Third-order dispersion drastically changes parametric gain in optical fiber systems

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We demonstrate that the third-order dispersion drastically changes the phase-sensitive parametric gain in cw-pumped fiber systems. We analytically calculate the phase-sensitive gain as a function of the frequency for amplification of two weak monochromatic fields with spectral components symmetric with respect to the pump field. In the absence of the third-order dispersion, the phase-sensitive gain is symmetric with respect to the frequency. When the third-order dispersion is present, the gain has an asymmetric shape. This asymmetry has its origin in the phase-sensitive nature of the parametric gain.

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Nonlinear systems are well known to manifest a large spectrum of dynamical behavior ranging from regular to chaotic regimes. Both regimes exhibit complex spatiotemporal solutions that have been subject to intensive research [1,2]. The key physical phenomenon characteristic for nonlinear systems is instability. Among the different types of instabilities, modulation instability (MI) is one of the most famous since it naturally appears in diverse physical areas, including hydrodynamics, plasma physics, and optics. The first studies of MI date back to the early sixties in such different fields as hydrodynamics [3], nonlinear optics [4], and plasma physics [5].

In optics, the first experimental observation of MI was reported in [6,7] where an intense quasicontinuous field injected in an optical fiber was converted into a train of ultrashort pulses. This ground-breaking experiment gave rise to rapid development of parametric amplification and ultrashort pulse generation that have become some of the most important branches of research in nonlinear fiber optics.

In hydrodynamics, MI was shown to be one of the fundamental mechanisms responsible for the formation of the so-called rogue waves [8]. Very recently, optical equivalents of the oceanic rogue waves have been discovered in optical fibers [9] and are called "optical rogue waves." In this context, the generalized nonlinear Schrödinger equation (NLSE) was shown to be successful in describing the formation of rogue waves both in the ocean and in fibers. Many authors have reported the important impact of higher order dispersive terms in the generalized NLSE on the main characteristics of rogue waves [8,10,11] and also on their non-Gaussian statistics [12]. In particular, it was demonstrated that the third-order dispersion is already sufficient to explain the optical rogue wave formation and, most importantly, their probability density function [13].

In this Brief Report, we show that the third-order dispersion drastically affects the MI gain even in situations in which only continuous-wave (cw) perturbations are considered. In our recent work [14], we have considered a pulsed input signal. Here, we take an input signal as two weak monochromatic cw's symmetrically detuned with respect to the pump wave. We show that the amplitude of the output signal depends on the third-order dispersion. We have obtained an exact analytical expression for the parametric gain over the whole MI frequency band. Our analytical formula is in excellent agreement with numerical simulations performed on the governing generalized NLSE.

Fiber-based phase-sensitive parametric amplifiers (PSAs) are very promising candidates for optical fiber communications. Indeed, it has been shown that PSAs can have a noise figure (NF) below the 3-dB quantum limit of the phase-insensitive amplifiers (PIA) [15]. Various configurations for practical implementation of PSAs in optical-fiber systems have been proposed recently in the literature (see, for example, [16] and references therein). The two key parameters characterizing the PSAs are the parametric gain and the NF. We expect therefore that our analytical expression for the parametric gain spectrum will be widely used by the community studying the PSAs in optical fibers.

For the MI community, we believe that the interesting aspect of our results is in the fact that we have demonstrated the dependence of the MI gain for the cw signals upon the odd-order dispersion terms in the generalized NLSE. This result stands out as a striking contrast to the widespread belief that MI is not affected by the odd-order dispersion terms and depends only on the even-order ones.

Light propagation in an optical fiber is well described by the following generalized NLSE for the slowly varying complex amplitude $A(z,\tau)$ of the electric field [17]:

$$i\frac{\partial A}{\partial z} - \frac{1}{2}\beta_2\frac{\partial^2 A}{\partial \tau^2} - i\frac{1}{6}\beta_3\frac{\partial^3 A}{\partial \tau^3} + \gamma|A|^2A = 0. \tag{1}$$

Here, $\tau = t - \beta_1 z$ is the time in the reference frame moving with the group velocity $v_g = 1/\beta_1$, β_2 and β_3 are the second- and the third-order dispersion coefficients, and γ is the nonlinear coefficient of the fiber.

Equation (1) has the following cw stationary (i.e., τ-independent) solution:

$$A_{\rm st}(z) = A_0\exp(i\gamma P_0 z), \tag{2}$$

where $P_0 = |A_0|^2$ is the cw intensity at the entrance of the fiber. We now perform a linearization of Eq. (1) around the stationary solution $A_{\rm st}(z)$ with respect to a small perturbation $a(z,\tau)$ such that $|a(z,\tau)| \ll |A_0|$. Namely, we look for a solution of Eq. (1) in the form

$$A(z,\tau) = A_{\rm st}(z) + a(z,\tau). \tag{3}$$

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Substituting Eq. (3) into Eq. (1) and keeping only linear terms in $a(z,\tau)$ and $a^\star(z,\tau)$, we obtain the following linearized equation:

$$i\frac{\partial a}{\partial z} - \frac{1}{2}\beta_2\frac{\partial^2 a}{\partial \tau^2} - i\frac{1}{6}\beta_3\frac{\partial^3 a}{\partial \tau^3} + 2\gamma|A_0|^2 a + \gamma|A_0|^2 e^{2i(\gamma|A_0|^2 z+\theta)}a^\star = 0, \quad (4)$$

where we have introduced the phase θ of the complex amplitude A_0 as $A_0 = |A_0|\exp(i\theta)$. In order to solve the linearized equation (4), we perform the Fourier transform of the perturbation $a(z,\tau)$ with respect to the temporal argument τ and introduce the complex Fourier amplitude $a(z,\Omega)$ as follows:

$$a(z,\Omega) = \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} a(z,\tau). \quad (5)$$

For the Fourier amplitude $a(z,\Omega)$, we obtain a linear differential equation, which can be solved exactly. We can therefore express the Fourier amplitude of the perturbation $a(z,\Omega)$ for an arbitrary z in terms of the initial values at $z=0$ of the Fourier amplitudes $a(0,\Omega)$ and the conjugate $a^\star(0,-\Omega)$. This solution can be written in the form

$$a(z,\Omega) = U(z,\Omega)a(0,\Omega) + V(z,\Omega)a^\star(0,-\Omega), \quad (6)$$

where the complex functions $U(z,\Omega)$ and $V(z,\Omega)$ read

$$U(z,\Omega) = e^{i(\gamma|A_0|^2 z+\frac{1}{6}\beta_3\Omega^3 z)} \times \left\{\cosh[g(\Omega)z] + i\frac{\gamma|A_0|^2 - \frac{1}{2}\beta_2\Omega^2}{g(\Omega)}\sinh[g(\Omega)z]\right\},$$

$$V(z,\Omega) = i e^{i(\gamma|A_0|^2 z+\frac{1}{6}\beta_3\Omega^3 z+2\theta)}\frac{\gamma|A_0|^2}{g(\Omega)}\sinh[g(\Omega)z], \quad (7)$$

with

$$g(\Omega) = |\Omega|\sqrt{\gamma|\beta_2||A_0|^2 - \left(\tfrac{1}{2}|\beta_2|\Omega\right)^2}. \quad (8)$$

In Eq. (8), we have assumed the anomalous dispersion, $\beta_2 < 0$, and have written $\beta_2 = -|\beta_2|$.

To our knowledge, the solution of the linearized NLSE with dispersion coefficients of arbitrary order (even and odd) was for the first time given in [18]. Unfortunately, the solution in [18] [Eqs. (23) and (24) in that reference] contains (presumably) a typographic misprint. Namely, the phase term, equivalent to our expression $\exp(i\frac{1}{6}\beta_3\Omega^3 z)$ is written without z and thus becomes independent of the propagation distance. As is clear from what follows, this term is crucial for our analysis. Therefore, we have solved independently the linearized NLSE and obtained the correct solution given by Eqs. (6)–(8).

The coefficients $U(z,\Omega)$ and $V(z,\Omega)$ have the following symmetry properties:

$$|U(z,\Omega)| = |U(z,-\Omega)|, |V(z,\Omega)| = |V(z,-\Omega)|, \quad (9)$$

$$U(z,\Omega)V(z,-\Omega) = U(z,-\Omega)V(z,\Omega), \quad (10)$$

which we use below. The expression for $g(\Omega)$ given by Eq. (8) is very well known in the literature on MI [17]. According to the standard theory of MI, after a sufficiently long distance of propagation L, the perturbation will be dominated by the most unstable Fourier modes corresponding to the maximum of the expression $g(\Omega)$ with respect to Ω. This maximum is reached for two symmetric frequencies $\pm\Omega_c$ such that $\Omega_c = \sqrt{2\gamma P_0/|\beta_2|}$ with $P_0 = |A_0|^2$, where the maximum gain $g_m = g(\pm\Omega_c) = \gamma P_0$ is attained. The instability extends to the frequencies $|\Omega| \leqslant \sqrt{2}\Omega_c$ where $g(\Omega) \geqslant 0$. This condition defines two MI side lobes. Since the third-order dispersion term β_3 does not appear in the expression for $g(\Omega)$, the general conclusion is that β_3 has no influence on the gain spectrum or on the bandwidth of the MI. However, one has to realize that conventional MI theory is restricted to monochromatic perturbations. In order to describe MI for realistic signals with a finite frequency band, one has to calculate the frequency-dependent gain of the system for different Fourier components Ω and not only for the most unstable modes with $\Omega = \pm\Omega_c$. In our recent work [14], we have considered the response of the system to a small pulsed input signal with Gaussian spectrum located around the most unstable mode with $\Omega = \Omega_c$. We have found that for such a signal the modulation instability gain depends on the third-order dispersion β_3.

The configuration considered in [14], with only one signal at Ω_c present at the input to the system, is known in the literature as phase-insensitive amplification [19]. In this Brief Report, we investigate a more general case with two monochromatic input signals around Ω_c and $-\Omega_c$, corresponding to the phase-sensitive amplification. Below we analytically calculate the phase-sensitive gain characterizing the amplified quadrature component of the field. We find that this phase-sensitive gain strongly depends on the third-order dispersion term β_3 even for the monochromatic input signals.

In order to calculate the phase-sensitive gain, we investigate the spatial evolution of the following small perturbation:

$$a(z,\tau) = a_1(z)e^{i\Omega_0\tau} + a_2(z)e^{-i\Omega_0\tau}, \quad (11)$$

where $a_1(z)$ and $a_2(z)$ are the complex amplitudes of the signal and idler and Ω_0 is their detuning frequency with respect to optical carrier frequency ω_0 of the pump field. Thus, the central frequencies of the signal ω_1 and the idler ω_2 are $\omega_1 = \omega_0 - \Omega_0$ and $\omega_2 = \omega_0 + \Omega_0$. We assume that in general Ω_0 is different from Ω_c discussed above. We evaluate the complex amplitudes of the signal and the idler $a_\mu(L), \mu = 1,2$ at $z = L$ with arbitrary initial conditions at $z=0$, $a_\mu(0) = a_\mu$. Let us write the complex amplitudes a_μ at the input of the system as follows:

$$a_1 = |a_1|e^{i\psi_1} = (|c_0| + |s_0|)e^{i\phi}e^{i\varphi},$$
$$a_2 = |a_2|e^{i\psi_2} = (|c_0| - |s_0|)e^{i\phi}e^{-i\varphi}. \quad (12)$$

Here, $|c_0|$ and $|s_0|$ are the amplitudes of the quadrature components of the total field (signal and idler) determined by the phase φ as

$$a_1 + a_2 = 2e^{i\phi}(|c_0|\cos\varphi + i|s_0|\sin\varphi), \quad (13)$$

The phase-sensitive gain $G_{\rm PSA}(\varphi)$ is defined as [20]

$$G_{\rm PSA}(\varphi) = \frac{|a_1(L) + a_2(L)|^2}{|a_1(0) + a_2(0)|^2}, \quad (14)$$

and depends on the phase φ of the amplified quadrature component.

For evaluating the complex amplitudes $a_\mu(L)$, we perform the Fourier transform of Eq. (11) and introduce the Fourier amplitudes of the signal $a_1(z,\Omega) = a(z,\Omega_0 + \Omega)$ and idler $a_2(z,\Omega) = a(z, -\Omega_0 + \Omega)$. Using a_μ from Eq. (12) as initial conditions, we have $a_\mu(0,\Omega) = a_\mu \delta(\Omega)$. These initial conditions guarantee that for $z = L$ we have $a_\mu(L,\Omega) = a_\mu(L)\delta(\Omega)$. The coefficients $a_\mu(L)$ are found by substitution of $a_\mu(0,\Omega)$ into Eq. (6) as

$$a_1(L) = U_1 a_1 + V_1 a_2^\star, \quad a_2(L) = U_2 a_2 + V_2 a_1^\star, \tag{15}$$

where $U_1 = U(L,\Omega_0)$, $V_1 = V(L,\Omega_0)$, $U_2 = U(L, -\Omega_0)$, and $V_2 = V(L, -\Omega_0)$.

Using the symmetry properties of the functions $U(z,\Omega)$ and $V(z,\Omega)$ from Eqs. (9) and (10), we can write the functions U_μ and V_μ in the following form:

$$U_\mu = \cosh(r) e^{\pm i\kappa} e^{i(\psi - \psi_0)}, \quad V_\mu = \sinh(r) e^{\pm i\kappa} e^{i(\psi + \psi_0)}, \tag{16}$$

with "+" corresponding to $\mu = 1$ and "−" corresponding to $\mu = 2$. Here $r(\Omega)$ is the squeezing parameter, defined as [21]

$$\exp[\pm r(\Omega)] = |U(z,\Omega)| \pm |V(z,\Omega)|, \tag{17}$$

and the arguments ψ_0, ψ, and κ are equal to

$$\begin{aligned} \psi_0(\Omega) &= -\arg[U(z,\Omega)V^\star(z,\Omega)]/2, \\ \psi(\Omega) &= \arg[U(z,\Omega)V(z,-\Omega)]/2, \\ \kappa(\Omega) &= \arg[V(z,\Omega)V^\star(z,-\Omega)]/2. \end{aligned} \tag{18}$$

Using Eqs. (16), (15), and (12), we can express the phase-sensitive gain $G_{\mathrm{PSA}}(\varphi)$ as a function of the squeezing parameter $r(\Omega)$ from Eq. (17) and the arguments from Eqs. (18) and (12). Here, for the sake of simplicity, we discuss the important case when the input intensities of the signal and idler are equal, that is, $|s_0| = 0$. In this particular situation, the phase-sensitive gain reads

$$G_{\mathrm{PSA}}(\varphi) = [e^{2r}\cos^2(\Delta\phi) + e^{-2r}\sin^2(\Delta\phi)]\frac{\cos^2(\varphi + \kappa)}{\cos^2(\varphi)}, \tag{19}$$

with $\Delta\phi = \psi_0 - \phi$. This expression is the main result of our paper. The first term in Eq. (19) is well known (see, for example, [19,21]). The originality of our work is in the appearance of the new multiplicative second term, which is a function of the frequency via the parameter $\kappa(\Omega)$. An inspection of the last equation in Eq. (18) shows that $\kappa(\Omega) = \frac{1}{6}\beta_3\Omega^3 z$ is proportional to β_3 and vanishes with it. This parameter provides a measure of the asymmetry in the spectrum that is widely observed in fiber systems when third-order dispersion has to be accounted for.

In Fig. 1, we have shown the phase-sensitive gain $G_{\mathrm{PSA}}(\varphi)$ as a function of the detuning frequency Ω_0 for different values of the relative phase φ and nonzero third-order dispersion coefficient β_3. One can clearly observe the asymmetry of these curves with respect to the frequency for the case of nonzero φ. It is worth noting that when $\varphi = 0$, the gain spectrum remains symmetric with respect to Ω_0 even for $\beta_3 \neq 0$. The explanation is easily seen from Eq. (19). Indeed, for $\varphi = 0$ the second factor becomes $\cos^2(\kappa)$ and remains symmetric with respect to frequency even for a nonzero third-order dispersion coefficient. We have compared the analytical expression given by Eq. (19) with the parametric gain obtained from numerical integration of the governing Eq. (1). The numerical results

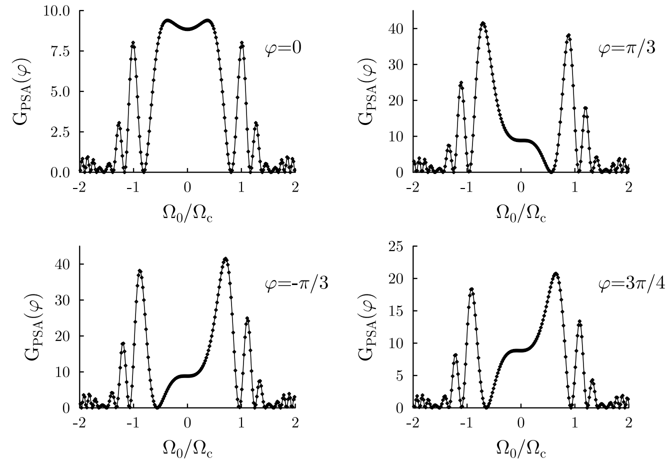

FIG. 1. Phase-sensitive gain $G_{\mathrm{PSA}}(\varphi)$ as a function of dimensionless frequency Ω_0/Ω_c for four different values of the phase φ: $\varphi = 0$, $\pi/3$, $-\pi/3$, and $3\pi/4$. The pump power $P_0 = 1$ W, the nonlinear coefficient $\gamma = 0.002\ \mathrm{W^{-1}m^{-1}}$, the fiber length $L = 700$ m, and the second- and third-order dispersion coefficients are $\beta_2 = -1 \times 10^{-28}\ \mathrm{s^2/m}$ and $\beta_3 = 1 \times 10^{-40}\ \mathrm{s^3/m}$. Solid curves are drawn using Eq. (19); diamonds correspond to numerical simulations of the NLSE.

are displayed in Fig. 1 by the diamonds. One can see that the agreement between the two results is excellent.

In Fig. 2, we have demonstrated the role of the third-order dispersion parameter β_3 on the parametric gain by plotting on the same graph the parametric gain spectra for β_3 that are different from zero (red) and equal to zero (black). We have chosen two different values of φ: $\varphi = 0$ and $\varphi = \pi/3$. In the first case, $\varphi = 0$, both curves are symmetric with respect to the detuning frequency in spite of the presence of β_3. The most striking feature in this case is that the picks of gain oscillations are limited by the gain curve corresponding to $\beta_3 = 0$. However, the curve with $\beta_3 \neq 0$ shows more oscillations due to the term $\kappa(\Omega) = \frac{1}{6}\beta_3\Omega^3 z$. Thus, even for $\varphi = 0$ the role of the third-order dispersion is very significant, since the maximum gain is localized around some specific frequency bands.

For $\varphi = \pi/3$, the presence of β_3 enhances the gain for all amplified frequency bands. Most important, the maximum peak gain can be several times (four times in Fig. 2) higher than the standard MI gain (black curve). This greatly contrasts with the case $\varphi = 0$ and opens the possibility for achieving high-gain noiseless amplification.

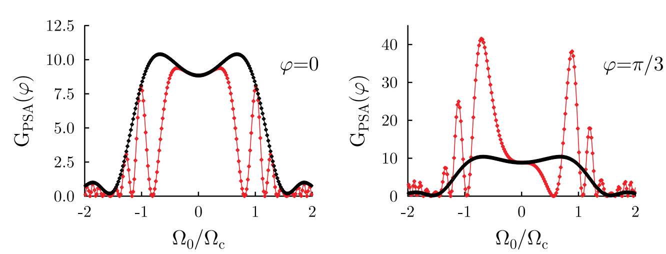

FIG. 2. (Color online) Phase-sensitive gain $G_{\mathrm{PSA}}(\varphi)$ as a function of dimensionless frequency Ω_0/Ω_c for $\beta_3 = 1 \times 10^{-40}\ \mathrm{s^3/m}$ (red curve) and $\beta_3 = 0$ (black curve with single minimum). All other parameters are as in Fig. 1.

In summary, our analysis establishes the important impact of the third-order dispersion on MI, resulting in a drastic change in the phase-sensitive gain spectrum. Our results are not specific to the third-order dispersion and can be extended to the higher order odd-dispersion terms. Because our solution is valid only at the initial stages of wave propagation in a fiber, it should be considered as a precursor for longer propagation distances.

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