Interpreting quantum discord through quantum state merging

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We present an operational interpretation of quantum discord based on the quantum state merging protocol. Quantum discord is the markup in the cost of quantum communication in the process of quantum state merging, if one discards relevant prior information. Our interpretation has an intuitive explanation based on the strong subadditivity of von Neumann entropy. We use our result to provide operational interpretations of other quantities like the local purity and quantum deficit. Finally, we discuss in brief some instances where our interpretation is valid in the single-copy scenario.

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I. INTRODUCTION

Quantum-information science is primarily aimed at harnessing the quantum structure of nature for information processing and computing tasks [1]. This quest has met with considerable success over the last decade, but there has been substantial progress in the other direction as well. Information theory has provided a novel framework for unraveling the intricacies of quantum mechanics. Quantum correlations, as well as classical ones, are now viewed as resources, whose interconvertability is governed by quantum-information theory [2]. Foremost among these is evidently entanglement, which provides enhanced performance in several important tasks like communication, computation, metrology, and others [3].

In the realm of mixed-state quantum-information, however, instances are known where quantum advantages are evidenced in the presence of little or no entanglement [4]. Recently, quantum discord was proposed as the source behind this enhancement and first steps toward a formal proof have been taken [5]. Quantum discord was originally suggested as a measure of quantumness of correlations [6], and has since been studied in variety of systems and settings [7–9]. Initial motivation for its definition arose in the context of pointer states and environment-induced decoherence [10]. It has since been related to quantum phase transitions [11] and the performance to quantum and classical Maxwell's demons [12]. Though satisfactory from a physical perspective, the benchmark for accepting some quantity as a resource in quantum-information science is that it appear as the solution to an appropriate asymptotic information processing task. It is this *information theoretic* interpretation that has been lacking for quantum discord, and we now provide in this article. This also addresses a more fundamental dichotomy in quantum-information science, where resources and their manipulations can have both thermodynamic and information theoretic interpretations independently, which are not intuitively or mathematically reconciled. Our article bridges this gap in the context of quantum discord, as well quantum deficit and local purity.

Quantum discord aims at capturing all quantum correlations in a quantum state, including entanglement [6,13,14]. Quantum mutual information is generally taken to be the measure of total correlations, classical and quantum, in a quantum state. For two systems, A and B, it is defined as $I(A:B)=H(A)+H(B)-H(A,B)$. Here $H(\cdot)$ denotes the Shannon entropy of the appropriate distribution. For a classical probability distribution, Bayes' rule leads to an equivalent definition of the mutual information as $I(A:B)=H(A)-H(A|B)$. This motivates a definition of classical correlation in a quantum state. Suppose Alice and Bob share a quantum state $\rho_{AB}\in\mathcal{H}_A\otimes\mathcal{H}_B$. If Bob performs the Positive operator valued measurements (POVM) set $\{\Pi_i\}$, the resulting state is given by the shared ensemble $\{p_i,\rho_{A|i}\}$, where

$$\rho_{A|i}=\mathrm{Tr}_B(\Pi_i\rho_{AB})/p_i,\quad p_i=\mathrm{Tr}_{A,B}(\Pi_i\rho_{AB}).$$

A quantum analog of the conditional entropy can then be defined as $\tilde{S}_{\{\Pi_i\}}(A|B)\equiv\sum_i p_i S(\rho_{A|i})$, and an alternative version of the quantum mutual information can now be defined as $\mathcal{J}_{\{\Pi_i\}}(\rho_{AB})=S(\rho_A)-\tilde{S}_{\{\Pi_i\}}(A|B)$, where $S(\cdot)$ denotes the von Neumann entropy of the relevant state. The above quantity depends on the chosen set of measurements $\{\Pi_i\}$. To capture all the classical correlations present in ρ_{AB}, we maximize $\mathcal{J}_{\{\Pi_i\}}(\rho_{AB})$ over all $\{\Pi_i\}$, arriving at a measurement-independent quantity

$$\mathcal{J}(\rho_{AB})=\max_{\{\Pi_i\}}[S(\rho_A)-\tilde{S}_{\{\Pi_i\}}(A|B)]. \quad (1)$$

Then, quantum discord is defined as [6]

$$\begin{aligned}\mathcal{D}(\rho_{AB})&=I(\rho_{AB})-\mathcal{J}(\rho_{AB})\\&=S(\rho_B)-S(\rho_{AB})+\min_{\{\Pi_i\}}\tilde{S}_{\{\Pi_i\}}(A|B).\end{aligned} \quad (2)$$

Since the conditional entropy is concave over the set of POVMs, which is convex, the minimum is attained on the extreme points of the set of POVMs, which are rank 1 [15]. In the asymptotic limit, when Alice and Bob share n copies of the state ρ_{AB}, we can define a regularized version of quantum discord as

$$\begin{aligned}\overline{\mathcal{D}}(\rho_{AB})&=\lim_{n\to\infty}\frac{\mathcal{D}(\rho_{AB}^{\otimes n})}{n}\\&\equiv I(\rho_{AB})-\overline{\mathcal{J}}(\rho_{AB}),\end{aligned} \quad (3)$$

where

$$\overline{\mathcal{J}}(\rho_{AB})=\lim_{n\to\infty}\frac{\mathcal{J}(\rho_{AB}^{\otimes n})}{n}. \quad (4)$$

The quantity $\overline{\mathcal{J}}(\rho_{AB})$ has an operational interpretation as a measure of classical correlations, as the distillable common randomness (DCR) with one-way classical communication [15], which is identical to the regularized version of the measure of classical correlations as defined by Henderson and Vedral [13]. Whether there exists a "single-letter" expression for discord depends on its additivity, which is equivalent to that of the entanglement of formation, since

$$\overline{\mathcal{D}}(\rho_{AC}) = E_C(\rho_{AB}) + S(\rho_C) - S(\rho_{AC}), \quad (5)$$

where ρ_{ABC} is pure and $E_C(\cdot)$ is the entanglement cost, the regularized version of the entanglement of formation [3]. This can be obtained using the monogamy between DCR and E_C [16]. Following the counterexample to the additivity of the minimum output entropy [17] and therefore the entanglement of formation, we can conclude that quantum discord is not additive either. In fact, the subadditivity of minimum output entropy implies that in general, quantum discord is subadditive. Our endeavor here is to provide an operational interpretation for quantum discord $\overline{\mathcal{D}}$ itself, without seeking recourse to its definition as the difference of total and classical correlations. To that end, we employ the process of quantum state merging, which we describe next. For brevity, in the remainder of the paper we suppress explicit mention of the state ρ_{AB} in the argument of quantities, denoting its von Neumann entropy as $S(A,B)$, its quantum discord when measurements are made on B as $\mathcal{D}(A|B)$, etc.

II. QUANTUM STATE MERGING AND DISCORD

Consider a party Bob having access to some incomplete information Y, and another party Alice having the missing the part X. We can think of X and Y as random variables. If Bob wishes to learn X fully, how much information must Alice send to him? Evidently, she can send $H(X)$ bits to satisfy Bob. However, Slepian and Wolf showed that she can do better by merely sending $H(X|Y) = H(X,Y) - H(Y)$, the conditional information [18]. Since $H(X|Y) \leqslant H(X)$, Alice can take advantage of correlations between X and Y to reduce the communication cost needed to accomplish the given task. The quantum state merging protocol is the extension of the classical Slepian-Wolf protocol into the quantum domain where Alice and Bob share the quantum state $\rho_{AB}^{\otimes n}$, with each party having the marginal density operators $\rho_A^{\otimes n}$ and $\rho_B^{\otimes n}$, respectively. Let $|\Psi_{ABC}\rangle$ be a purification of ρ_{AB}. Assume, without loss of generality, that Bob holds C. The quantum state merging protocol quantifies the minimum amount of quantum-information which Alice must send to Bob so that he ends up with a state arbitrarily close to $|\Psi\rangle_{B'BC}^{\otimes n}$, B' being a register at Bob's end to store the qubits received from Alice. It was shown that in the limit of $n \to \infty$, and asymptotically vanishing errors, the answer is given by the quantum conditional entropy [19,20]: $S(A|B) = S(A,B) - S(B)$. When $S(A|B)$ is negative, Bob obtains the full state with just local operations and classical communication, and distills $-S(A|B)$ ebits with Alice, which can be used to transfer additional quantum-information in the future.

An intuitive argument for our interpretation of quantum discord begins with strong subadditivity, which states that [20]

$$S(A|B,C) \leqslant S(A|B). \quad (6)$$

From the point of view of the state merging protocol, the above has a very clear interpretation: having more prior information makes state merging cheaper. Or in other words, throwing away information will make state merging more expensive. Thus, if Bob discards system C, it will increase the cost of quantum communication needed by Alice in order to merge her state with Bob. Our intent here shall be to relate this increase in the cost of state merging to quantum discord between A and B.

To that end, we expand the size of the Hilbert space so that an arbitrary measurement (with forgetting) can be modeled by coupling to the auxiliary subsystem and then discarding it. This permits us to apply strong subadditivity to the problem in question. We assume C to initially be in a pure state $|\mathbf{0}\rangle$, and a unitary interaction U between B and C. Letting primes denote the state of the system after U has acted, we have $S(A,B) = S(A,BC)$ as C starts out in a product state with AB. We also have $I(A:BC) = I(A':B'C')$. As discarding quantum systems cannot increase the mutual information, we get $I(A':B') \leqslant I(A':B'C')$. Now consider the state merging protocol between A and B in the presence of C. We have $S(A|B) = S(A) - I(A:B) = S(A) - I(A:BC) = S(A|BC)$. After the application of the unitary U, but before discarding the subsystem C, the cost of merging is still given by $S(A'|B'C') = S(A|B)$. This implies that one can always view the cost of merging the state of system A with B as the cost of merging A with the system BC, where C is some ancilla (initially in a pure state) with which B interacts coherently through a unitary U. Such a scheme does not change the cost of state merging, as shown, but helps us in counting resources. Discarding system C yields

$$I(A':B') \leqslant I(A':B'C') = I(A:BC) = I(A:B), \quad (7)$$

or alternatively,

$$S(A'|B') \geqslant S(A'|B'C') = S(A|B). \quad (8)$$

Computing the markup in the price in the state merging on discarding information gives $D = I(A:B) - I(A':B')$. This quantity D is equal to quantum discord when our quantum operations are quantum measurements maximizing $I(A':B')$. Thus, discord is the minimum possible increase in the cost of quantum communication in performing state merging, with a measurement on the party receiving the final state. This also addresses the asymmetry that is inherent in quantum discord. This is exhibited operationally in our interpretation since the state merging protocol is not invariant under exchanging the parties.

We now show that the minimum of D over all possible measurements is the quantum discord. The state ρ_{AB}, under measurement of subsystem B, changes to $\rho'_{AB} = \sum_j p_j \rho_{A|j} \otimes \pi_j$, where $\{\pi_j\}$ are orthogonal projectors resulting from a Neumark extension of the POVM elements. The unconditioned post-measurement states of A and B are

$$\rho'_A = \sum_j p_j \rho_{A|j} = \rho_A, \quad \rho'_B = \sum_j p_j \pi_j.$$

Computing the value of $I(A':B')$, we get

$$\begin{aligned} I(A':B') &= S(A') + S(B') - S(A',B'), \\ &= S(A') + H(p) - \left\{ H(p) + \sum_j p_j S(\rho_{A|j}) \right\}, \\ &= S(A) - \sum_j p_j S(\rho_{A|j}). \end{aligned} \tag{9}$$

After maximization, it reduces to $\mathcal{J}(\rho_{AB})$, as in Eq. (1). The reduction to rank 1 POVMs follows as stated earlier.

We can also rewrite the expression for D using Eq. (8) instead of Eq. (7) as the increase of the conditional entropy $D = S(A'|B') - S(A|B)$. The above expression makes our interpretation even more transparent. Quantum measurements on B destroy quantum correlations between A and B. This increases the average cost of quantum communication needed by A to merge her post-measurement state with B. Since $S(A'|B') = \sum_j p_j S(\rho_{A|j}) \geqslant S(A|B)$, there is always a markup in the cost of state merging.

III. EXAMPLE AND DISCUSSIONS

As an example, consider the separable state that has nonzero discord, $\rho_{AB} = (|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + |1\rangle_A\langle 1| \otimes |+\rangle_B\langle +|)/2$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The cost incurred by A to merge her state with B is $S(A|B) = 0.399124$ ebits, which is equal to the number of EPR pairs that A and B need to share per copy in order to merge this state. After measuring B using the projectors $(I \pm \frac{\sigma_x - \sigma_z}{\sqrt{2}})/2$, $S(A'|B') = 0.600876$ ebits. This means that one now needs these many EPR pairs to perform state merging (SM). The markup in the cost of state merging is $S(A'|B') - S(A|B) = 0.201752$ ebits, which is equal to the additional EPR pairs needed to perform SM after measuring B. This is exactly the quantum discord of the state ρ_{AB}, as per our interpretation. Hence, any information lost through the measurement results in making the quantum state merging more expensive by exactly the same amount.

We can now use our quantum state merging perspective to derive the various properties of discord. Since measurements on system B will always result in either discarding of some information or at best preserving the original correlations, we will always get a price hike in state merging or at best we can hope to just break even. Hence, discord, which is the markup, will always be greater than zero [6,21].

Quantum discord of a state is zero if and only if the density matrix is of the form $\rho_{AB} = \sum_i p_i \rho_{A|i} \otimes |\lambda_i\rangle\langle\lambda_i|$, in the basis which diagonalizes ρ_B. Measuring the projectors $|\lambda_i\rangle\langle\lambda_i|$ and discarding the measurement results on such a state yields $\rho_{AB}^M = \sum_i P_j \rho_{AB} P_j = \rho_{AB}$. Thus we have a measurement which causes no loss of information and retains all the correlations between A and B. Hence there is no markup in the cost of merging a zero discord state.

The converse can be seen through the application of strong subadditivity in Eq. (6). The equality of mutual information, $I(A:B)$, of the initial state and that of the state after quantum operations on B, $I(A':B')$ coincides with the equality condition for strong subadditivity. But this is exactly the condition for the nullity of quantum discord [21]. Thus a zero markup in the cost of state merging implies zero discord.

An upper bound on discord is decided by an upper bound on the markup we can get. Since Bob cannot lose more information than there is at his disposal, the entropy of the state at Bob's end, $S(B)$, is an upper bound on quantum discord.

Finally, for pure states, quantum discord reduces to entanglement and $S(A|B) = S(A) - I(A:B) = -S(A) \leqslant 0$. From our perspective, measurement destroys all the correlations present between A and B. Though the post-measurement state merging of the state of A with that of B occurs at zero cost, they lose the $-S(A|B)$ potential Bell pairs, which could have been put to some use. This provides us a novel way of measuring entanglement, as the markup in merging a pure state, when B is measured.

IV. OTHER MEASURES

We now use our result to provide operational interpretations for a couple of other quantities that were introduced to capture the quantumness of correlations, with motivations different from those of discord. Since the entropy of a closed system cannot decrease, thermodynamically, the purity of quantum states is a resource that needs quantification. The allowed set of operations in this paradigm are called closed local operations and classical communications (CLOCC), which is a modification of the local operations and the classical communications (LOCC) paradigm without free pure ancilla. The central task in this setup then is local purity distillation. If one-way communication is allowed from Bob to Alice, the rate for this task is given by [22]

$$\kappa(A|B) = \log_2(d_{AB}) - S(A,B) - \overline{\mathcal{D}}(A|B), \tag{10}$$

where $d_{AB} = \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$. This immediately provides an operational interpretation for local purity.

Another measure of quantumness of correlations in the CLOCC framework is the quantum deficit, which is also thermodynamically motivated and can be intuitively thought of as a form of nonlocality without entanglement, but with distinguishability [23]. Like quantum discord, unlike entanglement, it can be nonzero for separable states. The corresponding measure of classical correlations, the classical deficit, is known to be equivalent to the DCR [22] in the asymptotic limit. So the quantum deficit actually coincides with the quantum discord in this regime and has the same operational interpretation as discord. For a finite number of copies, the quantum deficit is always a lower bound on the quantum discord [23], provided the measurements are restricted to von Neumann projections instead of POVMs, because free pure ancilla are not allowed. Finally, operational interpretations can easily be provided for other discord-like measures, for instance, measurement-induced disturbance (MID) [24] using quantum state merging, by simple variations of our argument.

The end product of our information theoretic interpretation is the regularized form of quantum discord. This was necessitated since the single-copy version of state merging does not lead to the conditional von Neumann entropy [25]. There are, however, several interesting cases in which the rate of asymptotic state merging can be identified with the quantum discord of a single copy. Evidently, pure states are one such class, since in that case quantum discord reduces to entanglement. Since the DCR is additive for separable states [15], we have a "single-letter" definition of discord for such states as well. A more interesting set of states for which discord is additive are the Bell-diagonal states, since their DCR is additive too [26]. Quantum discord of Bell diagonal states of two qubits is among the best understood [7], and we have now shown that this understanding can be exported to the asymptotic regime without further effort.

In conclusion, this article places quantum discord squarely in the midst of quantum informational concepts and opens up the way for its manipulation as a resource in quantum-information processing. We also hope that our work will serve as a stepping stone for a more comprehensive and unified understanding of quantum physics, thermodynamics, and information theory.

Note added. Recently a manuscript [28] appeared which interprets quantum discord in terms of the entanglement consumption in an extended state merging of ρ_{AB}. The interpretation is through the entanglement of formation of ρ_{AC}, $\rho_{AC} = \text{Tr}_B[|\psi_{ABC}\rangle\langle\psi_{ABC}|]$, where $|\psi_{ABC}\rangle$ is the purification of ρ_{AB}.

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