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Quasistatic limit of the strong-field approximation describing atoms in intense laser fields: Circular polarization

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In the recent work of Vanne and Saenz [Phys. Rev. A **75**, 063403 (2007)] the quasistatic limit of the velocity gauge strong-field approximation describing the ionization rate of atomic or molecular systems exposed to linearly polarized laser fields was derived. It was shown that in the low-frequency limit the ionization rate is proportional to the laser frequency ω (for a constant intensity of the laser field). In the present work I show that for circularly polarized laser fields the ionization rate is proportional to ω^4 for H($1s$) and H($2s$) atoms, to ω^6 for H($2p_x$) and H($2p_y$) atoms, and to ω^8 for H($2p_z$) atoms. The analytical expressions for asymptotic ionization rates (which become nearly accurate in the limit $\omega \to 0$) contain no summations over multiphoton contributions. For very low laser frequencies (optical or infrared) these expressions usually remain with an order-of-magnitude agreement with the velocity gauge strong-field approximation.

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The name "the strong-field approximation" (SFA) is frequently used to call one of two well-known versions of the S-matrix theory which describes nonresonant multiphoton ionization of atoms and ions in intense laser fields [1,2]. In principle, this name (SFA) fits well to both theories where the main approximation is connected with the use of the Gordon-Volkov wave function [3,4] as a final state of an outgoing electron. The basic difference between these two versions of the S-matrix theory is the Hamiltonian form of the laser-atom interaction. Keldysh used it in the length gauge (LG) [1], while Reiss used this Hamiltonian in the velocity gauge (VG) [2]. The common feature of both approaches [1,2] is the application of nonrelativistic and dipole approximations to a description of atoms (or ions) in intense laser fields. In superstrong laser fields first nondipole (i.e., connected with a magnetic-field component of an electromagnetic plane wave) and then relativistic effects have to be taken into account [5–8]. Let us note that the magnetic-field component of the strong but nonrelativistic laser field is less essential for a circular polarization (CP) than for a linear polarization (LP) of the field [7,8]. This is because a classical free electron in the circularly polarized plane-wave electromagnetic field, even in the fully relativistic regime, moves along a circle lying in the polarization plane (in the simplest frame of reference, see Sec. 48, p. 134 of Ref. [9] and Ref. [7]). In contrast, in the fully relativistic linearly polarized plane-wave field the motion takes place along the "figure-8" path in the plane determined by the polarization direction and the propagation direction (in the simplest frame of reference, see Refs. [6–9]). However, even in the case of the LP, nonrelativistic nondipole tunneling theories [5,10,11] proved correct in experiments for higher laser intensities than expected [12,13].

In the recent work of Vanne and Saenz [14] the quasistatic limit of the VG SFA in linearly polarized laser fields was derived. It appears that in the quasistatic limit ($\omega \to 0$) the ionization rate is proportional to the laser frequency. Naturally the question then arises how the VG SFA ionization rate behaves in the quasistatic limit when the laser field is circularly polarized. This is the main purpose of my present work. The expressions which describe ionization rates in the VG SFA were given in Ref. [2] [for the H($1s$) atom] and in Ref. [15] [for the H($1s$), H($2s$), H($2p_x$), H($2p_y$), and H($2p_z$) atoms]. Of course the nonrelativistic Gordon-Volkov wave function was used to derive the ionization rates in Refs. [2,15]. On the other hand, in the works of Krainov and Shokri [16,17] an improved low-frequency theory was presented. It is based on multiplying the Gordon-Volkov wave function by the quasiclassical Coulomb correction factor of $(2Z^2/n^2Fr)^n$ (Z is the nuclear charge, n is the effective principal quantum number of the initial atomic or ionic state, F is the amplitude of the electric field vector of the laser, and r is the distance between the electron and the nucleus). [In the present work I use atomic units (a.u.): $\hbar = e = m_e = 1$, and I substitute explicitly -1 for the electronic charge.] The results of Refs. [16,17] are valid not only when

$$F < F_{\mathrm{BSI}} = Z^3/16n^4, \tag{1}$$

where F_{BSI} is the so-called barrier-suppression field strength {see, e.g., Eqs. (4) and (5) in Ref. [16]}. F_{BSI} is a critical field strength above which the atom should, in principle, ionize immediately in the quasistatic limit. Equation (1) gives $F_{\mathrm{BSI}} = 0.0625$ a.u. for the H($1s$) atom. Another critical field strength for the H($1s$) atom was given in Refs. [18,19]. Namely, it was $F_{\mathrm{cr}} \approx 0.15$ a.u., which seems to be a more realistic value. For a sufficiently low frequency of the laser, the Keldysh parameter γ [1] obeys

$$\gamma = Z\omega/nF \ll 1. \tag{2}$$

I will assume that Eqs. (1) and (2) are satisfied, although my numerical results [20] (see also Figs. 1–4 here) indicate that for $F \geqslant F_{\mathrm{BSI}}$ the new formulas are also useful if ω is sufficiently low.

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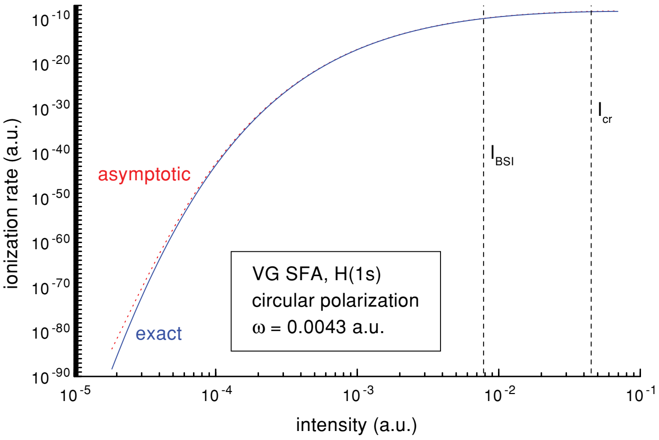

FIG. 1. (Color online) The VG SFA ionization rates of the H(1s) atom in the circularly polarized laser field for $\omega = 0.0043$ a.u. vs intensity of the field (see text for more detail).

The present notation resembles the one from Refs. [16,17]. In this notation Eqs. (A.11)–(A.14) from Ref. [15] take the form

$$\Gamma_{1s} = \sum_{N=N_0}^{\infty} a_N \int_0^{\pi} d\vartheta \, \sin\vartheta \, J_N^2(p_N F \sin\vartheta/\omega^2), \tag{3}$$

$$\Gamma_{2s} = \sum_{N=N_0}^{\infty} b_N \int_0^{\pi} d\vartheta \, \sin\vartheta \times J_N^2(p_N F \sin\vartheta/\omega^2), \tag{4}$$

$$\Gamma_{2p_x} = \Gamma_{2p_y} = \sum_{N=N_0}^{\infty} c_N \int_0^{\pi} d\vartheta \, \sin^3\vartheta \times J_N^2(p_N F \sin\vartheta/\omega^2), \tag{5}$$

$$\Gamma_{2p_z} = \sum_{N=N_0}^{\infty} 2c_N \int_0^{\pi} d\vartheta \, \sin\vartheta \, \cos^2\vartheta \times J_N^2(p_N F \sin\vartheta/\omega^2), \tag{6}$$

where $a_N = 8Z^5 p_N/d_N$, $b_N = Z^5 p_N(-2E_B + p_N^2)^2/d_N^2$, $c_N = Z^7 p_N^3/(2d_N^2)$, $d_N = (2E_B + p_N^2)^2$ and $p_N = \sqrt{2(N\omega - U_P - E_B)}$ is the asymptotic momentum of the outgoing electron. $E_B = Z^2/(2n^2)$ is the binding energy of the initial state of the atom, $U_P = F^2/(2\omega^2)$ is the ponderomotive potential, N is the number of photons absorbed, and $N_0 = [(U_P + E_B)/\omega] + 1$ is the minimal value of N. {The symbol $[x]$ denotes the integer part of x. For the laser field intensity I and the CP the following relation holds: $I = 2F^2$.} It appears that in each of Eqs. (3)–(6) the main contribution to the respective sum comes from the terms where the final (asymptotic) kinetic energy ($E_N = p_N^2/2$) of the outgoing electron is close to U_P. For the ordinary Bessel function J_N [from Eqs. (3)–(6)] the following asymptotic expansion will be used [21] [Eq. (9.3.2)]:

$$J_N\left(\frac{N}{\cosh\alpha}\right) \approx (2\pi N \tanh\alpha)^{-1/2} \exp\left[N(\tanh\alpha - \alpha)\right], \tag{7}$$

which is valid for $\alpha > 0$. [I have numerically verified that both Eqs. (7) and (8) become more and more accurate as $\omega \to 0$. Then $\alpha \to 0$ and $N \to \infty$, but $N\alpha^3 \to$ const for those values of N which give the largest contribution to the sum over N.] One substitutes $N/\cosh\alpha = p_N F \sin\vartheta/\omega^2$. Then in Eq. (7)

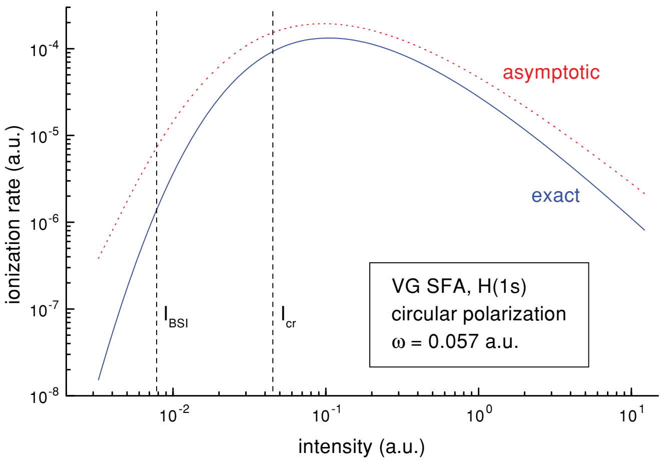

FIG. 2. (Color online) As Fig. 1 but for $\omega = 0.057$ a.u.

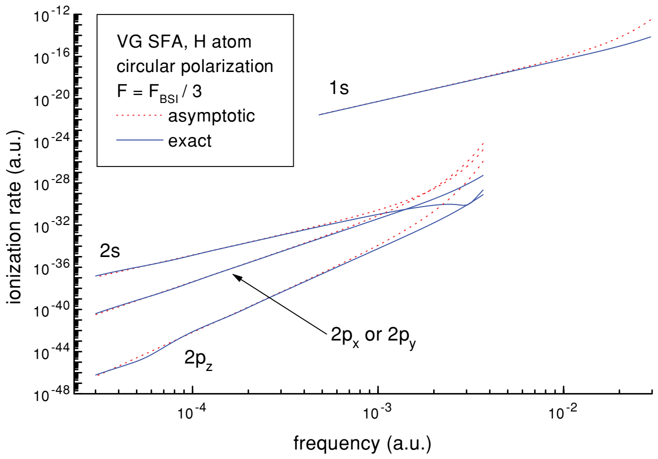

FIG. 3. (Color online) The VG SFA ionization rates of the H atom in the circularly polarized laser field for $F = F_{\mathrm{BSI}}/3$ vs ω. The data shown in the plot are for four different initial states of this atom, namely 1s, 2s, 2p_x or 2p_y, and 2p_z. (For $n = 1$, $F_{\mathrm{BSI}} = 0.0625$ a.u., and for $n = 2$, $F_{\mathrm{BSI}} = 0.00391$ a.u.)

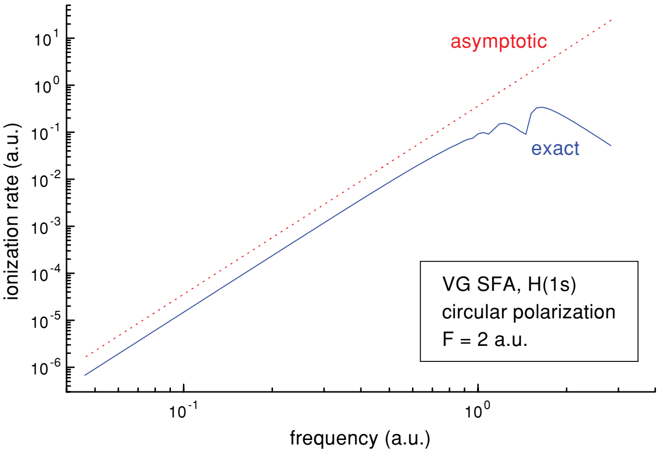

FIG. 4. (Color online) The VG SFA ionization rates of the H(1s) atom in the circularly polarized laser field for $F = 2$ a.u. vs ω.

one applies the Taylor expansion to the argument of J_N and the argument of an exponential function, retaining each time two nonzero terms of the lowest order. After solving the resulting algebraic equation for the small quantity α, one substitutes this α to the exponent in Eq. (7). Since $\alpha \ll 1$ and $\alpha \approx \gamma$ assumption (2) is important. In the pre-exponential factor the approximation $\tanh \alpha \approx \alpha$ is sufficient. (For more detail see Sec. 4 of Ref. [17].) As a result, one obtains

$$J_N^2\left(\frac{p_N F \sin\vartheta}{\omega^2}\right) \approx \frac{n\omega}{2\pi Z p_N} \exp\left[-\frac{2Z^3}{3n^3F}\left(1-\frac{\gamma^2}{15}\right) - \frac{ZF}{n\omega^2}\left(\frac{\pi}{2}-\vartheta\right)^2 - \frac{Z\omega^4}{nF^3}\delta N'^2\right], \tag{8}$$

where $\delta N' = \delta N - F^2/(6n^2\omega)$ and $\delta N = N - F^2/\omega^3 - E_B/\omega$. When Eqs. (1) and (2) are satisfied, angular distributions of outgoing electrons are strongly peaked at $\vartheta = \pi/2$, that is, electrons are mainly emitted in the polarization plane. Equation (8) can be used now in Eqs. (3)–(6), leading to simple Gaussian integrals upon $\vartheta' = \pi/2 - \vartheta$, where only a very narrow vicinity of $\vartheta' = 0$ matters. Therefore, in the integrals upon ϑ' it is enough to take into account only the first nonzero term of the Taylor expansion of trigonometric functions, which appear in the integrands from Eqs. (3)–(6). For the H($1s$) atom (one puts $n = 1$) the respective integral is

$$\int_0^\pi d\vartheta \sin\vartheta\, J_N^2\left(\frac{p_N F \sin\vartheta}{\omega^2}\right) \approx \frac{\omega^2}{2\sqrt{\pi Z^3 F}\, p_N} \exp\left[-\frac{2Z^3}{3F}\left(1-\frac{\gamma^2}{15}\right) - \frac{Z\omega^4}{F^3}\delta N'^2\right]. \tag{9}$$

For H($2s$), H($2p_x$) [or H($2p_y$)], and H($2p_z$) atoms (one puts $n = 2$) the respective integrals are

$$\int_0^\pi d\vartheta \sin\vartheta\, J_N^2\left(\frac{p_N F \sin\vartheta}{\omega^2}\right) \approx \frac{\sqrt{2}\omega^2}{\sqrt{\pi Z^3 F}\, p_N} \exp\left[-\frac{Z^3}{12F}\left(1-\frac{\gamma^2}{15}\right) - \frac{Z\omega^4}{2F^3}\delta N'^2\right], \tag{10}$$

$$\int_0^\pi d\vartheta \sin^3\vartheta\, J_N^2\left(\frac{p_N F \sin\vartheta}{\omega^2}\right) \approx \frac{\sqrt{2}\omega^2}{\sqrt{\pi Z^3 F}\, p_N} \exp\left[-\frac{Z^3}{12F}\left(1-\frac{\gamma^2}{15}\right) - \frac{Z\omega^4}{2F^3}\delta N'^2\right], \tag{11}$$

$$\int_0^\pi d\vartheta \sin\vartheta \cos^2\vartheta\, J_N^2\left(\frac{p_N F \sin\vartheta}{\omega^2}\right) \approx \frac{\sqrt{2}\omega^4}{\sqrt{\pi Z^5 F^3}\, p_N} \exp\left[-\frac{Z^3}{12F}\left(1-\frac{\gamma^2}{15}\right) - \frac{Z\omega^4}{2F^3}\delta N'^2\right]. \tag{12}$$

Substituting Eqs. (9)–(12) in Eqs. (3)–(6), respectively, one obtains asymptotic ionization rates in the form $\Gamma^{\text{asympt}} = \sum_{N=N_0}^{\infty} \Gamma_N(Z,\omega,F)$, where $\Gamma_N(Z,\omega,F)$ denote partial ionization rates corresponding to absorption of exactly N photons. When $F = \text{const}$ and $\omega \to 0$ one obtains that $N_0 \to \infty$, but one can change the index of summation to $\delta N'$ [as defined right below Eq. (8)]. In the limit $\omega \to 0$ $\delta N'$ changes from $-\infty$ to $+\infty$, but Eqs. (9)–(12) clearly show that the main contribution to each sum comes from terms with $\delta N' \approx 0$. Furthermore, one can transform each sum over $\delta N'$ to a Gaussian integral. Since p_N is large and changes much slower with $\delta N'$ than the exponential factor, one can put $p_N^2 \approx F^2/\omega^2 \gg E_B$ and neglect E_B in the resulting integrand. This makes the integration upon $\delta N'$ trivial. Finally, one obtains the following asymptotic expressions describing the VG SFA ionization rates:

$$\Gamma_{1s}^{\text{asympt}} = \frac{4Z^3\omega^4}{F^3} \exp\left[-\frac{2Z^3}{3F}\left(1-\frac{\gamma^2}{15}\right)\right], \tag{13}$$

$$\Gamma_{2s}^{\text{asympt}} = \frac{Z^3\omega^4}{F^3} \exp\left[-\frac{Z^3}{12F}\left(1-\frac{\gamma^2}{15}\right)\right], \tag{14}$$

$$\Gamma_{2p_x}^{\text{asympt}} = \Gamma_{2p_y}^{\text{asympt}} = \frac{Z^5\omega^6}{2F^5} \exp\left[-\frac{Z^3}{12F}\left(1-\frac{\gamma^2}{15}\right)\right], \tag{15}$$

$$\Gamma_{2p_z}^{\text{asympt}} = \frac{Z^4\omega^8}{F^6} \exp\left[-\frac{Z^3}{12F}\left(1-\frac{\gamma^2}{15}\right)\right]. \tag{16}$$

Equations (13)–(16) are the main result of this work. For the H($1s$), H($2s$), and H($2p_z$) atoms their quantum numbers (n,l,m) are the following: (1,0,0), (2,0,0), and (2,1,0), respectively. Wave functions of the atoms H($2p_x$) and H($2p_y$) are linear combinations of wave functions (normalized to unity) with the quantum numbers (2,1,−1) and (2,1,1), namely

$$\Phi_{2p_x} = \frac{1}{\sqrt{2}}(\Phi_{2,1,-1} - \Phi_{2,1,1}), \qquad \Phi_{2p_y} = \frac{i}{\sqrt{2}}(\Phi_{2,1,-1} + \Phi_{2,1,1}). \tag{17}$$

One can show [20] that in the VG SFA the ionization rate is the same for all the initial states given by

$$\Phi_{2p} = \alpha\Phi_{2,1,-1} + \beta\Phi_{2,1,1}, \tag{18}$$

where α and β are arbitrary complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. In particular one has

$$\Gamma_{2,1,-1}^{\text{asympt}} = \Gamma_{2,1,1}^{\text{asympt}} = \Gamma_{2p_x}^{\text{asympt}} = \Gamma_{2p_y}^{\text{asympt}}. \tag{19}$$

When the electron is initially bound ($E_B = \kappa^2/2$) in the zero-range potential with the initial-state wave function (normalized to unity) given by

$$\Phi(\vec{r}) = \sqrt{\kappa/2\pi}\exp(-\kappa r)/r, \tag{20}$$

the ionization rate for the circularly polarized laser field in the VG SFA is [2]

$$\Gamma_{\text{zero-range}} = \sum_{N=N_0}^{\infty} \kappa p_N \int_0^\pi d\vartheta \sin\vartheta\, J_N^2\left(p_N F \sin\vartheta/\omega^2\right). \tag{21}$$

If one utilizes Eq. (9) (with $Z = \kappa$) in Eq. (21), one finally obtains

$$\Gamma^{\text{asympt}}_{\text{zero-range}} = \frac{F}{2\kappa} \exp\left[-\frac{2\kappa^3}{3F}\left(1 - \frac{\gamma^2}{15}\right)\right]. \tag{22}$$

The result (22) is in an agreement with the old result for the ionization in a static electric field [22] if one puts $\gamma = 0$ in the last equation (what corresponds to $\omega = 0$). Therefore, for the zero-range binding potential, the ionization rate in the circularly polarized laser field with $\omega \to 0$ is exactly the same as in the static field of the same amplitude F. This is not true in the VG SFA if the binding potential is the Coulomb one. For example, instead of the well-known [22–24] static-field expression for the H(1s) atom: $\Gamma_{\text{stat}} = (4Z^5/F)\exp(-2Z^3/3F)$, Eq. (13) for $\gamma = 0$ gives $\Gamma^{\text{asympt}}_{1s} \approx (4Z^3\omega^4/F^3)\exp(-2Z^3/3F)$. Only exponential factors are the same. The proportionality coefficient ω^4, which exists in Eq. (13), is the counterpart (for the CP) of the proportionality coefficient ω found formerly by Vanne and Saenz in Ref. [14] (for the LP). However, unlike for the LP, the power of ω in the coefficient depends on the (n,l,m) quantum numbers of the initial state for the CP [cf. Eqs. (13)–(16)]. As a result, for both polarizations of the laser field $I =$ const and $\omega \to 0$ lead to nulling of the ionization rate. The authors of Ref. [14] state that "This evidently unphysical result indicates a breakdown of the SFA." (see Sec. IV of Ref. [14]) and they propose an application of a Coulomb correction factor (based on the results of Ref. [10]) to the VG SFA ionization rate formula. For the H(1s) atom and the VG SFA such Coulomb-corrected theory has been recently proposed for both polarizations [25].

In Figs. 1–4 there are the VG SFA ionization rates as a function of intensity (Figs. 1 and 2) or frequency (Figs. 3 and 4) for the H atom in the strong circularly polarized laser field. In each plot I compare the exact and the asymptotic results. Solid lines correspond to the exact ones [from Eqs. (3)–(6)] and dotted lines correspond to the asymptotic ones [from Eqs. (13)–(16)]. In Figs. 1–4 the field parameters (ω, I) cover a total range of a validity of the nonrelativistic SFA. In Figs. 1 and 2 the laser frequencies are fixed. Both frequencies are of an experimental interest. $\omega = 0.0043$ a.u. conforms with CO$_2$ laser radiation ($\lambda = 10.6$ μm), and $\omega = 0.057$ a.u. conforms with Ti : sapphire laser radiation ($\lambda = 800$ nm). There are also two vertical lines (in each of Figs. 1 and 2) which show $I_{\text{BSI}} = 2F^2_{\text{BSI}}$ and $I_{\text{cr}} = 2F^2_{\text{cr}}$ for the H(1s) atom. The agreement between exact and asymptotic ionization rates particularly for the lower frequency $\omega = 0.0043$ a.u. is satisfactory. For $\omega = 0.057$ a.u. it is hard to obey both the conditions (1) and (2). Therefore, in the case of Fig. 2, Eq. (13) can be treated rather as an useful approximation (an upper bound) to Eq. (3). Nevertheless, if $F \sim F_{\text{BSI}}$ (or lower) is fixed, the agreement between exact and asymptotic results is very good for sufficiently low frequencies (see Fig. 3, where $F = F_{\text{BSI}}/3$). I have evaluated that in the limit $\omega \to 0$ a relative error introduced by the approximation (7) is of the order of $5n^3F/(24Z^3)$. {This estimation is based on Eqs. (9.3.7) and (9.3.9) from Ref. [21].} Therefore, if condition (1) is satisfied this error is of the order of 1% or less. In Fig. 4 $F = 2$ a.u., which is well above F_{BSI}. For $\omega \to 0$ there are two parallel lines in the log-log plot. This fact indicates that also for $F > F_{\text{BSI}}$ (in the limit $\omega \to 0$) the VG SFA ionization rate for the H(1s) atom is proportional to ω^4.

In conclusion, I have derived approximate formulas for the VG SFA ionization rate for the hydrogenic atom in the initial state with n either equal to 1 or 2. It appears that the respective ionization rate is proportional to ω^4 [for $(n,l,m) = (1,0,0)$ or $(2,0,0)$], to ω^6 [for $(2,1,\pm1)$], and to ω^8 [for $(2,1,0)$]. These asymptotic expressions become nearly exact in the quasistatic limit. For finite but low frequencies these expressions may be treated as very simple upper bounds to the exact VG SFA expressions. The latter become much more time consuming (in numerical calculations) in the infrared or far-infrared frequency regime.

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