Effects of periodic potentials on the critical velocity of superfluid Fermi gases in the BCS-BEC crossover

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We study the effects of an external periodic potential on the critical velocity of a superfluid Fermi gas in the crossover between the Bardeen-Cooper-Schrieffer (BCS) phase and Bose-Einstein condensation (BEC). We numerically solve the Bogoliubov–de Gennes equations to model a three-dimensional gas of ultracold atoms in the superfluid phase flowing through a one-dimensional optical lattice. We find that when the recoil energy is comparable to the Fermi energy, the presence of the periodic potential reduces the effect of pair-breaking excitations. This behavior is a consequence of the peculiar band structure of the quasiparticle energy spectrum in the lattice. When the lattice height is much larger than the Fermi energy, the periodic potential makes pairs of atoms to be strongly bound, even in the BCS regime, and pair-breaking excitations are further suppressed. We have also found that when the recoil energy is comparable to or larger than the Fermi energy, the critical velocity due to long-wavelength phonon excitations shows a nonmonotonic behavior along the BCS-BEC crossover.

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I. INTRODUCTION

Ultracold atom gases in optical lattices have been continuously attracting great interest for the last 10 years [1–3]. Recent developments in the field of ultracold atom gases provide a new research arena in the physics of quantum fluids: by using Feshbach resonances of ultracold Fermi atoms, one can study the crossover from the Bardeen-Cooper-Schrieffer (BCS) phase to a Bose-Einstein condensate (BEC) of molecules [4]. In the current research frontier, superfluidity of ultracold Fermi gases in optical lattices is a very intriguing problem which has interesting connections with similar issues in solid-state physics, nuclear physics, and astrophysics.

The critical velocity of superflow due to energetic instability is one of the most important properties of superfluids, an area of research pioneered by Landau [5]. If the velocity of superflow exceeds some critical value, the kinetic energy of the superfluid can be dissipated by creating excitations [5–8]. In uniform superfluid Fermi gases in the BCS-BEC crossover, excitations which cause the energetic instability are of two types: fermionic pair-breaking excitations in the BCS regime and long-wavelength phonon excitations in the BEC regime [9,10]. In the unitary regime both mechanisms are suppressed and the critical velocity shows a maximum value [9–11].

Recently, the effects of periodic potentials on the critical velocity of Fermi superfluids have been studied experimentally [12]. This experiment has stimulated theoretical investigations of this problem [13–16]. Most of them have focused on the BCS regime in tight-binding approximation [13–15]. The purpose of the present work is to obtain an understanding of the critical velocity from a unified point of view covering all regions along the BCS-BEC crossover and both the strong and weak lattice regime. To this purpose, we use the Bogoliubov–de Gennes (BdG) equations. This theory accounts for both types of excitations which are relevant in this problem. In our previous work [16] we used it for a gas at

unitarity; here we extend the calculations in order to explore the whole crossover region. As a main result, we find that when the lattice height is comparable to or much larger than the Fermi energy, the periodic potential reduces the effect of pair-breaking excitations. This is due to the periodic structure of the quasiparticle energy spectrum in the Brillouin zone and the formation of the bound molecules induced by the lattice. Another main result is that when the recoil energy is comparable to or larger than the Fermi energy, the critical velocity due to long-wavelength phonon excitations shows a nonmonotonic behavior along the BCS-BEC crossover. These effects are unique for Fermi superfluids in periodic potentials and do not exist in the case of single barrier potentials [16–18].

This paper is organized as follows. In Sec. II we explain the basic formalism employed in the present work, and we show the results in Sec. III. Finally, a summary and outlook are given in Sec. IV.

II. BASIC FORMALISM

We want to study the effect of the periodic potential on the Landau critical velocity of Fermi superfluids in the whole BCS-BEC crossover in situations where the Fermi energy is larger or smaller than the lattice height. Toward this end, we need to use a theoretical framework which can account for the formation of bound molecules induced by the periodic potential, which is important when the lattice height is larger than the Fermi energy [19,20]; the same formalism must also account for pair-tunneling processes, which are important on the BEC side of the resonance [20,21]. A suitable approach consists of the numerical solution of the BdG equations [22]:

$$\begin{pmatrix} H'(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H'(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \epsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}, \quad (1)$$

where u_i and v_i are quasiparticle amplitudes and ϵ_i the corresponding eigenenergies. The single-particle Hamiltonian

is $H'(\mathbf{r}) = -\hbar^2\nabla^2/2m + V_{\mathrm{ext}} - \mu$, where m is the atom mass and $V_{\mathrm{ext}}(\mathbf{r})$ is the external potential. The order parameter (or gap parameter) $\Delta(\mathbf{r})$ and the chemical potential μ, appearing in Eq. (1), are variational parameters determined from the gap equation,

$$\Delta(\mathbf{r}) = -g \sum_i u_i(\mathbf{r})v_i^*(\mathbf{r}), \tag{2}$$

together with the constraint

$$n_0 = \frac{2}{V} \sum_i \int |v_i(\mathbf{r})|^2 \, d\mathbf{r}, \tag{3}$$

enforcing the conservation of the average density n_0. Here g is the coupling constant for the contact interaction and V is the volume of the system. The BdG eigenfunctions obey the normalization condition $\int d^3r[u_i^*(\mathbf{r})u_j(\mathbf{r}) + v_i^*(\mathbf{r})v_j(\mathbf{r})] = \delta_{i,j}$. Finally, the energy density e can be calculated as

$$e = \frac{1}{V} \int d\mathbf{r} \sum_i [2(\mu - \epsilon_i)|v_i(\mathbf{r})|^2 + \Delta^*(\mathbf{r})u_i(\mathbf{r})v_i^*(\mathbf{r})]. \tag{4}$$

In the present study we consider a three-dimensional (3D) superfluid Fermi gas which is uniform in the x and y directions and subject to a one-dimensional (1D) optical lattice along z:

$$V_{\mathrm{ext}}(z) = s E_{\mathrm{R}} \sin^2 q_{\mathrm{B}} z \equiv V_0 \sin^2 q_{\mathrm{B}} z \ . \tag{5}$$

Here $V_0 \equiv s E_{\mathrm{R}}$ is the lattice height, s is the laser intensity in dimensionless units, $E_{\mathrm{R}} = \hbar^2 q_{\mathrm{B}}^2/2m$ is the recoil energy, $q_{\mathrm{B}} = \pi/d$ is the Bragg wave vector, and d is the lattice constant. For practical reasons, throughout this paper we set $s = 1$ except for special cases, which we mention explicitly. The ratio between the Fermi energy and the lattice height is then varied by changing the average density of the gas.

In the presence of a supercurrent with wave vector $Q = P/\hbar$ moving in the direction of the periodic potential, one can write the gap parameter in the form

$$\Delta(\mathbf{r}) = e^{i2Qz}\tilde{\Delta}(z), \tag{6}$$

where $\tilde{\Delta}(z)$ is a complex function with period d. Therefore, from the gap equation we see that the eigenfunctions of Eq. (1) must have the Bloch form $u_i(\mathbf{r}) = \tilde{u}_i(z)e^{iQz}e^{i\mathbf{k}\cdot\mathbf{r}}$ and $v_i(\mathbf{r}) = \tilde{v}_i(z)e^{-iQz}e^{i\mathbf{k}\cdot\mathbf{r}}$. The wave vector k_z lies in the first Brillouin zone, and $\tilde{u}_i$ and $\tilde{v}_i$ are periodic in z with period d. We also define the quasimomentum P_{edge} and quasiwavenumber Q_{edge} at the edge of the Brillouin zone as $P_{\mathrm{edge}} = \hbar Q_{\mathrm{edge}} \equiv \hbar q_{\mathrm{B}}/2$. This Bloch decomposition transforms Eq. (1) into the following BdG equations for $\tilde{u}_i$ and $\tilde{v}_i$:

$$\begin{pmatrix} \tilde{H}'_Q(z) & \tilde{\Delta}(z) \\ \tilde{\Delta}^*(z) & -\tilde{H}'_{-Q}(z) \end{pmatrix} \begin{pmatrix} \tilde{u}_i(z) \\ \tilde{v}_i(z) \end{pmatrix} = \epsilon_i \begin{pmatrix} \tilde{u}_i(z) \\ \tilde{v}_i(z) \end{pmatrix}, \tag{7}$$

where

$$\tilde{H}'_Q(z) \equiv \frac{\hbar^2}{2m}\left[k_x^2 + k_y^2 + (-i\,\partial_z + Q + k_z)^2\right] + V_{\mathrm{ext}}(z) - \mu. \tag{8}$$

Here, the label i represents the wave vector $\mathbf{k}$ as well as the band index. In order to remove the ultraviolet divergences in the BdG equations with contact potentials, we use the regularization scheme proposed by Refs. [23,24]. Since we

need to calculate the second derivatives of the energy with respect to the density and the quasimomentum, we use large values of the cutoff energy E_C, especially in the BEC side, where the size of the pair is much smaller than the average interatomic distance (details are described in Sec. III).

As discussed in Refs. [9,10], the energetic instability of superfluids of dilute Fermi gases can be caused by two processes [25]: the creation of long-wavelength superfluid phonon excitations or fermionic pair-breaking excitations. The critical velocity by the former process can be determined by the hydrodynamic analysis of the excitations [7,16,28–30]. Starting from the continuity equation and the Euler equation and linearizing with respect to the perturbations of the density and the velocity fields, we obtain the dispersion relation of the long-wavelength phonon,

$$\omega(q) = \frac{\partial^2 e}{\partial n_0 \partial P} \, q + \sqrt{\frac{\partial^2 e}{\partial n_0^2} \frac{\partial^2 e}{\partial P^2}} \, |q|. \tag{9}$$

Here, $\hbar\omega$ and q are the energy and the wavenumber of the excitations, and n_0 and P are the average density and the quasimomentum of the superfluids. The energetic instability occurs when $\omega(q)$ becomes negative:

$$\frac{\partial^2 e}{\partial n_0 \partial P} = \sqrt{\frac{\partial^2 e}{\partial n_0^2} \frac{\partial^2 e}{\partial P^2}}. \tag{10}$$

In practice, we calculate the energy density $e(n_0, P)$ for a given n_0 and P from Eq. (4) using the solution of the BdG equations (7). Then the critical quasimomentum P_c at which the energetic instability occurs is determined by Eq. (10) [31]. We finally obtain the critical velocity v_c from

$$v_c = \frac{1}{n_0}\left(\frac{\partial e}{\partial P}\right)_{P_c}. \tag{11}$$

On the other hand, the critical velocity due to the pair-breaking fermionic excitations can be determined by looking at the quasiparticle energy spectrum ϵ_i. The energetic instability by the pair-breaking excitations occurs when some quasiparticle energy ϵ_i starts to be negative:

$$\epsilon_i \leqslant 0. \tag{12}$$

From Eq. (11), evaluated at the critical quasimomentum determined by this condition, we obtain a critical velocity for the pair-breaking excitations. The actual critical velocity of the system is the lowest between the ones obtained from the above two conditions [32]. We finally note that the gas becomes unstable also when some excitation energy starts to have a nonzero imaginary part. This corresponds to a dynamical instability which causes an exponential growth of the amplitude of the perturbation. To address the problem of dynamical instability, short-wavelength bosonic excitations also should be included. This is beyond the scope of the present work, in which we instead focus on the energetic instability. Results of the critical velocity for dynamical instability due to long-wavelength excitations are given in the Appendix.

III. RESULTS

We study the three cases of $E_{\mathrm{F}}/E_{\mathrm{R}} = 2.5$, 1, and 0.1 with a fixed value of $s = 1$, except for a few cases which
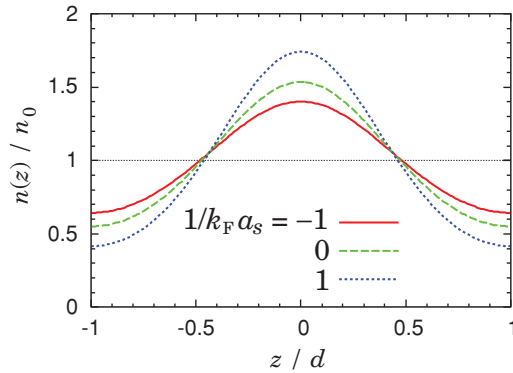
FIG. 1. (Color online) Density profile $n(z)$ at $P = 0$ in the BCS-BEC crossover for $E_F/E_R = 1$ and $s = 1$. Here, $n(z)$ is normalized by the average density n_0.

we shall mention explicitly. Here $E_F = \hbar^2 k_F^2/(2m)$ and $k_F = (3\pi^2 n_0)^{1/3}$ are the Fermi energy and momentum, respectively, of a uniform noninteracting Fermi gas of density n_0. For each value of E_F/E_R, we solve the BdG equations for several values of the parameter $1/k_F a_s$ along the crossover from the BCS to the BEC side, namely, $1/k_F a_s = -1$, -0.5, 0, 0.5, and 1, where a_s is the s-wave scattering length of atoms.

In the x and y directions, we assume periodic boundary conditions with period $L_\perp = 12\pi k_F^{-1}$. We set the cutoff energy E_C as follows: for $E_F/E_R = 2.5$, $E_C = 40E_F$ in the BCS side and $E_C = 100E_F$ at unitarity and in the BEC side; for $E_F/E_R = 1$, $E_C = 50E_F$ in the BCS side and $E_C = 100E_F$ at unitarity and in the BEC side; for $E_F/E_R = 0.1$, $E_C = 250E_F$ in the BCS side, $E_C = 350E_F$ at unitarity, and $E_C = 500E_F$ in the BEC side.

A. Density profiles and gap parameter

In Fig. 1 we show the density profile $n(z) = 2 \sum_i |\tilde{v}_i(z)|^2$ of Fermi atoms at $P = 0$ along the BCS-BEC crossover for $E_F/E_R = 1$. Moving from the BCS regime ($1/k_F a_s = -1$) to the BEC regime ($1/k_F a_s = 1$), the density $n(z)$ becomes more inhomogeneous. This behavior is consistent with the fact that the compressibility of a uniform Fermi gas is known to increase monotonically with $1/k_F a_s$. We observe the same qualitative behavior for the other values of E_F/E_R.

In Fig. 2 we show the amplitude $|\Delta(z)|$ of the order parameter at $P = 0$ for different values of $1/k_F a_s$ and E_F/E_R. From this figure one can see that, especially in the BCS regime ($1/k_F a_s = -1$), the order parameter $|\Delta|$ is enhanced when the Fermi energy is smaller than the lattice strength, as shown by the blue curves for $E_F/E_R = 0.1$ (i.e., $E_F/V_0 = 0.1$). We understand this fact as due to the formation of bosonic molecules induced by the external periodic potential. This process is indeed expected to become significant when the lattice is strong [20].

B. Critical velocity

Before presenting the numerical results for the critical velocity, let us discuss the conditions upon which the periodic potential can produce significant effects on the behavior of long-wavelength phonons and pair-breaking excitations. For long-wavelength phonons the condition is $E_F/E_R \gtrsim 1$ or $s \gg 1$. In fact, in the opposite case, $E_F/E_R \ll 1$ and $s \lesssim 11$,
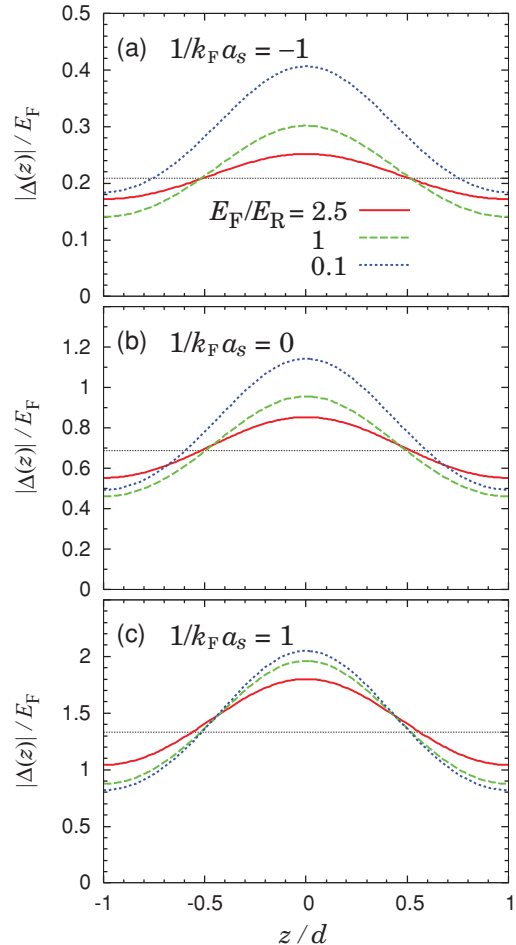

FIG. 2. (Color online) Amplitude $|\Delta(z)|$ of the gap parameter at $P = 0$ in the BCS-BEC crossover: $1/k_F a_s = -1$ (a), 0 (b), and 1 (c). The horizontal dotted lines show the amplitude of the gap parameter for the uniform system at the same value of $1/k_F a_s$.

the right-hand side of Eq. (10), which coincides with the sound speed at $P = 0$, is much smaller than the maximum value of the left-hand side of Eq. (10), which is of order q_B/m (see also Fig. 5 in Ref. [16]). Thus Eq. (10) is satisfied at $P \simeq 0$ and the critical velocity is basically determined by the sound speed at $P = 0$, which is also close to the sound speed in the uniform system. Consequently, the critical velocity due to long-wavelength phonons is almost unaffected by the presence of the lattice if $E_F/E_R \ll 1$ and $s \lesssim 1$, even though the lattice height V_0 is large compared to the Fermi energy E_F. On the other hand, the sufficient condition for pair-breaking excitations to be affected by the periodic potential due to the formation of bound molecules is $E_F/V_0 \ll 1$. This condition can be satisfied either by decreasing E_F/E_R or by increasing s. When $E_F/E_R \simeq 1$, the critical velocity for pair-breaking excitations is also affected by a peculiar band structure of the quasiparticle spectrum.

Our results for the critical velocity v_c are shown in Fig. 3 for $s = 1$. Let us first concentrate on the results at high density, $E_F/E_R = 2.5$, in panel (a). The open circles correspond to the critical velocity for long-wavelength phonons, which exhibits a nonmonotonic behavior. In particular, in the BCS regime (negative $1/k_F a_s$) this critical velocity is strongly reduced
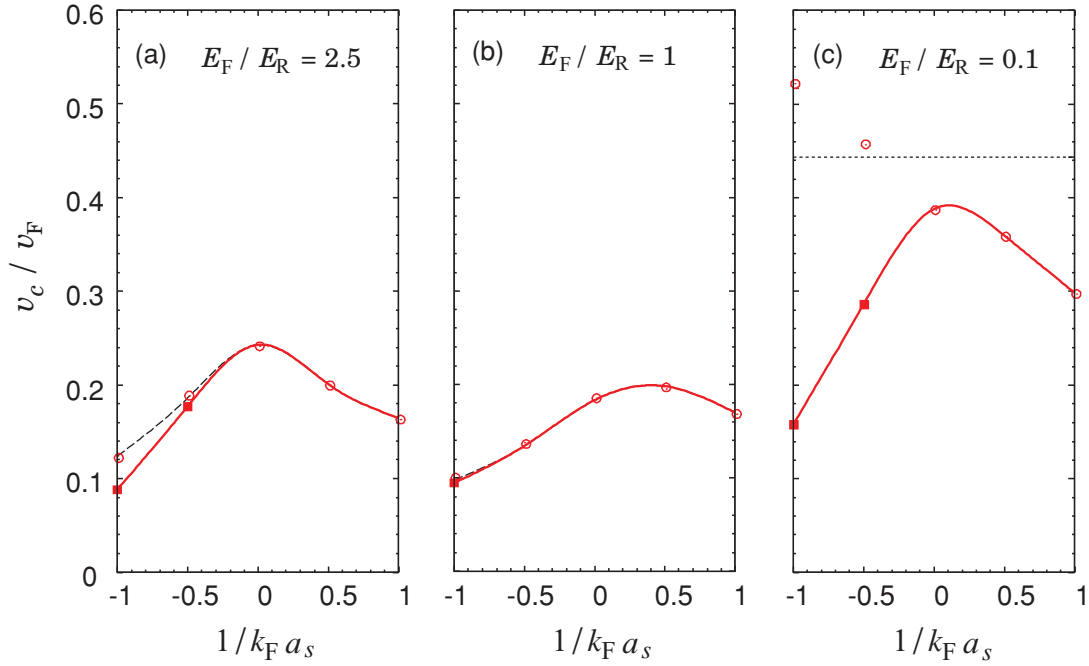
FIG. 3. (Color online) Critical velocity v_c of the energetic instability for $E_F/E_R = 2.5$ (a), 1 (b), and 0.1 (c) with $s = 1$ in the BCS-BEC crossover. Open circles and filled squares show the critical velocity due to long-wavelength phonons and fermionic pair-breaking excitations, respectively. The horizontal dotted line in panel (c) represents the value of the sound velocity $c_s^{(0)}$ of uniform system at unitarity, $c_s^{(0)}/v_F = (1 + \beta)^{1/2}/\sqrt{3} \simeq 0.443$. The red solid lines and the black dashed lines are guides to the eye.

compared to the one in a uniform gas (see, e.g., Fig. 8 in Ref. [10]). This is a peculiar effect of the lattice. However, for the parameters of Fig. 3(a), the actual critical velocity in the BCS regime is still given by fermionic pair-breaking excitations (filled squares). The latter are almost unaffected by the lattice and therefore, the overall behavior of the critical velocity in the crossover is qualitatively similar to that of a uniform gas, already discussed in Ref. [10]. Namely, in the BCS regime v_c increases when approaching unitarity ($1/k_F a_s = 0$), because the intrapair attraction becomes stronger and thus the amplitude of the gap parameter increases; in the opposite BEC regime ($1/k_F a_s > 0$) the critical velocity is given by long-wavelength phonons and an increase of the interpair repulsion leads to a larger sound speed and, again, a critical velocity v_c increases toward unitarity. As a consequence, v_c takes a maximum value at $1/k_F a_s \simeq 0$.

When the recoil energy is comparable to the Fermi energy, the periodic potential causes qualitative changes in the results of the critical velocity. For $E_F/E_R = 1$ [Fig. 3(b)] we observe that at $1/k_F a_s = -0.5$, the critical velocity is given by long-wavelength phonon excitations rather than pair-breaking excitations even in the BCS regime. We understand this effect as mainly due to a peculiar band structure of the quasiparticle energy spectrum. In Fig. 4 we show the lowest band of the quasiparticle energy spectrum ϵ_i for the first radial branch with $k_\perp^2 \equiv k_x^2 + k_y^2 = 0$ at $P = 0$ and $1/k_F a = -0.5$. In general, the quasiparticle spectrum near the center of the Brillouin zone, at $|k_z| \simeq 0$, is only weakly affected by the periodicity of the system and hence the change of ϵ_i with increasing P is close to that in the uniform system, given by the Doppler term $P\hbar k_z/m$. On the other hand, close to the zone edge, at $k_z \simeq \pm Q_{\rm edge}$,

the change of ϵ_i with increasing P is much smaller than $P\hbar k_z/m$ because of the periodicity of the Brillouin zone (ϵ_i at $k_z = \pm Q_{\rm edge}$ must be identical). In the case of $E_F/E_R = 1$, the minimum of ϵ_i is indeed located close to the edge of the Brillouin zone, unlike the other two cases of $E_F/E_R = 2.5$ and 0.1. Therefore, the reduction of the minimum value of ϵ_i with increasing P is relatively small for $E_F/E_R = 1$, and this is why we find that v_c is determined by phononic instead of fermionic excitations in this case.

For smaller density ($E_F/E_R = 0.1$), we observe a significant increase of v_c in the whole crossover [Fig. 3(c)]. In our previous article [16] we showed that in a unitary Fermi superfluid with $E_F/E_R \ll 1$, the phononic critical velocity is almost unaffected by the lattice if $s \lesssim 1$ and remains close to
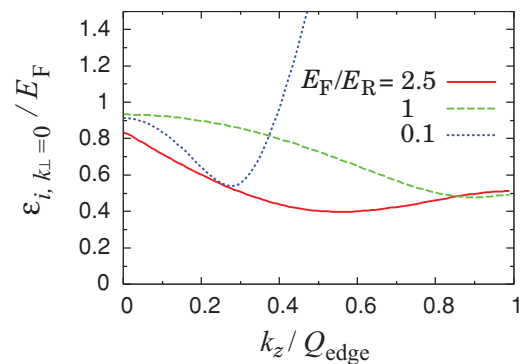

FIG. 4. (Color online) Lowest band of the quasiparticle energy spectrum ϵ_i for $P = 0$ and $1/k_F a = -0.5$. Here, we show the first radial branch with $k_\perp^2 \equiv k_x^2 + k_y^2 = 0$. For $E_F/E_R = 1$, the minimum of ϵ_i is located close to the Brillouin zone edge $k_z = Q_{\rm edge} \equiv q_{\rm B}/2$.
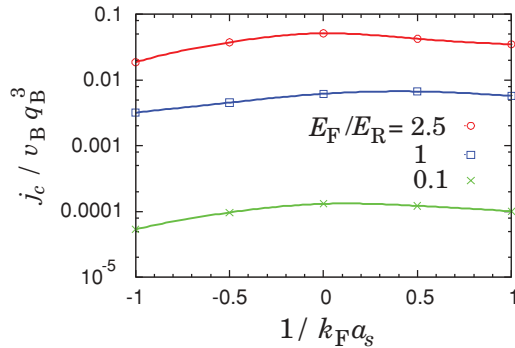
FIG. 5. (Color online) Critical current $j_c = n_0 v_c$ for the same cases in Fig. 3. Here we show the lowest value of j_c, given either by long-wavelength phonon excitations or by fermionic pair-breaking excitations. The curves connecting symbols are guides to the eye.

the speed of sound of a uniform gas with the same density [33]. This can be understood by recalling that phonons always have wavelength larger than the healing length of the superfluid, which is the order of k_F^{-1} or greater. When $E_F/E_R \ll 1$, the healing length becomes much larger than the lattice spacing $d = \pi/q_B$, and this makes the phonons insensitive to the lattice itself. In the present work, we find that the same is true even away from unitarity, leading to a larger critical velocity in the whole crossover [empty circles in Fig. 3(c)]. The critical velocity due to pair-breaking excitations (filled squares) is also increased because a lattice strength V_0 much larger than the Fermi energy gives a stronger attraction between paired atoms.

In Fig. 5 we show the critical current $j_c = n_0 v_c$ for the same cases of Fig. 3. Due to the low density at $E_F/E_R = 0.1$, the critical current is much smaller than the other cases even though v_c in units of $v_B = q_B/m$ for $E_F/E_R = 0.1$ is comparable to that of $E_F/E_R = 1$.

C. Dependence on lattice height

All results shown in Figs. 1–5 have been obtained by fixing the lattice height to $s = 1$ and varying the density in order to change the key parameter E_F/E_R. If we increase s keeping the average density fixed, the superfluid flow is suppressed and v_c is also reduced in general.

A systematic analysis as a function of s, which would be natural from an experimental viewpoint, is computationally very demanding and is beyond the scope of this work. The choice of $s = 1$ is not accidental, however. It turns out, in fact, that around $s = 1$ the effects of the lattice on the critical velocity are the most pronounced as far as the interplay between pair-breaking and long-wavelength bosonic excitations is concerned. For lower values of s these two types of excitations behave qualitatively the same as in a uniform superfluid, as a function of $1/k_F a$, being scarcely affected by the lattice, at least within the range of E_F/E_R considered in this work. On the other hand, at larger s values the pair-breaking instability is quickly suppressed and long-wavelength excitations become dominant along the crossover.

The reason a strong lattice prevents the pair-breaking processes can be understood by looking at Fig. 6, where we show the quasiparticle energy spectra at various values of P in the case of $s = 5$, $E_F/E_R = 0.1$ (i.e., $E_F/V_0 = 0.02$),
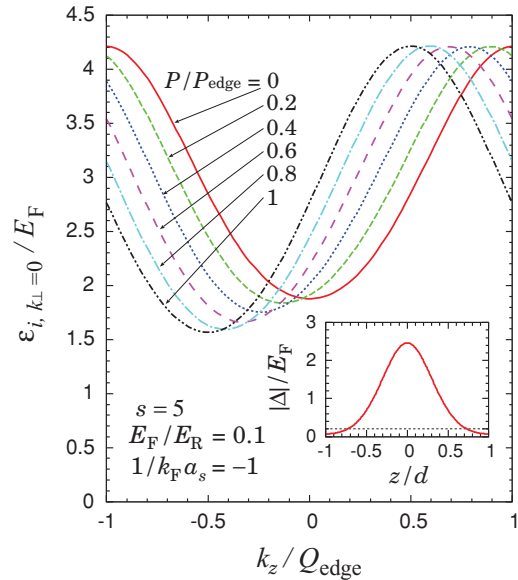

FIG. 6. (Color online) Lowest band of the quasiparticle energy spectrum ϵ_i for large lattice height with $s = 5$ and $E_F/E_R = 0.1$ (i.e., $E_F/V_0 = 0.02$) in the BCS regime at $1/k_F a_s = -1$. Here, we show the first radial branch with $k_\perp^2 \equiv k_x^2 + k_y^2 = 0$, which always gives the smallest values of ϵ_i in this case. The inset shows the amplitude $|\Delta(z)|$ of the order parameter at $P = 0$. The horizontal dotted line shows the amplitude of the order parameter for the uniform system at the same values of $1/k_F a_s = -1$.

and $1/k_F a_s = -1$. The spectrum for $P = 0$ shows a quadratic dependence of k_z with a positive curvature at around $k_z = 0$ and there are no minima at $k_z \neq 0$. Even though the figure represents a case in the deep BCS regime, the structure of ϵ_i is consistent with the formation of bound pairs. In the inset of the same figure, we show the amplitude $|\Delta(z)|$ of the gap parameter at $P = 0$. First, we note that the minimum value of $|\Delta(z)|$ at $z/d = \pm 1$ is smaller than but still comparable to the value of $|\Delta|$ in the uniform case, suggesting that the system is indeed in the superfluid phase. More importantly, one sees a large enhancement of $|\Delta(z)|$, near $z = 0$, compared to the uniform system, which shows the formation of bosonic bound molecules. A consequence of this lattice-induced molecular formation is that the energetic instability due to pair-breaking excitations is suppressed and does not occur at any values of P.

For the same parameters of Fig. 6, the energetic instability due to long-wavelength phonons instead occurs at $P = 0.226 \hbar q_B = 0.452 P_{edge}$ and the corresponding critical velocity at $1/k_F a_s = -1$ is $v_c = 0.0662 v_F$. We also find $v_c = 0.0429 v_F$ at unitarity and $v_c = 0.0302 v_F$ at $1/k_F a_s = 1$. This means that in the whole crossover, the critical velocity at $s = 5$ is largely reduced compared to the red line in Fig. 3(c) for $s = 1$ and does not exhibit a maximum anymore. The behavior of v_c as a function of s at unitarity has been already discussed in Ref. [16].

IV. SUMMARY AND OUTLOOK

We have studied the effects of a periodic potential on the Landau critical velocity of a Fermi superfluid in the

BCS-BEC crossover. We have considered a 3D superfluid Fermi gas flowing in a 1D periodic potential produced by an optical lattice. Using the Bogoliubov–de Gennes equations, we have obtained a unifying picture both for weak and strong lattices and in the whole BCS-BEC crossover. We have found that when the recoil energy is comparable to the Fermi energy, energetic instability due to fermionic pair-breaking excitations can be less effective as a consequence of the periodic structure of the quasiparticle energy spectrum. When the lattice height is much larger than the Fermi energy, pair-breaking excitations are prevented because the lattice potential gives a stronger attraction between paired atoms, eventually forming bound bosonic molecules. We have also found that when the recoil energy is comparable to or larger than the Fermi energy, the critical velocity due to the long-wavelength phonon excitations is drastically reduced by the lattice in the BCS regime, leading to its nonmonotonic behavior along the BCS-BEC crossover.

A further interesting issue is the possible existence of roton-like minima in the bosonic dispersion curve. This excitations are obtained at low filling fractions and within a tight-binding attractive Hubbard model [14,15,26,27]. The rotonlike minima arise from strong charge-density wave fluctuations. These fluctuations are expected to be less favored in our system, where the gas is uniform in the transverse directions. However, if such rotonlike excitations exist also in the our case (3D gas in a 1D lattice), they would lower the critical velocity in the BCS regime and for strong lattices. To address this issue, one should use, for instance, a quasiparticle random phase approximation (QRPA) on top of the stationary solution of the Bogoliubov–de Gennes equations. This is an interesting challenge for future investigations.

Finally, we would like to discuss a similarity between the present system and nuclear "pasta" phases [35–37] in crusts of neutron stars. The pasta nuclei are those of exotic shapes such as rodlike and slablike structures. In neutron star crusts, the pasta nuclei are immersed in background electrons and a gas of dripped neutrons, which is regarded to be in the superfluid phase. The setup considered in the present work resembles superfluid neutrons in the pasta phase with slablike nuclei which are in the normal phase and provide a 1D periodic potential for superfluid neutrons [38].
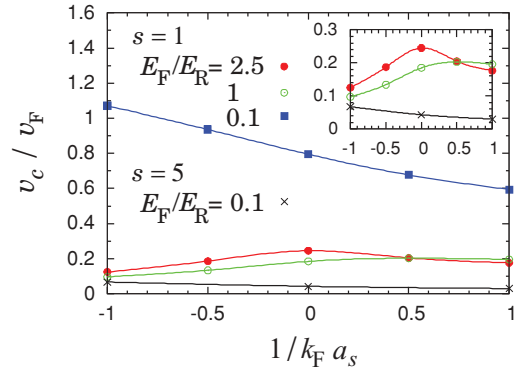
FIG. 7. (Color online) Critical velocity v_c for dynamical instability due to long-wavelength excitations in the BCS-BEC crossover. The solid lines are guides to the eye. The inset shows the magnification of the region of $v_c/v_F = 0 - 0.3$.

APPENDIX: DYNAMICAL INSTABILITY WITHIN HYDRODYNAMIC ANALYSIS

In this appendix we show the results of the critical velocity v_c for dynamical instability due to long-wavelength excitations. This instability occurs when a nonzero imaginary part appears in the excitation energy (9). The condition is

$$\frac{\partial^2 e}{\partial n_0^2} \frac{\partial^2 e}{\partial P^2} < 0. \qquad (A1)$$

Even though the long-wavelength excitations in general do not give the lowest critical velocity of the dynamical instability, this condition becomes useful at $V_0 \gg E_F$, when the P dependence of e is almost sinusoidal and the critical velocity depends rather weakly on the wavelength of the excitations. In this case the above condition gives an onset of the dynamical instability at $P = P_{\text{edge}}/2$, which coincides with the condition $e(P + \hbar q) - e(P) = e(P) - e(P - \hbar q)$ for $V_0 \gg |\Delta|$ [39]. This corresponds to the energy and momentum conservation for two particles decaying into two different Bloch states with $e(P \pm \hbar q)$.

In Fig. 7 we show the critical velocity for the dynamical instability determined by the condition (A1). The most striking feature is the large values of v_c in the case of $s = 1$ and $E_F/E_R = 0.1$. As in the case of the energetic instability, this is because phonons are insensitive to the lattice when $s \lesssim 1$ and $E_F/E_R \ll 1$, i.e., the healing length is much larger than the lattice spacing. Note that even for the same value of $E_F/E_R = 0.1$, the critical velocity can be rather small provided $s \gg 1$. This tendency is confirmed by the results for $s = 5$ and $E_F/E_R = 0.1$ shown by crosses in Fig. 7.

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[31] Note that the partial derivatives in Eq. (10) should be done by keeping a_s and d constant.

[32] We notice that the meaning of Landau stability is different for the bosonic and fermionic branches of the spectrum. A violation of the Landau stability condition for bosonic excitations makes thermal equilibrium impossible when $v > v_c$, due to boundless creation of the excitations. Conversely, in the case of fermionic excitations, the flow cannot create more than one excitation for a given state due to the Pauli principle. Thus a velocity greater than the critical one results in the appearance of a normal part of the fluid even at $T = 0$. How this normal part manifests itself depends on the concrete experimental realization. For example, for a finite body moving in the fluid, the fermionic excitations produced *via* Landau instability would move away from the body, causing a friction. The investigation of this problem, however, is beyond the scope of the present work.

[33] Note that the sound velocity at $P = 0$ basically decreases with increasing $1/k_F a_s$ for all values of E_F/E_R. However, the velocity at which the energetic instability occurs due to the long-wavelength phonon excitations does not show a monotonic behavior with increasing $1/k_F a_s$ for $E_F/E_R = 2.5$ and 1, unlike $E_F/E_R = 0.1$.

[34] Here we use $\beta = -0.41$ predicted by the mean-field BdG theory. We notice that *ab initio* Monte Carlo simulations instead give $\beta = -0.58 \pm 0.01$. See, J. Carlson, S.-Y. Chang, V. R. Pandharipande, and K. E. Schmidt, Phys. Rev. Lett. **91**, 050401 (2003); G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, *ibid.* **93**, 200404 (2004); J. Carlson and S. Reddy, *ibid.* **95**, 060401 (2005).

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[38] In the case of the nuclear pasta phase in neutron stars, an estimate of the critical velocity can be obtained as follows. For the pasta phase with slablike nuclei, the average neutron density $n_0 \simeq 0.08$ fm^{-3} and the lattice constant $d \simeq 15 - 20$ fm and thus $E_F/E_R = (k_F/q_B)^2 \sim 40 - 70$. The strength V_0 of the optical lattice corresponds to the depth of the bottom of the conduction band of neutrons measured from the average potential energy outside nuclei [40]. Since the neutron Fermi energy $E_{F,in}$ for the density of neutrons inside nuclei is $E_{F,in} \sim 35 - 40$ MeV and the neutron chemical potential μ_{in} inside nuclei is $\mu_{in} \sim 10 - 15$ MeV, we estimate $V_0 \sim |\mu_{in} - E_{F,in}| \sim 25$ MeV. Thus, we obtain $s \sim 25 - 50$. Since the absolute value of the neutron scattering length $a_s \simeq -18.5$ fm is much larger than the radius r_s of the average volume per neutron, $r_s \simeq 1.5$ fm, superfluid neutrons in neutron star crusts are roughly approximated as a unitary Fermi gas [41,43,44] (see also Refs. [40,45,46]). Here, because $E_F/E_R \gg 1$, we can safely employ the hydrodynamic theory with the local density approximation given in Ref. [16] to estimate the critical velocity. Using Eq. (3) of Ref. [16] together with $\beta = -0.58$ [34] and the above parameters, we estimate $v_c \sim 10^{-3} - 10^{-2} v_F$ with $v_F \simeq 0.3c$, where c is the speed of light.

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