[a_2a_3 - a_2^*a_3^* + a_1a_4 - a_1^*a_4^*]\Gamma_R - i[(a_2 + a_2^*)(a_3 + a_3^*) + (a_1 + a_1^*)(a_4 + a_4^*)]\Gamma_I\},$$
$$\langle v_3\rangle_u = -i\frac{1}{2}\eta\{[a_1^2 - (a_1^*)^2 + a_2^2 - (a_2^*)^2 - a_3^2 + (a_3^*)^2 - a_4^2 + (a_4^*)^2]\Gamma_R - i[(a_1 + a_1^*)^2 + (a_2 + a_2^*)^2 - (a_3 + a_3^*)^2 - (a_4 + a_4^*)^2]\Gamma_I\}, \tag{54}$$

$$\langle X\rangle_{\rm os} = \frac{1}{2\Gamma_R}[a_1a_2^* - a_1^*a_2 + a_3^*a_4 - a_3a_4^*], \tag{55}$$

where

$$\xi = \frac{1}{2\Gamma_R^2}\frac{1 + 4g}{1 + 2g}, \quad \eta = \frac{\hbar}{\mu A^2(1 + 2g)}, \quad g = \frac{4\Gamma_R}{A^2}. \tag{56}$$

The dimensionless function g characterizes the quantum fluctuations of the classical orbit. This is seen from the probability density of the coherent state (39) which is localized at $\mathbf{u_m} = (\mathbf{a}+\mathbf{a}^*)/(2\Gamma_R)$ with a mean spread Δu of the order of $\sqrt{1/\Gamma_R}$. Thus, the distance operator $r=\mathbf{u}^2$ is localized at $r_m = A^2/(4\Gamma_R^2)$ with a spread of the order $1/\Gamma_R$, which tells us that $g = 1/(r_m\Gamma_R)$ is the mean spread in units of r_m. It follows from (28) and (35) with $\gamma_0 = 1$ that $1/\Gamma_R = \cosh(\Upsilon)/\Gamma_\Omega$, which, by definition (27), is proportional to $\hbar$ and vanishes in the classical limit $\hbar \to 0$.

At time $\sigma = 0$, the imaginary and real parts of Γ amount to $\Gamma_I(0) = 0$ and $\Gamma_R(0) = \Gamma_\Omega$, respectively. The initial parameters μ_i and ν_i, with $a_i(0) = \mu_i + i\nu_i$, are conveniently written in terms of plane polar coordinates as follows:

$$\begin{aligned} \mu_1 &= \rho_{12}\cos(\varphi_{12}), \quad \mu_2 = \rho_{12}\sin(\varphi_{12}),\\ \mu_3 &= \rho_{34}\cos(\varphi_{34}), \quad \mu_4 = \rho_{34}\sin(\varphi_{34}),\\ \nu_1 &= R_{12}\cos(\phi_{12}), \quad \nu_2 = R_{12}\sin(\phi_{12}),\\ \nu_3 &= R_{34}\cos(\phi_{34}), \quad \nu_4 = R_{34}\sin(\phi_{34}). \end{aligned} \tag{57}$$

Inserting (57) into (53) to (55), we obtain at time $\sigma = t = 0$

$$\begin{aligned} r_0 &= \langle x_1(0)\rangle_u = 4\xi_0\rho_{12}\rho_{34}\cos(\varphi_{12}+\varphi_{34}),\\ 0 &= \langle x_2(0)\rangle_u = 4\xi_0\rho_{12}\rho_{34}\sin(\varphi_{12}+\varphi_{34}),\\ 0 &= \langle x_3(0)\rangle_u = 4\xi_0\big(\rho_{12}^2-\rho_{34}^2\big)/2,\\ 0 &= \langle v_1(0)\rangle_u = 2\eta_0[\rho_{12}R_{34}\cos(\varphi_{12}\\ &\quad +\phi_{34}) + \rho_{34}R_{12}\cos(\varphi_{34}+\phi_{12})],\\ v_0 &= \langle v_2(0)\rangle_u = 2\eta_0[\rho_{12}R_{34}\sin(\varphi_{12}+\phi_{34})\\ &\quad + \rho_{34}R_{12}\sin(\varphi_{34}+\phi_{12})],\\ 0 &= \langle v_3(0)\rangle_u = 2\eta_0[\rho_{12}R_{12}\cos(\varphi_{12}-\phi_{12})\\ &\quad - \rho_{34}R_{34}\cos(\varphi_{34}-\phi_{34})],\\ 0 &= \langle X(0)\rangle_{\text{os}} = -\frac{i}{\Gamma}[R_{12}\rho_{12}\sin(\phi_{12}-\varphi_{12})\\ &\quad - R_{34}\rho_{34}\sin(\phi_{34}-\varphi_{34})], \end{aligned} \tag{58}$$

where $\xi_0 = \xi(\sigma = 0)$ and $\eta_0 = \eta(\sigma = 0)$.

Since $\langle x(0)_3\rangle_u = 0$, we infer from (58) that

$$\rho_{12} = \rho_{34} \equiv \rho_0. \tag{59}$$

Furthermore, $\langle x(0)_2\rangle_u = 0$ and $r_0 \equiv \langle x(0)_1\rangle_u > 0$ imply $\varphi_{34} = -\varphi_{12}$. And $\langle v(0)_1\rangle = \langle v(0)_3\rangle = \langle X(0)\rangle = 0$, together with the assumption $v_0 > 0$, gives rise to the unique conditions

$$\begin{aligned} R_{34} &= R_{12} \equiv \nu\,\rho_0, \quad \phi_{34} = -\varphi_{12} + \pi/2,\\ \phi_{12} &= \varphi_{12} + \pi/2, \quad \varphi_{34} = -\varphi_{12}, \end{aligned} \tag{60}$$

where, instead of R_{12}, we introduce the dimensionless parameter $\nu > 0$, which will turn out to be related to the eccentricity e. Therewith, the right-hand side expressions of (58) imply the relations

$$\begin{aligned} \langle \mathbf{r(0)}\rangle_u &= (r_0,0,0), \quad r_0 = 4\xi_0\rho_0^2,\\ \langle \mathbf{v(0)}\rangle_u &\equiv (0,v_0,0), \quad v_0 = 4\eta_0\nu\rho_0^2. \end{aligned} \tag{61}$$

From the mean initial condition we infer, in particular,

$$\rho_0^2 = \frac{r_0\gamma_0^2\Gamma_\Omega^2}{2}\frac{1+2g_0}{1+4g_0}, \quad g_0 = g(\sigma = 0). \tag{62}$$

With the notation $\Phi = \varphi_{12}$, the parameters μ_i and ν_i, as given in (57), attain the special form

$$\begin{aligned} &\mu_1 = \rho_0\cos(\Phi), \quad \mu_2 = \rho_0\sin(\Phi), \quad \mu_3 = \rho_0\cos(\Phi),\\ &\mu_4 = -\rho_0\sin(\Phi); \quad \nu_1 = -\nu\rho_0\sin(\Phi),\\ &\nu_2 = \nu\rho_0\cos(\Phi), \quad \nu_3 = \nu\rho_0\sin(\Phi), \quad \nu_4 = \nu\rho_0\cos(\Phi). \end{aligned} \tag{63}$$

We combine (63), $a_k(0) = \mu_k + i\nu_k$, and (36) for the time-dependent coefficients $a_k(\sigma)$ in order to write

$$\begin{aligned} a_1(\sigma) &= \frac{\rho_0[\cos(\Phi) - i\nu\sin(\Phi)]}{\cosh(\Upsilon/2) + i\gamma_0\sinh(\Upsilon/2)},\\ a_2(\sigma) &= \frac{\rho_0[\sin(\Phi) + i\nu\cos(\Phi)]}{\cosh(\Upsilon/2) + i\gamma_0\sinh(\Upsilon/2)},\\ a_3(\sigma) &= \frac{\rho_0[\cos(\Phi) + i\nu\sin(\Phi)]}{\cosh(\Upsilon/2) + i\gamma_0\sinh(\Upsilon/2)},\\ a_4(\sigma) &= \frac{\rho_0[-\sin(\Phi) + i\nu\cos(\Phi)]}{\cosh(\Upsilon/2) + i\gamma_0\sinh(\Upsilon/2)}. \end{aligned} \tag{64}$$

The open phase Φ is related to the rotation invariance (3) of the KS transformation. This can be seen by considering the relevant exponent of the wave function (39), $\mathbf{a}(\Phi)\cdot\mathbf{u}$. Inserting (64) for $\mathbf{a}$ and applying the KS rotation $\mathbf{u} = \mathrm{T}(-\Phi)\mathbf{u}'$ one observes that

$$\mathbf{a}(\Phi)\cdot\mathbf{u} = \mathbf{a}(\Phi)\cdot[\mathrm{T}(-\Phi)\mathbf{u}'] = \mathbf{a}(0)\cdot\mathbf{u}'. \tag{65}$$

As a consequence, the mean values of operators consisting of $\mathbf{x(u)}$ or $\mathbf{v(u)}$ and taken with the state (39) do not depend on the KS phase.

When the time-dependent parameters $a_k(\sigma)$, see (64), are assigned to the mean values (53) to (55), it turns out that the orbit is confined to the (x_1,x_2) plane for all time values with

$$\langle x_3(\sigma)\rangle = \langle v_3(\sigma)\rangle = 0 \quad \text{for} \quad \sigma \geqslant 0, \tag{66}$$

and also the mean constraint operator reads $\langle X(\sigma)\rangle_{\text{os}} = 0$ for $\sigma \geqslant 0$. Setting now the arbitrary parameter $\gamma_0 = 1$, and with the aid of the abbreviations

$$\begin{aligned} \xi_1 &\equiv \xi_1(\sigma) = \frac{(1+4\,g(\sigma))(1+2\,g_0)}{(1+2\,g(\sigma))(1+4\,g_0)},\\ \eta_1 &\equiv \eta_1(\sigma) = \frac{\hbar}{\mu[1+2\,g(\sigma)]}, \quad \gamma_0 = 1, \end{aligned} \tag{67}$$

we obtain after straightforward simplifications

$$\begin{aligned} x(\sigma) &\equiv \langle x_1(\sigma)\rangle_u = \frac{\xi_1}{2}r_0[1+\nu^2+(1-\nu^2)\cosh(\Upsilon)],\\ x(0) &= r_0; \quad y(\sigma) \equiv \langle x_2(\sigma)\rangle_u = \xi_1 r_0\nu\sinh(\Upsilon),\\ y(0) &= 0; \quad z(\sigma) \equiv \langle x_3(\sigma)\rangle_u = 0; \end{aligned} \tag{68}$$

$$\begin{aligned} v_x(\sigma) &\equiv \langle v_1(\sigma)\rangle_u = \frac{\Gamma_\Omega}{2}\eta_1\frac{(1-\nu^2)\sinh(\Upsilon)}{1-\nu^2+(1+\nu^2)\cosh(\Upsilon)},\\ v_x(0) &= 0; \quad v_y(\sigma) \equiv \langle v_2(\sigma)\rangle_u\\ &= \Gamma_\Omega\eta_1\frac{\nu\cosh(\Upsilon)}{1-\nu^2+(1+\nu^2)\cosh(\Upsilon)}, \quad v_y(0) = v_0;\\ v_z(\sigma) &\equiv \langle v_3(\sigma)\rangle_u = 0. \end{aligned} \tag{69}$$

We list further mean values needed below:

$$S^{-1} = \left\langle \frac{1}{r} \right\rangle_u = r_0 Z(\Upsilon)(1+2\,g)\frac{1+2g_0}{1+4g_0}, \tag{70}$$

$$Z(\Upsilon) = \tfrac{1}{2}[1-\nu^2+(1+\nu^2)\cosh(\Upsilon)], \tag{71}$$

$$\langle r\rangle_u = S\,\frac{1+6g+6g^2}{(1+2g)^2}, \tag{72}$$

$$g = \frac{4\Gamma_R}{A^2} = g_0\frac{\cosh(\Upsilon)}{Z(\Upsilon)}, \quad g_0 = \frac{1}{r_0\Gamma_\Omega}\frac{1+4g_0}{1+2g_0}. \tag{73}$$

Obviously,

$$S = \langle r\rangle_u[1+O(g)] \quad \text{and} \quad g_0 = \frac{\theta}{r_0\Gamma_\Omega}, \quad 1 \leqslant \theta \leqslant 2, \tag{74}$$

which in the limit of vanishing quantum corrections, $g_0 \to 0$, supports the recipe $dt = \langle r\rangle\, d\sigma$ as was proposed in [3].

VI. ELEMENTS OF HYPERBOLIC ORBIT

In this section, we anticipate a hyperbolic orbit and work out its elements, including quantum corrections. Moreover, a kind of Kepler equation is derived from the connection between pseudotime and real time. From the explicit mean values of the velocity components, in Sec. VI D, the hodograph is established and related to recent discussions [14] concerning one of Feynman's "Lost Lectures." Strictly, the orbit is hyperbolic only in the macroscopic limit $\hbar \to 0$. With the quantum parameter $g \neq 0$, we actually describe the orbit by a family of osculating hyperbolas.

A. Semimajor axis and eccentricity

We show that the mean coordinates x and y, defined by (67) and (68), lie on a hyperbola with axes a and b. This is true even with quantum corrections and also applies for parameters $\gamma_0 \neq 1$ which is not shown. However, the axes depend on the quantum correction g, which means, they are time dependent, in principle. We compare the results (68) with the parametric representation of a hyperbolic orbit as given on p. 43 of [15], with notation $\xi \to \Upsilon$,

$$\begin{aligned} x &= a[e-\cosh(\Upsilon)],\\ y &= a\sqrt{e^2-1}\,\sinh(\Upsilon), \quad \Upsilon \in \mathbb{R}, \end{aligned} \tag{75}$$

and readily connect our parameters ν and r_0 with the eccentricity $e > 1$ and the semimajor axis a of the hyperbola [the origin of the coordinates (x,y) is the center of the heavy mass, the focus of the given hyperbola branch]. From the above x coordinate, one finds

$$\nu^2 = \frac{e+1}{e-1} \quad \text{and} \quad r_0 = a(e-1)/\xi_1, \quad e > 1. \tag{76}$$

When this is inserted into the expression of y in (68), one verifies the second equation of (75). Clearly, (75) implies the canonical equation for a hyperbola,

$$\begin{aligned} \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1, \quad \text{with} \quad X = x - e\,a, \; Y = y,\\ b^2 = a^2(e^2-1), \end{aligned} \tag{77}$$

which also contains the right-hand side branch:

$$\begin{aligned} x &= a[e+\cosh(\Upsilon)],\\ y &= a\sqrt{e^2-1}\,\sinh(\Upsilon), \quad \Upsilon \in \mathbb{R}. \end{aligned} \tag{78}$$

With respect to the Cartesian system (X,Y), the vertices of the two branches lie at $\pm a$, and the two focal points at $F_{1,2} = \mp e\,a$.

By the definition in (60), the parameter ν is constant in time and, thus, is the eccentricity e. The axes a and b, on the other hand, depend on the function $\xi_1(\sigma)$ defined in (67). Since g and $1/\xi_1$ monotonically decrease with increasing σ and g, respectively, the semimajor axis is monotonically decreasing from $a(0) = r_0/(e-1)$ to an asymptotic value as follows:

$$a(0) \geqslant \lim_{\sigma\to\infty} a(\sigma) = a(0)\frac{1+2g_\infty}{1+4g_\infty}\,\frac{1+4\,g_0}{1+2\,g_0} \geqslant a(0)\,\frac{1+2g_0}{1+4g_0}, \tag{79}$$

$$g_\infty = g_0\frac{e-1}{e}.$$

If the quantum parameter g_0 is small, then the axis a, practically, is constant; quantum corrections are quantitatively discussed in Sec. VIII.

B. Choosing the energy parameter E

The parameter E is an energy eigenvalue of the stationary Schrödinger equation (13). We choose it in such a way that $\mu/2(v_x^2+v_y^2) - \kappa\langle 1/r\rangle_u$ is constant in time, up to quantum corrections of order g. We remark that $v_x^2 + v_y^2 = \langle v_1^2 + v_2^2\rangle_u[1+O(g_0)]$, see (B16) and (B18). Using (67), (69), (70), (71), (76), and (27), one obtains

$$\begin{aligned} E_{\rm kin} &= \frac{1}{2}\mu\left(v_x^2+v_y^2\right) = \Gamma_\Omega^2\eta_1^2\frac{1+e\cosh(\Upsilon)}{8[e\cosh(\Upsilon)-1]}\\ &= E\frac{1+e\cosh(\Upsilon)}{e\cosh(\Upsilon)-1}\frac{1}{(1+2g)^2}; \quad E_{\rm pot} = -\kappa\langle 1/r\rangle_u\\ &= -\kappa\frac{e-1}{r_0[e\cosh(\Upsilon)-1]}\;\frac{1+4g_0}{(1+2g)(1+2g_0)}. \end{aligned} \tag{80}$$

We set

$$E = (e-1)\frac{\kappa}{2r_0} \tag{81}$$

and obtain

$$\begin{aligned} &E_{\rm kin} + E_{\rm pot} = E + \Delta E,\\ \Delta E &= -4E\frac{g_0(1+2g)+g(1+2g_0)[-g+e(1+g)\cosh(\Upsilon)]}{(1+2g)^2(1+2g_0)[e\cosh(\Upsilon)-1]}. \end{aligned} \tag{82}$$

It is seen that $\Delta E/E$ is of the order of g, and the setting (81) for E does not depend on time, as required. For negligible g, one obtains $E = \kappa/(2a)$, which is a well-known formula in the Kepler problem, see p. 87 of [16].

The contribution of the constraint operator in H_u, see (18), is calculated in Appendix C with the result

$$\begin{aligned} \langle H_X\rangle_u &\equiv \frac{\hbar^2}{8\mu}\left\langle \frac{1}{r^2}X^2\right\rangle_u = -g_0 E\frac{e(e-1)}{[e\cosh(\Upsilon)-1]^2}K(g),\\ K(g) &= \frac{1-g+g\exp[-1/g]}{1+2g}. \end{aligned} \tag{83}$$

This tells that the relative correction of the constraint to the energy, H_X/E, is of order g_0 for small g. The function $K(g)$, $g \geqslant 0$, is monotonically decreasing, when g increases with $K(g) \leqslant K(0) = 1$ and the asymptotic behavior amounts to $K(g) \overset{g\to\infty}{\longrightarrow} 1/(4g^2)$.

One can show, with some effort, that the mean-square deviation of the energy is of order g: $(\Delta H_u)^2/E^2 = O(g)$.

We conclude this subsection by deriving from the previous results a further standard-type relation between eccentricity, angular momentum $L = \mu\, r_0\, v_0$, and energy parameter E,

$$e = \sqrt{1 + \frac{2EL^2}{\mu\kappa^2}(1+2g_0)^2}, \tag{84}$$

which for $g_0 \to 0$ corresponds to the result on p. 41 of [15]. To prove this, we use (27), (67), (69), and (76) and express the initial speed $v_0 = v_y(\sigma = 0)$ as follows:

$$v_0 = \Gamma_\Omega \eta_1(0)\nu/2 = 2\Omega\nu/(1+2g_0). \tag{85}$$

After squaring, introducing $L = \mu r_0 v_0$, and setting $r_0 = (e-1)\kappa/(2E)$ from (81), we obtain

$$\begin{aligned} v_0^2 &\equiv \frac{L^2}{\mu^2 r_0^2} = \frac{4L^2E^2}{\mu^2\kappa^2(e-1)^2} \overset{!}{=} 4\Omega^2\nu^2\frac{1}{(1+2g_0)^2} \\ &= 2\frac{E}{\mu}\frac{e+1}{e-1}\frac{1}{(1+2g_0)^2}, \end{aligned} \tag{86}$$

which amounts to the following relation which is equivalent to (84):

$$e^2 - 1 = \frac{2EL^2}{\mu\kappa^2}(1+2g_0)^2. \tag{87}$$

C. Kepler's equation in the hyperbolic case

According to (48) and (49), the connection between pseudotime σ and real time t is mediated by the function S. With the aid of (70), (71), (73), and (76), we can write

$$\begin{aligned} dt &= S(\sigma)d\sigma \\ &= r_0\frac{1+2g_0}{1+4g_0}\left[\frac{e\cosh(\Upsilon)-1}{e-1} + 2g_0\cosh(\Upsilon)\right]d\sigma. \end{aligned} \tag{88}$$

Integration, with $t = \sigma = 0$, leads to

$$t = \frac{r_0}{2(e-1)\Omega}\{-\Upsilon + [e + 2g_0(e-1)]\sinh(\Upsilon)\}\frac{1+2g_0}{1+4g_0}. \tag{89}$$

In the limit $g_0 \to 0$, we get

$$\frac{2\Omega}{a}t = [-\Upsilon + e\sinh(\Upsilon)], \quad \Upsilon = 2\Omega\sigma, \tag{90}$$

which is consistent with [15] (see p. 43). Equation (90) is analogous to Kepler's equation in the case of elliptic orbits,

$$M = \Upsilon - e\sin(\Upsilon), \quad e < 1, \tag{91}$$

where M and Υ denote the mean and eccentric anomaly, respectively. The inversion of (90) for $\Upsilon = \Upsilon(t)$ encounters similar difficulties as the elliptic case, because of the "heterogeneous nature of arc and sinus" as Kepler remarks [17].

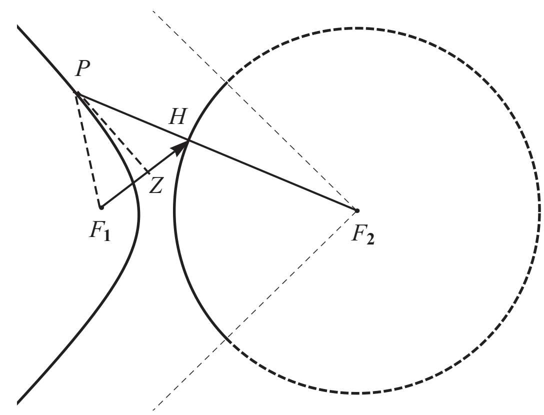

FIG. 1. Hodograph of a hyperbolic orbit. The hodograph of the given hyperbola branch covers only part of the circle (solid curve). F_1 is the center of the heavy mass in the Kepler problem, F_2 is the second focus point, P indicates an arbitrary location of the "planet," and the corresponding (rotated) velocity is indicated by the arrow $\overrightarrow{F_1H}$, which is vertical to the true velocity at P, in analogy to Maxwell's construction of the hodograph for an elliptic orbit [18]. The dashed line PZ is tangent at P and the bisecting normal to $\overrightarrow{F_1H}$. The dashed lines through the circle center are parallel to the asymptotes of the hyperbola.

D. Hodograph

The hodograph is the curve traced out by the end points of the velocity vectors when drawn from a fixed origin. As was shown by Maxwell in [18], the hodograph of the Kepler motion is a circle. By relating the hodograph to the elliptic orbit by means of Kepler's laws, Maxwell derived the law of gravitation. Later on, Feynman apparently used Maxwell's design, Fig. 16 in [18], in one of his "Lost Lectures," in order to construct the elliptic orbit from the hodograph in an elementary geometric way [14]. As was shown in [19–21], the hodograph of a hyperbolic orbit is a circle too.

In the following, we prepare our results for drawing the hodograph in Fig. 1. In the expressions of the velocity components (69), we insert Γ_Ω from (27), η_1 from (67), and ν^2 from (76) and obtain, neglecting quantum corrections,

$$v_x = -\frac{2\Omega\sinh(\Upsilon)}{e\cosh(\Upsilon)-1}; \quad v_y = \frac{2\Omega\sqrt{e^2-1}\cosh(\Upsilon)}{e\cosh(\Upsilon)-1}. \tag{92}$$

It is elementary to show that

$$v_x^2 + (v_y - v_0)^2 = \frac{4\Omega^2}{e^2-1}, \quad v_0 = \frac{2e\Omega}{\sqrt{e^2-1}}. \tag{93}$$

This proves that the hodograph is a circle, or part of it. In his Fig. 16, Maxwell [18] combined the hodograph and the orbit. To this end, he mapped the velocities to space vectors, $\mathbf{v} \to \mathbf{R}$, by scaling with the constant of motion f, which here in the hyperbolic case is given as

$$\mathbf{v} = f\mathbf{R}; \quad f = \frac{\Omega}{a\sqrt{e^2-1}}. \tag{94}$$

The equation for the hodograph now attains the form

$$R_x^2 + (R_y - R_0)^2 = (2a)^2, \quad R_0 = 2ea. \tag{95}$$

We are still free to orient the scaled velocity space with respect to orbit space. We rotate the vectors $\mathbf{R}$ clockwise by $\pi/2$: $\{R_1,R_2\} \to \{R_2,-R_1\}$, so that the rotated vector is perpendicular to the velocity; simultaneously, we have to shift $\{0,R_0\} \to \{R_0,0\}$. As a consequence, the center of the hodograph, with radius $2a$, falls into the focus F_2 at $(0,ea)$. This is in full analogy to Maxwell's method in the case of elliptic orbits [18].

To find the speed at an orbit point P, one draws a line from P to the focus F_2, which intersects the circle at point H (see Fig. 1). The vector $\overrightarrow{F_1H}$ is normal to the tangent at P, and is, therefore, the (scaled) velocity vector we are looking for; by construction, it is normal to the true velocity. To see that $\overrightarrow{F_1H}$ is normal to the tangent, one proves that the triangles F_1ZP and ZHP are congruent: (1) they have in common the side PZ; (2) they have equal angles $\angle F_1PZ = \angle ZPH$, since the tangent bisects the focus lines (see, e.g., [22]); (3) the lengths $PH = PF_1$. The latter follows from the hyperbola property that the difference of the distance between P and the two focus points $PF_2 - PF_1 = 2a$. Now, by construction, $F_2H = 2a$ and, therefore, $PF_2 = 2a + PH = 2a + PF_1$.

By applying elementary geometry, one can construct the hyperbola from the hodograph, see also [14]. According to Fig. 1, one carries out the following steps: (1) draw a circle with radius $2a$ around some point, F_2; (2) choose a second point F_1 at distance $2ea > 2a$ from F_2; (3) draw a line through F_2 and some arbitrary point H on the circle; (4) draw the line F_1H with midpoint Z; then, the normal in Z and the line F_2H intersect each other in a hyperbola point P.

VII. PARABOLIC LIMIT

The limit of eccentricity $e \to 1$ leads to a parabolic orbit, which constitutes a separatrix between elliptic and hyperbolic orbits and is expected to be particularly sensitive to fluctuations. As it turns out, g diverges for $e \to 1$ and leads to nonphysical velocity components v_x and v_y, together with a singular relation between pseudotime and real time. As a matter of fact, the constraint contribution to the energy, H_X, is not negligible in the parabolic limit, indicating the limitation of our wave function (39) which fulfills the constraint in the mean only. In the next section we discuss a necessary condition which is consistent with our approximate treatment of the constraint. By (27) and (81), with $e \to 1$, the energy parameter E vanishes as does the pseudofrequency Ω:

$$E = \frac{\kappa}{2r_0}(e-1); \quad \Omega \equiv \sqrt{\frac{E}{2\mu}} = \frac{1}{2}\sqrt{\frac{\kappa}{r_0\mu}}\sqrt{e-1}, \tag{96}$$

which implies, since $\Upsilon = 2\Omega\sigma$,

$$\begin{aligned} \cosh(\Upsilon) &= 1 + \frac{\kappa\sigma^2}{2r_0\mu}(e-1)[1+O(e-1)]; \\ \sinh(\Upsilon) &= \sigma\sqrt{\frac{\kappa}{r_0\mu}}\sqrt{e-1}[1+O(e-1)]. \end{aligned} \tag{97}$$

Furthermore, one infers from (71) and (73) that g gets infinite in the parabolic limit:

$$\begin{aligned} g &= g_0\frac{1}{1+\kappa_1\sigma^2}[1+O(e-1)], \\ g_0 &\xrightarrow{e\to1} \frac{2}{r_0\Gamma_\Omega} = \frac{\kappa_2}{\sqrt{e-1}}, \end{aligned} \tag{98}$$

where

$$\kappa_1 = \frac{\kappa}{2r_0\mu}, \quad \kappa_2 = \frac{\hbar}{\sqrt{r_0\mu\kappa}}. \tag{99}$$

The above limits lead to the following asymptotic coefficients of the position and velocitiy components [see (67)]:

$$\xi_1 \xrightarrow{e\to1} 1, \quad \eta_1 \xrightarrow{e\to1} \frac{\hbar}{2\mu g}, \quad \Gamma_\Omega\eta_1 \xrightarrow{e\to1} \frac{\kappa}{\hbar}(1+\kappa_1\sigma^2)(e-1). \tag{100}$$

In view of (97) and (100), the coordinates x and y, defined in (68), attain the following forms in the parabolic limit:

$$x = r_0 - \frac{\kappa}{2\mu}\sigma^2, \quad y = \sqrt{\frac{2\kappa r_0}{\mu}}\sigma, \tag{101}$$

which implies the standard parabola equation $x = x(y)$ with

$$x = r_0\left[1 - \frac{1}{4r_0^2}y^2\right]. \tag{102}$$

This result holds true with the inclusion of quantum corrections.

The parabolic limit of "Kepler's" equation (90), which includes quantum corrections, at first leads to

$$t = r_0\left[1 + \frac{4g_0^2}{1+4g_0}\right]\sigma + \frac{\kappa}{6\mu}\frac{1+2g_0}{1+4g_0}\sigma^3. \tag{103}$$

Without quantum corrections, with $g_0 = 0$, one would obtain the classical result (see p. 45 of [15], or p. 91 of [23]), where scaled parameters are used instead of σ. However, since g_0 diverges for $e \to 1$, the time relation gets singular in the linear term of σ.

In the case of the velocity components, defined in (69), one obtains in the parabolic limit

$$v_x \to \frac{2\kappa_1\sigma}{1+\kappa_1\sigma^2}\frac{1}{1+2g}, \quad v_y \to 2\frac{\sqrt{\kappa_1}}{1+\kappa_1\sigma^2}\frac{1}{1+2g}. \tag{104}$$

Now, by (98) the function g diverges proportional to $1/\sqrt{e-1}$ which causes $v_{x,y}$ to vanish. A further nonphysical limit occurs in the relation (87), where the right-hand side, due to the factor $(2+2g_0)^2$, converges to a finite value in the limit $e \to 1$, whereas the left-hand side vanishes. Indeed, the parabolic limit signals that the constraint becomes effective beyond the mean value approximation adopted.

VIII. QUANTUM CORRECTIONS AND CONSTRAINT

The relevance of the constraint in the parabolic limit can be directly seen by examining the contribution $\langle H_X\rangle_u$ of the constraint to the energy. As stated in (83), the relative contribution can be written as

$$\begin{aligned} h_X &\equiv \langle H_X\rangle_u/E = -g_0 J(\Upsilon)K(g), \\ J(\Upsilon) &= e(e-1)/[e\cosh(\Upsilon)-1]^2. \end{aligned} \tag{105}$$

At zero time, with $\Upsilon = 0$, one derives the following limiting behavior for $e \to 1$:

$$J(\Upsilon) \to 1/(e-1), \quad K(g_0) \to 1/\big(4g_0^2\big) \to \text{const.}\,(e-1), \tag{106}$$

which implies that the product JK stays finite for $e \to 1$. As a consequence, $|h_X|$ diverges with g_0, i.e., proportional to $1/\sqrt{e-1}$. The smallness of $|h_X|$, however, is a necessary condition underlying our calculations.

It is convenient to express g_0 in terms of the energy E. By using the definition of g_0 in (74) and of Γ_Ω in (27), and taking into account that the elliptic case with $E \leqslant 0$ leads to the same quantum parameter g_0 with $E \to -E$, see [24], we write

$$g_0 = \frac{\theta_E}{2}\sqrt{\frac{E_\hbar}{|E|}}, \quad E_\hbar = \frac{\hbar^2}{2r_0^2\mu}, \quad 1 \leqslant \theta_E \leqslant 2, \quad E \in \mathbb{R}. \tag{107}$$

The constant $E_\hbar$, clearly, is a microscopic parameter. As an example, consider a space vehicle which after engine stop starts its flight at a distance $r_0 = 10^7$ m from the Earth center with mass $\mu = 10^3$ kg, then $E_\hbar \approx 10^{-86}$ J. This appears to be small beyond physical imagination: an electron which would be shifted by 1 Å in the Earth field at the distance r_0 would cause a change in potential energy $\Delta E \approx 10^{-40}$ J, which practically is infinitely larger than $E_\hbar$. However, as is discussed below, the smallness of g_0 does not imply negligible $|h_X|$.

On the other hand, in the case of the hydrogen atom being in a state with principle quantum number n, the distance amounts to $r_0 = n^2 a_B$ (distance between position of maximal radial probability density and origin), where a_B is the Bohr radius. With the electron mass μ, one gets $E_\hbar = 3.40/n^4$ eV, and $E = E_n = -13.6/n^2$ eV. This gives rise to $g_0 = 0.50\theta_n/n$, where the indeterminacy parameter θ_n can be easily obtained by solving the third equation of (73) for g_0. For $n = 1$, one finds $g_0 = 0.81$, and for the experimentally accessible Rydberg state with $n = 72$, see [25], $g_0 = 0.007$.

In the following, we derive a criterion for the smallness of $|h_X|$. By definition (105), and since $K(g) \leqslant 1$, we have the upper bound $h_{\max} := g_0 e/(e-1) \geqslant |h_X|$. We express the eccentricity e by the energy E using (81) and (107) to obtain

$$h_{\max} = \sqrt{\frac{E_\hbar}{E}}\left(1 + \frac{E_P}{E}\right), \quad E_P = \frac{\kappa}{2r_0} > 0, \quad E > 0. \tag{108}$$

To estimate a lower limit $E_{\min}$ which corresponds to $h_{\max} = 1$, we set $E_{\min} = \epsilon\, E_P$ and solve for ϵ:

$$\frac{1}{\sqrt{\epsilon}}\left(1 + \frac{1}{\epsilon}\right) = \sqrt{\frac{E_P}{E_\hbar}}. \tag{109}$$

Assuming $E_P/E_\hbar \gg 1$, we get $\epsilon \to (E_\hbar/E_P)^{1/3}$, and, thus, the following criterion for the smallness of $|h_X|$:

$$E \gg E_{\min} = \left(\frac{E_\hbar}{E_P}\right)^{1/3} E_P. \tag{110}$$

For the above example of the space vehicle, we find $E_{\min} \approx 2 \times 10^{-3}$ eV $\approx 3 \times 10^{-22}$ J. We note that, when $|h_X|$ is small, also the relative energy deviation $|\Delta E|/E <$ in (82) has a small magnitude, and, on the other hand, the latter diverges together with h_X.

In the case of elliptic orbits with $E < 0$ and $e < 1$, the corresponding constraint contribution is $|h_X| = (1-e)\, g_0 K(g)/(1+2\,g)/[1 - e\cos(\Upsilon)]^2$ [24] (without the additional factor e in the numerator). We get the same expression for $E_{\min}$ as in (110), and the criterion $|E| \gg E_{\min}$. Obviously, if $|E| \gg E_{\min}$, then we also have $|E| \gg E_\hbar$ and, therefore, the quantum parameter $|g|$ is much smaller than 1, too.

In the case of the hydrogen atom in a state with principal quantum number n, one finds $E_{\min} = 13.6\, n^{-8/3}$ eV. This tells us that the criterion (110), certainly, is not fulfilled in the ground state with $n = 1$. For general quantum number n, the condition $|E_n| \gg E_{\min}$ amounts to $n^{2/3} \gg 1$. For the Rydberg state considered in [2] with $n = 40$, one gets $40^{2/3} \approx 11.7$, which is not impressively larger than 1. Thus, the present theory, which deals approximately with the constraint, cannot be safely applied to this state. Nauenberg's theory [2] of Rydberg states, from the outset, is confined to the original configuration's space $x \in \mathbb{R}^3$ and, thus, is free of a constraint.

There are, of course, manifold perturbations to the Kepler orbit of a near-Earth artificial satellite, such as, e.g., oblateness of the Earth, higher terms of the Earth's gravitational field, atmospheric drag, radiation pressure, and gravitational force of the Moon and Sun (see [26]). These effects undermine a sharp definition of the energy E defined in (81). In praxis, we may define the fuzziness of E by the uncertainty of measuring the position and velocity of a satellite. For near-Earth satellites one has $\delta r \approx 1$ m and $\delta v \approx 0.5 \times 10^{-4}$ m/s [27]. If we suppose an exact parabolic orbit in the case of our space vehicle example, we find a measurement uncertainty of the energy $\delta E \approx 4 \times 10^3$ J, which is far above $E_{\min} \approx 10^{-22}$ J.

IX. FLUCTUATIONS AND UNCERTAINTY PRODUCTS

In this section, we discuss the msrd of position and velocity components, which are calculated in Appendix B, together with the position-momentum uncertainty products. As it turns out, the results are similar to the corresponding ones of a propagating Gaussian wave packet in free space, insofar as the msrd of the velocity components stay finite in the limit $t \to \infty$, whereas the msrd of the space components increase linearly at sufficiently large time. However, as will be seen, the scaling factors behave differently.

In view of (B6) and (B9) we have for all three space components the following result:

$$(\Delta x_i)^2 = 2g_0 r_0^2 f_0(\Upsilon), \qquad f_0 = \cosh(\Upsilon)\frac{e\cosh(\Upsilon) - 1}{e-1}[1 + O(g_0)]. \tag{111}$$

For large $\Upsilon > 0$, the function $f_0(\Upsilon)$ is conveniently expressed in terms of the real time t. To this end, we make use of the property $\sinh(\Upsilon) = \cosh(\Upsilon)[1 + O(\exp(-2\Upsilon))]$. Then, using "Kepler's" equation (90), we can write for large Υ

$$\cosh[\Upsilon] \approx \sin[\Upsilon] \approx \frac{2(e-1)\Omega}{e r_0} t, \tag{112}$$

which leads to (neglecting g_0 as compared to 1)

$$f_0 = \frac{2}{e}\tau[2(e-1)\tau - 1] \approx 4\frac{e-1}{e}\tau^2, \quad \tau = \frac{\Omega}{r_0}t \gg 1. \tag{113}$$

Thus, for sufficiently large time, we obtain using (107)

$$\Delta x_i = 2r_0\sqrt{\frac{e-1}{e}}\sqrt{\frac{E_\hbar}{E}}\tau, \quad E>0, \quad \tau \gg 1. \tag{114}$$

And in view of (27), we can also write

$$\Delta x_i = r_0\sqrt{\frac{e-1}{e}}\,\frac{\hbar}{\mu r_0^2}t. \tag{115}$$

For comparison, in the case of a spreading Gaussian wave packet in free space, one finds in textbooks

$$(\Delta x_i)_{\text{Gauss}} = d\sqrt{1+\Delta^2} \approx d\Delta, \quad \Delta = \frac{\hbar}{2\mu d^2}t \gg 1, \tag{116}$$

where d denotes the initial localization length of the wave packet. Thus, the two expressions have the same structure with the initial localization length d of the Gaussian case replaced by the initial distance r_0; we remind that the initial localization length of a hyperbolic orbit is given by $1/\Gamma_\Omega = \hbar/\sqrt{8\mu E}$.

In the example of the space vehicle mentioned above, in addition to $\mu = 10^3$ kg and $r_0 = 10^7$ m, we assume eccentricity $e = 2$ and obtain, with the aid of (81), the energy $E \approx 2\times 10^{10}$ J. Then, the condition $\tau \gg 1$ is fulfilled for $t \gg 1$ h. On the other hand, in order that $\Delta x = 1$ m, the travel time should be about 10^{36} a, which precludes the observation of a quantum effect in the given example.

The mean-square deviations of the velocity components are stated in (B16), (B19), and (B21) in the form

$$(\Delta v_i)^2 = \frac{2E}{\mu}\, g_0\, f_i(\Upsilon)[1 + O(g_0) + O(\exp(-1/g_0))], \quad i = 1,2,3. \tag{117}$$

The functions f_i, defined in Appendix B, are bounded functions of Υ, and, thus, of time t. The initial and asymptotic values are

$$f_1(0) = (19e-18)/(e-1), \quad f_2(0) = 4(3e-1)/(e-1), \quad f_3(0) = e/(e-1); \tag{118}$$

$$f_1(\infty) = \frac{4(e-1)(e^2+1)}{e^3}, \quad f_2(\infty) = \frac{4(e-1)(3e^2-1)}{e^3}, \quad f_3(\infty) = \frac{2(e-1)}{e}. \tag{119}$$

For the above example, the asymptotic values of Δv are of the order of 10^{-20} m/s.

For the uncertainty product, we use the relations (27), (73), (111), and (117) to write

$$\Delta\pi_i \equiv \Delta x_i(\mu\Delta v_i) = \hbar\sqrt{f_0 f_i/2}[1 + O(g_0)], \quad i = 1,2,3. \tag{120}$$

We remind that the above results are not allowed to be extended to the parabolic limit $e \to 1$. With the aid of (118) and with $f_0(0) = 1$, we find at time $t = \Upsilon = 0$, up to terms of order $O(g_0)$,

$$\pi_1 \geqslant \sqrt{19/2}\hbar, \quad \pi_2 \geqslant \sqrt{6}\hbar, \quad \pi_3 \geqslant \sqrt{1/2}\hbar. \tag{121}$$

In the limit of large eccentricities, the uncertainty product $\pi_3(t=0)$ differs from the quantum mechanical minimum, $\hbar/2$, only be a factor $\sqrt{2}$, and at $e = 2$, for instance, it is only by a factor of 2 larger than the minimum.

X. CONCLUSIONS

The continuous transition from quantum mechanics to classical physics was corroborated for the nonrelativistic Hamiltonian of the H atom in the case of unbounded orbits with positive energy. We supplemented studies in the negative energy region which were carried out some time ago (see, e.g., [2,3]) and which for macroscopic parameters verify that elliptical orbits of negligible quantum corrections are solutions of the Schrödinger equation. Analogously, in this article we found quantum states which lead to hyperbolic orbits in the macroscopic limit, provided the orbits are not too close to a parabolic one. Practically, for instance, in space vehicle dynamics, the corresponding energy interval is predicted to be unaccessibly small in view of the measurement uncertainties.

The method used, basically, was that of [3]: The Hamiltonian was transformed by the Kustaanheimo-Stiefel transformation into a four-dimensional configuration space, and the wave function was built from harmonic oscillator coherent states, which, however, had to be newly constructed for imaginary oscillator frequencies. Moreover, it was necessary to introduce a modified scalar product in $\mathbb{R}^4$, in order to determine the contribution of the constraint operator to the mean energy, and also in view of our method to connect pseudotime and real time. The constraint was taken into account within an averaging approximation. The validity of the latter was examined from the relative effect of the constraint on the mean energy. As a necessary criterion for its smallness, we found that the energy E should have the property

$$|E| \gg \left(\frac{E_\hbar}{|E_P|}\right)^{1/3}|E_P|, \quad E_\hbar = \frac{\hbar^2}{2r_0^2\mu},$$

where μ denotes the reduced mass, r_0 the initial distance of the "planet," and E_P its initial potential energy. As an unpublished result [24], the criterion also holds for negative values of the energy when treated with the wave function (43) and thus can be applied, in principle, to the bound states of the H atom, in particular to Rydberg states. If the criterion is fulfilled, then also quantum corrections to the classical orbit, as msrd of position and velocity, are small, and this is true for most macroscopic situations. In some energy interval around the parabolic orbit, where the above condition is violated, the present theory gives nonphysical results.

Our approach to connect pseudotime and real time is based on the the equality of the mean velocity, $\langle \mathbf{w}(\sigma)\rangle_{\text{os}} = \langle \mathbf{v}(t)\rangle_u$, at $t = t(\sigma)$. This equality gives rise to the same scalar relation (49) for all three components with an outcome which is consistent with the classical limit for hyperbolic orbits which are not too close to the parabolic one. The choice of the velocity as observable appears to be natural, since the function $t(\sigma)$ is defined in the tangent space of the orbit. The function emerges in the form of Kepler's equation, see (90) and (91), and is different for different orbits analogously to the path integral treatment [5] where the reparametrization of time, $t \to \sigma$, is path dependent.

Ideally, the coherent state (39), $\psi_{\rm coh}$, should fulfill the Schrödinger equation in u space:

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H_u\psi.$$

A full discussion of the error terms, when $\psi_{\rm coh}$ is inserted for ψ, requires one to examine both diagonal and nondiagonal matrix elements with the aid of a suitable complete set of states, which, however, is outside the scope of this paper. Restricting to mean values with $\psi_{\rm coh}$, we have shown in Sec. VI B., that the contribution of the constraint part of H_u is negligible for sufficiently small quantum parameter g. That leaves us to examine (we use the definition (22) of $\psi_{\rm coh}$)

$$-\frac{\hbar}{i}\left\langle\psi_{\rm coh}\left|\frac{\partial}{\partial t}\psi_{\rm coh}\right.\right\rangle_u \equiv \left\langle\psi_{\rm coh}\left|\frac{d\sigma}{dt}H_{\rm os}\psi_{\rm coh}\right.\right\rangle_u \stackrel{?}{=} \left\langle\psi_{\rm coh}\left|\frac{1}{r(\mathbf{u})}H_{\rm os}\psi_{\rm coh}\right.\right\rangle_u,$$

which can be justified in the limit $g\to 0$, where the operator $1/r$ can be replaced by the mean value $\langle 1/r\rangle_u$ in consistency with the definition of $S^{-1} = d\sigma/dt$ in (49) and with (70).

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APPENDIX A: NORMALIZATION OF COHERENT STATE

As a check of the explicit solutions (30) to (34), we calculate the norm of ψ, as given in (23), by integration with respect to u. This is done in the oscillator metric,

$$\langle\psi|\psi\rangle_{\rm os} = C_0^2\exp[\beta+\beta^*]\sqrt{\frac{\pi}{\Gamma_R}}\exp\left[\frac{(a+a^*)^2}{4\Gamma_R}\right]. \quad \text{(A1)}$$

From (35) to (37), one finds with the abbreviation $D = \cosh^2(\Upsilon/2)+\gamma_0^2\sinh^2(\Upsilon/2)$

$$\begin{aligned} A_1 &\equiv [\gamma+\gamma^*]^{-1/2} = \left[\frac{D}{2\gamma_0}\right]^{1/2},\\ A_2 &\equiv \frac{(a+a^*)^2}{4\Gamma_R} = \frac{\alpha_0^2}{\gamma_0}\frac{\cosh^2(\Upsilon/2)}{D},\\ \beta+\beta^* &\equiv 2C_3 - \frac{1}{2}\log(B_1) + B_2,\\ B_1 &\equiv \cosh(\Upsilon/2+iC_1)\cosh(\Upsilon/2-iC_1) = \frac{D}{1+\gamma_0^2},\\ B_2 &\equiv -\frac{\alpha_0^2(\gamma+\gamma^*)}{2(1+\gamma_0^2)} = -\frac{\alpha_0^2\gamma_0}{1+\gamma_0^2}\frac{1}{D}. \end{aligned} \quad \text{(A2)}$$

From this, we obtain

$$\frac{A_1}{\sqrt{B_1}} = \sqrt{\frac{1+\gamma_0^2}{2\gamma_0}};\quad A_2+B_2 = \frac{\alpha_0^2}{\gamma_0(1+\gamma_0^2)}. \quad \text{(A3)}$$

Combining the above results we arrive at

$$\begin{aligned}\langle\psi|\psi\rangle_{\rm os} &= C_0^2\sqrt{2\pi\Gamma_\Omega}\sqrt{\frac{1+\gamma_0^2}{2\gamma_0}}\exp\left[2C_3+\frac{\alpha_0^2}{\gamma_0(1+\gamma_0^2)}\right]\\ &= C_0^2\sqrt{2\pi\Gamma_\Omega}\sqrt{\frac{1+\gamma_0^2}{2\gamma_0}} = 1. \end{aligned} \quad \text{(A4)}$$

APPENDIX B: MEAN VALUES

1. Mean-square deviations of position components

We start with the calculation of

$$\langle x_1^2\rangle_u = C_u^2\, 4\int du_1\dots du_4\, r(u)\,(u_1u_3-u_2u_4)^2\,|\psi_0(\mathbf{u})|^2. \quad \text{(B1)}$$

After the u polynomial has been multiplied out, the components u_i of each monomial are replaced by the first rule of (50), the factor r is dealt with by means of the second rule. This is done with the aid of a short MATHEMATICA [13] program. The result is

$$\begin{aligned}\langle x_1^2\rangle_u = \frac{1}{4(A^2+8\Gamma_R)\Gamma_R^4}\big(&-8\Gamma_R^2(A^2+8\Gamma_R) + A^2\{-2(a_1+a_1^*)(a_2+a_2^*)(a_3+a_3^*)(a_4+a_4^*)\\ &+[(a_1+a_1^*)^2+2\Gamma_R][(a_3+a_3^*)^2+2\Gamma_R]+[(a_2+a_2^*)^2+2\Gamma_R][(a_4+a_4^*)^2+2\Gamma_R]\}\\ &+24\Gamma_R\{-2(a_1+a_1^*)(a_2+a_2^*)(a_3+a_3^*)(a_4+a_4^*)+[(a_1+a_1^*)^2+2\Gamma_R][(a_3+a_3^*)^2+2\Gamma_R]\\ &+[(a_2+a_2^*)^2+2\Gamma_R][(a_4+a_4^*)^2+2\Gamma_R]\}\big). \end{aligned} \quad \text{(B2)}$$

We subtract $\langle x_1\rangle_u^2$ using (53) and (56) and obtain after some simplifications

$$\begin{aligned}(\Delta x_1)^2 \equiv \langle x_1^2\rangle_u - \langle x_1\rangle_u^2 = \frac{1}{2(A^2+8\Gamma_R)^2\Gamma_R^3}\{&\big(A^2+8\Gamma_R\big)\big(A^4+24A^2\Gamma_R+64\Gamma_R^2\big) - 32\Gamma_R[(a_1+a_1^*)(a_3+a_3^*)\\ &-(a_2+a_2^*)(a_4+a_4^*)]^2\}. \end{aligned} \quad \text{(B3)}$$

We introduce $g = 4\,\Gamma_R/A^2$ and replace $(a_1+a_1^*)(a_3+a_3^*)-(a_2+a_2^*)(a_4+a_4^*)$ by $2\langle x_1\rangle_u\Gamma_R^2(1+2g)/(1+4g)$ to write

$$(\Delta x_1)^2 = \Delta_1+\Delta_2,\quad \Delta_1 = \frac{2(1+6g+4g^2)}{g(1+2g)\Gamma_R^2},\quad \Delta_2 = -\frac{4g^2}{(1+4g)^2}\langle x_1\rangle_u^2. \quad \text{(B4)}$$

To evaluate Δ_1, we make use of (73), (35), and (76) to write

$$\frac{1}{(\Gamma_R)^2 g} = g_0 r_0^2 \frac{e\cosh(\Upsilon) - 1}{e-1}\cosh(\Upsilon)\frac{(1+2g_0)^2}{(1+4g_0)^2}, \tag{B5}$$

which gives rise to

$$(\Delta x_1)^2 = \Delta_1[1+O(g_0)] = 2g_0 r_0^2 \frac{e\cosh(\Upsilon)-1}{e-1}\cosh(\Upsilon)[1+O(g_0)], \tag{B6}$$

where the term Δ_2 could be omitted since it is by a factor of g smaller than Δ_1.

For the two other components, we obtain similarly

$$\begin{aligned}(\Delta x_2)^2 = {} & \frac{1}{2(A^2+8\Gamma_R)^2\Gamma_R^3}\{(A^2+8\Gamma_R)(A^4+24\,A^2\Gamma_R \\ & +64\Gamma_R^2) - 32\Gamma_R[(a_1+a_1^*)(a_4+a_4^*) \\ & +(a_2+a_2^*)(a_3+a_3^*)]^2\}. \end{aligned}\tag{B7}$$

$$\begin{aligned}(\Delta x_3)^2 = {} & \frac{1}{2(A^2+8\Gamma_R)^2\Gamma_R^3}\{(A^2+8\Gamma_R)(A^4+24A^2\Gamma_R \\ & +64\Gamma_R^2) - 8\Gamma_R[(a_1+a_1^*)^2+(a_2+a_2^*)^2 \\ & -(a_3+a_3^*)^2-(a_4+a_4^*)]^2\}. \end{aligned}\tag{B8}$$

This allows for the analogous writing as (B4) with Δ_1 being the same for all three components:

$$(\Delta x_i)^2 = \Delta_1 - \frac{4g^2}{(1+4g)^2}\langle x_i\rangle_u^2, \quad i = 1,2,3. \tag{B9}$$

Thus, we have for all three components the same result (B6).

2. *Mean-square deviations of velocity components*

In the following we derive the mean-square deviations (msd) up to first order in g_0 as stated in (117). For the rather involved calculations we used the computer software MATHEMATICA [13]. We remark that $0 < g \leqslant g_0$, so we consider small order $O(g)$ being equivalent with $O(g_0)$.

We start with the first component by using the operator w_1 from (45), and apply partial integration

$$\begin{aligned}\langle v_1^2\rangle_u = {} & -C_u^2\int du_1\ldots du_4 r\left[\frac{1}{r}w_1\psi_0^*\right]\left[\frac{1}{r}w_1\psi_0\right] \\ = {} & \frac{\hbar^2}{4\mu}C_u^2\int du_1\ldots du_4\frac{1}{r}[a_3^*u_1 - a_4^*u_2 + a_1^*u_3 \\ & -a_2^*u_4 - 2(\Gamma_R - i\Gamma_I)(u_1u_3 - u_2u_4)][a_3u_1 - a_4u_2 \\ & +a_1u_3 - a_2u_4 - 2(\Gamma_R + i\Gamma_I)(u_1u_3-u_2u_4)] \\ & \times|\psi_0(\mathbf{u},\Gamma_R\to s)|^2. \end{aligned}\tag{B10}$$

The third rule (50) applies to $|\psi_0|^2$ only, leaving Γ_R's within the square brackets untouched. After the square brackets are multiplied out, the first rule of (50) is applied to each monomial in $\mathbf{u}$. For the factor 1/r, the integration rule of (50) is carried out. The latter produces terms proportional to $\exp[-A^2/(4\Gamma_R)] \equiv \exp[-1/g]$, which we omit, since we assume small quantum corrections g. Then we subtract $\langle v_1\rangle_u^2$.

In the manipulations below we use the substitutions (64) and (35) for the parameters $\mathbf{a}_i$ and $\Gamma_R + i\Gamma_I = \Gamma_\Omega\gamma$, both with $\gamma_0 = 1$. We also make use of the relations $\hbar^2\Gamma_\Omega^2/(8\mu) = E$, $\rho_0^2 = \Gamma_\Omega/(2g_0)$, and (73) for g; furthermore, we use the substitution (76) to express v^2 in terms of the eccentricity e.

For dealing with the $1/r$ term by means of the third rule (50), we order by powers of $\Gamma_R \to s$ stemming from $|\psi_0|^2$:

$$\begin{aligned}&\langle v_1^2\rangle_u = \sum_{n=1}^{4}K_n^{(1)}, \quad K_n^{(1)} = J_n k_n^{(1)}, \\ &J_n = C_u^2\int_{\Gamma_R}^{\infty} ds\frac{1}{s^{n+2}}\exp[A^2/(4\,s)]. \end{aligned}\tag{B11}$$

The integrals are elementary, for instance, J_1 can be brought into the form

$$J_1 = \frac{\Gamma_R g^2}{\pi^2(1+2g)}[1 - g + g\exp(-1/g)]. \tag{B12}$$

Omitting the exponential terms, we find

$$\begin{aligned}&J_1 = \frac{\Gamma_R g^2}{\pi^2}[1-3g+O(g^2)], \quad J_2 = \frac{g^2}{\pi^2}[1-4g+O(g^2)], \\ &J_3 = \frac{4g}{\pi^2A^2}[1-5g+O(g^2)], \quad J_4 = \frac{16}{\pi^2A^4}[1-6g+O(g^2)]. \end{aligned}\tag{B13}$$

After straightforward manipulations, we obtain to the two lowest orders

$$\begin{aligned}K_1^{(1)} = {} & \frac{2E}{\mu}g_0\frac{e(e-1)}{[e\cosh(\Upsilon)-1]^2} + O(g_0^2), \\ K_2^{(1)} = {} & \frac{2E}{\mu}\frac{1+e^2-2e/\cosh(\Upsilon)}{[e\cosh(\Upsilon)-1]^2} - \frac{2E}{\mu}g_0\frac{e-1}{[e\cosh(\Upsilon)-1]^3} \\ & \times\{6(1+e^2)\cosh(\Upsilon) - e[11+\cosh(2\Upsilon)]\} + O(g_0^2), \\ K_3^{(1)} = {} & -\frac{2E}{\mu}\frac{2(-1+e/\cosh(\Upsilon))}{e\cosh(\Upsilon)-1} + \frac{2E}{\mu}g_0\frac{e-1}{[e\cosh(\Upsilon)-1]^2} \\ & \times\{-12\cosh(\Upsilon) + e[11+\cosh(2\Upsilon)]\} + O(g_0^2), \\ K_4^{(1)} = {} & \frac{2E}{\mu}\frac{(e-\cosh(\Upsilon))^2}{[e\cosh(\Upsilon)-1]^2} + \frac{2E}{\mu}g_0\frac{e-1}{[e\cosh(\Upsilon)-1]^2} \\ & \times\{-12\cosh(\Upsilon)+e[11+\cosh(\Upsilon)]\} + O(g_0^2). \end{aligned}\tag{B14}$$

For the msd we have to subtract

$$\langle v_1\rangle_u^2 = 2\frac{E}{\mu}\frac{\sinh^2(\Upsilon)}{[e\cosh(\Upsilon)-1]2} - 2\frac{2E}{\mu}g_0\frac{e-1}{[e\cosh(\Upsilon)-1]^3}\sinh(\Upsilon)\sinh(2\Upsilon)+O\left(g_0^2\right). \tag{B15}$$

As it turns out, the zero order terms of $K_2^{(1)} + K_3^{(1)} + K_4^{(1)} - \langle v_1\rangle_u^2$ cancel, and we are left with

$$\begin{aligned}(\Delta v_1)^2 &\equiv \left\langle v_1^2\right\rangle_u - \langle v_1\rangle_u^2 = \frac{2\,E}{\mu}\,g_0\,f_1(\Upsilon)[1+O(g_0)],\\ f_1 &= \frac{e-1}{[e\cosh(\Upsilon)-1]^3}[-24e + (17+18e^2)\cosh(\Upsilon) - 13e\cosh(2\Upsilon) + (1+e^2)\cosh(3\Upsilon)].\end{aligned} \tag{B16}$$

The function f_1, obviously, is bounded in time with the special values at zero and infinite time stated in (118) and (119).

The msd $(\Delta v_2)^2$ is found analogously with the functions $K_j^{(1)}$ of (B14) replaced as

$$\begin{aligned}K_1^{(2)} &= \frac{2E}{\mu}g_0\frac{e-1}{[e\cosh(\Upsilon)-1]^2}[1+O(g_0)],\\ K_2^{(2)} &= \frac{2E}{\mu}\frac{e^2-1}{[e\cosh(\Upsilon)-1]^2} - \frac{2E}{\mu}g_0\frac{e-1}{[e\cosh(\Upsilon)-1]^3}[(-2+6e^2)\cosh(\Upsilon) - e(3+\cosh(2\Upsilon)][1+O(g_0)],\\ K_3^{(2)} &= \frac{2E}{\mu}g_0\frac{2(e-1)\cosh(\Upsilon)}{e\cosh(\Upsilon)-1}[1+O(g_0)],\\ K_4^{(2)} &= \frac{2E}{\mu}\frac{(e^2-1)\sinh^2(\Upsilon)}{[e\cosh(\Upsilon)-1]^2} + \frac{2E}{\mu}g_0\frac{2(e-1)\cosh(\Upsilon)}{e\cosh(\Upsilon)-1}[1+O(g_0)].\end{aligned} \tag{B17}$$

With the expression

$$\langle v_2\rangle_u^2 = \frac{2E}{\mu}\frac{(e^2-1)\cosh^2(\Upsilon)}{[e\cosh(\Upsilon)-1]^2} - \frac{2E}{\mu}\,g_0\,\frac{4(e-1)^2(e+1)\cosh^3(\Upsilon)}{[e\cosh(\Upsilon)-1]^3}[1+O(g_0)], \tag{B18}$$

the zero order terms of the msd cancel out, once more, and we obtain to next higher order

$$\begin{aligned}(\Delta v_2)^2 &= \frac{2\,E}{\mu}g_0 f_2(\Upsilon)[1+O(g_0)],\\ f_2 &= \frac{2(e-1)\cosh(\Upsilon)}{[e\cosh(\Upsilon)-1]^3}[3(1+e^2) - 8e\cosh(\Upsilon) + (3e^2-1)\cosh(2\Upsilon)].\end{aligned} \tag{B19}$$

In the case of the third velocity component, we obtain with $K_4^{(3)} = 0$:

$$\begin{aligned}K_1^{(3)} &= \frac{2E}{\mu}g_0\frac{e(e-1)}{[e\cosh(\Upsilon)-1]^2}[1+O(g_0)], \quad K_2^{(3)} = -\frac{2E}{\mu}g_0\frac{2(e-1)[e-\cosh(\Upsilon)]}{[e\cosh(\Upsilon)-1]^2}[1+O(g_0)],\\ K_3^{(3)} &= \frac{2E}{\mu}g_0\frac{2(e-1)\cosh(\Upsilon)}{e\cosh(\Upsilon)-1}[1+O(g_0)],\end{aligned} \tag{B20}$$

which, with $\langle v_3\rangle_u = 0$, gives rise to

$$(\Delta v_3)^2 = \frac{2E}{\mu}g_0 f_3(\Upsilon)[1+O(g_0)], \quad f_3 = \frac{e(e-1)\cosh(2\Upsilon)}{[e\cosh(\Upsilon)-1]^2}. \tag{B21}$$

3. Mean angular momentum

The mean value of the angular momentum

$$\boldsymbol{L} = C_u^2 \mu \int du_1 du_2 du_3 du_4 r(u) \psi^* (\mathbf{x} \times \mathbf{v}) \psi \tag{B22}$$

can be written in the form

$$\boldsymbol{L} = \langle \mathbf{x} \rangle_{\mathrm{O}s} \times \langle \mathbf{v} \rangle_u + \boldsymbol{B}, \tag{B23}$$

where we used the abbreviations

$$\begin{aligned} B_1 &= i \frac{2\hbar}{A^2 + 8\Gamma_R} [a_1^* a_4 - a_1 a_4^* + a_2^* a_3 - a_2 a_3^*], \\ B_2 &= i \frac{2\hbar}{A^2 + 8\Gamma_R} [a_1 a_3^* - a_1^* a_3 + a_2^* a_4 - a_2 a_4^*], \\ B_3 &= i \frac{2\hbar}{A^2 + 8\Gamma_R} [a_1 a_2^* - a_1^* a_2 + a_3 a_4^* - a_3^* a_4]. \end{aligned} \tag{B24}$$

We remark that the momentum and position operators obey the canonical commutation relations in u space:

$$[\mu v_i, x_k]_u = \frac{\hbar}{i} \delta_{ik}. \tag{B25}$$

After assignment of the initial values according to (64), we find that the mean values are conserved in time:

$$L_1(\sigma) = L_2(\sigma) = 0; \quad L_3(\sigma) = \frac{1}{2} \hbar \nu (r_0 \Gamma_\Omega) \frac{1 + 2g_0}{1 + 4g_0}. \tag{B26}$$

APPENDIX C: CONTRIBUTION OF CONSTRAINT TO MEAN ENERGY

By (18), the constraint part of the Hamiltonian reads

$$H_X = \frac{\hbar^2}{8\mu} \frac{\mathrm{X}^2}{r^2}; \quad X = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_4 \frac{\partial}{\partial u_3} + u_3 \frac{\partial}{\partial u_4}. \tag{C1}$$

The operator H_X is Hermitian, since X commutes with r. The mean value is taken with respect to the wave function (39). Using partial integration, we can write

$$H_X = \hbar^2/(8\mu) M, \quad M = \langle \psi | X^2/r^2 \psi \rangle_u = -\langle X/r\psi | \mathrm{X}/r\psi \rangle_u. \tag{C2}$$

One immediately obtains

$$M = -C_u^2 \left\langle \psi_0 \middle| \frac{1}{r^2} (a_2^* u_1 - a_1^* u_2 - a_4^* u_3 + a_3^* u_4)(a_2 u_1 - a_1 u_2 - a_4 u_3 + a_3 u_4) \psi_0 \right\rangle_u. \tag{C3}$$

After the polynomial in $\mathbf{u}$ has been multiplied out and the first and third rule of (50) are applied, one obtains (we remind of the metric factor r which cancels in $1/r^2$ one r)

$$M = -C_u^2 \int_{\Gamma_R}^{\infty} \mathrm{d}s \left[P_9^2 - 2P_2 s \right] G(A^2, s), \tag{C4}$$

where $P_2 = \mathbf{a} \cdot \mathbf{a}^*$ and $P_9 = -1/(2\Gamma_R)\langle X \rangle_{\mathrm{os}}$. The assignments (64) make $P_9 \equiv 0$, see (55) and (58). With this simplification, integration of (C.4) leads to

$$M = -\frac{8}{1 + 2g} \frac{P_2 \Gamma_R^3}{A^4} \{1 - g + g \exp[-1/g]\}; \quad g = \frac{4\Gamma_R}{A^2}. \tag{C5}$$

After the assignments (64) and (35) to P_2/A^4 and $\Gamma_R \equiv \Gamma_\Omega \gamma(\sigma)$, respectively, in addition with (62) to ρ_0^2 with $\gamma_0 = 1$, we find

$$H_X = -\frac{\hbar^2 \Gamma_\Omega}{8\mu r_0} \frac{e(e-1)}{[e \cosh(\Upsilon) - 1]^2} \frac{(1 - g + g \exp[-1/g])(1 + 4g_0)}{(1 + 2g)(1 + 2g_0)}. \tag{C6}$$

This expression can be brought into a more transparent form by eliminating $\hbar$ in terms of Ω and Γ_Ω by means of (27), by using $\Omega^2 = E/(2\mu)$, and, finally, by expressing Γ_Ω in terms of r_0 and g_0 by means of (73):

$$H_X = -g_0 E \frac{e(e-1)}{[e \cosh(\Upsilon) - 1]^2} \frac{1 - g + g \exp[-1/g]}{1 + 2g}. \tag{C7}$$

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