

Density of defects and the scaling law of the entanglement entropy in quantum phase transition of one-dimensional spin systems induced by a quench

Banasri Basu^{*} and Pratul Bandyopadhyay[†]

Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700 108, India

Priyadarshi Majumdar[‡]

Jyotinagar Bidyasree Niketan H.S. School, 41 Jyotinagar, Kolkata 700 108, India

(Received 12 September 2010; published 21 March 2011)

We have studied quantum phase transition induced by a quench in different one-dimensional spin systems. Our analysis is based on the dynamical mechanism which envisages nonadiabaticity in the vicinity of the critical point. This causes spin fluctuation which leads to the random fluctuation of the Berry phase factor acquired by a spin state when the ground state of the system evolves in a closed path. The two-point correlation of this phase factor is associated with the probability of the formation of defects. In this framework, we have estimated the density of defects produced in several one-dimensional spin chains. At the critical region, the entanglement entropy of a block of L spins with the rest of the system is also estimated which is found to increase logarithmically with L . The dependence on the quench time puts a constraint on the block size L . It is also pointed out that the Lipkin-Meshkov-Glick model in point-splitting regularized form appears as a combination of the XXX model and Ising model with magnetic field in the negative z axis. This unveils the underlying conformal symmetry at criticality which is lost in the sharp point limit. Our analysis shows that the density of defects as well as the scaling behavior of the entanglement entropy follows a universal behavior in all these systems.

DOI: [10.1103/PhysRevA.83.032312](https://doi.org/10.1103/PhysRevA.83.032312)

PACS number(s): 03.67.Mn, 03.65.Ud, 64.70.Tg

I. INTRODUCTION

Kibble [1] has pointed out that in second-order phase transitions if the system is driven at a fixed rate characterized by a quench time its evolution cannot be adiabatic close to the critical point. The nonadiabatic evolution in the critical region produces defects such that the system becomes a mosaic of ordered domains whose finite size depends on the transition rate. In the cosmological scenario Kibble considered relativistic causality to set the size of the domains. Later, Zurek [2] suggested the dynamical mechanism based on the universality of critical slowing down and predicted that the size of the ordered domains scales with the transition time τ_q as τ_q^χ where χ is a critical exponent. The Kibble-Zurek mechanism (KZM) in second-order phase transition essentially involves thermal fluctuation which initiates symmetry breaking. In recent times, the zero-temperature quantum phase transition (QPT) has been studied in light of KZM and the formation of defects in one-dimensional transverse Ising model has been investigated in detail by several authors [3–5]. In QPT of the spin chains, apart from the study of the formation of defects it is also important to examine how entangled various parts of the system are with each other. Recently, it has been shown that for a quantum spin system in a mixed state the entanglement of formation of a pair of nearest neighbor spins, i.e., concurrence is related to the Berry phase factor acquired by a spin state when the ground state of the system evolves in a closed path [6–8]. When a bipartite quantum system is in a pure state the measure of entanglement between two subsystems is

given by von Neumann entropy. When the bipartite system is in a mixed state the entanglement of formation given by concurrence has the property that it reduces to the von Neumann entropy in a pure state [9]. It has been observed that the entanglement entropy of a block of L spins with the rest of the system at criticality follows a logarithmic scaling law in one-dimensional spin systems. The basic ingredient behind this result is that at criticality the correlation length diverges and the system becomes conformal invariant. This conformal symmetry leads to the logarithmic scaling law of the entanglement entropy at criticality [10]. When the criticality is induced by a quench the scaling behavior changes. The formation of defects as well as the entanglement entropy of a block of L spins with the rest of the system in transverse Ising model, when QPT is induced by a quench, has been studied in detail by Cincio *et al.* [5].

The behavior of the geometric phase in an XY spin chain during a linear quench caused by a gradual turning off of the magnetic field was studied earlier [11]. In a recent note [12] we formulated a dynamical mechanism for QPT induced by a quench based on the fact that spin fluctuation in the vicinity of the critical point leads to the fluctuation of the Berry phase acquired by the ground state of the system when it evolves in a closed path. The probability of the formation of defects is determined by the two-point correlation of this geometric phase factor. We have studied the formation of defects in transverse Ising model during critical slowing down based on this formalism and the results are found to be identical with those derived by other authors [3–5]. We have also studied the scaling law of the entanglement entropy in the transverse Ising model at criticality induced by a quench and it is shown that the prefactor now depends on the quench time and there is a restriction on the block size L [13]. In this paper, we study the formation of defects as well as the scaling law of

^{*}banasri@isical.ac.in

[†]b_pratul@yahoo.co.in

[‡]majumdar-priyadarshi@yahoo.com

the entanglement entropy at criticality induced by a quench in some other one dimensional spin systems on the basis of this dynamical mechanism of QPT induced by a quench.

The organization of the paper is as follows. In Sec. II we briefly review of our work [12] related to the isotropic XY model (transverse Ising model) for comprehension of the formulation we have adopted. In Secs. III and IV we study the XX and XXX models, respectively. In Sec. V we deal with the Lipkin-Meshkov-Glick (LMG) model. We make a comparative analysis of the results obtained for different spin systems in Sec. VI. Finally, we discuss results in Sec. VII.

II. THE XY MODEL

The Hamiltonian for the XY model is given by

$$H_{XY} = - \sum_i \left(\frac{1+\gamma}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\gamma}{2} \sigma_i^y \sigma_{i+1}^y + \lambda \sigma_i^z \right). \quad (1)$$

Here λ is the external magnetic field, γ is the anisotropy parameter and σ_i^a ($a = x, y, z$) are the Pauli matrices. The XY model with ($\gamma \neq 0$) falls into the free fermion universality class and is critical for $\lambda = 1$. The isotropic XY model ($\gamma = 1$) corresponds to the transverse Ising model. We consider the Hamiltonian for the transverse Ising model

$$H = - \sum_i (\sigma_i^z \sigma_{i+1}^z + \lambda \sigma_i^x). \quad (2)$$

For $\lambda \gg 1$ all the spins are aligned along the x axis and the system is in the paramagnetic state. In the region $\lambda \ll 1$ the system is in the ferromagnetic state with all spins either up or down. During critical slowing down we introduce a linear quench

$$\lambda(t < 0) = -\frac{t}{\tau_q}, \quad (3)$$

where τ_q is the quench time. In the vicinity of the critical region, the nonadiabaticity induces spin fluctuation. This eventually causes fluctuation of the Berry phase acquired by a spin state when the ground state evolves in a closed Path. The Berry phase acquired by a spin state is given by $\Gamma = \pi(1 - \cos \theta)$, where θ is the angle of deviation of the spin axis from the quantization axis [14–17]. In the critical region, due to nonadiabaticity some spins get excited and defects (kinks) are formed. The Berry phase factor associated with such a spin is given by

$$\phi_{k_0} = \frac{\Gamma_{k_0}}{2\pi} = \frac{1}{2} (1 - \cos \theta_{k_0}), \quad (4)$$

where k_0 is the momentum mode of the quasiparticle corresponding to the spin undergoing excitation and θ_{k_0} is the corresponding angle of deviation of the spin axis from the z axis. In our formulation it is assumed that during criticality ϕ_{k_0} undergoes stochastic fluctuation and it follows the simplest stochastic differential equation

$$d\phi_{k_0}(t) = -\omega_{k_0} \phi_{k_0}(t) dt + d\eta(t), \quad (5)$$

where ω_{k_0} is the frequency related to the energy ϵ_{k_0} near criticality and $\eta(t)$ is a Gaussian white noise satisfying the moments

$$\begin{aligned} \langle d\eta(t) \rangle &= 0, \\ \langle d\eta(t) d\eta(t') \rangle &= \delta(t - t') dt'. \end{aligned} \quad (6)$$

This gives rise to the correlation

$$\langle \phi_{k_0}(t) \rangle = 0, \quad \langle \phi_{k_0}(t) \phi_{k_0}(t') \rangle = \frac{1}{2} e^{-\omega_{k_0}(t-t')}. \quad (7)$$

At criticality during the quench time τ_q , the spin states transit from the paramagnetic state to the ferromagnetic state so that the angle of deviation $[\theta_{k_0}(\tau_q)]$ of a spin axis is $\pi/2$. The spin state undergoing excitation settles down at $t = 0$ with the spin axis reversed so that $\theta_{k_0}(0)$ takes the value π with certain probability p_{k_0} . This implies that the random variable $\phi_{k_0}(t)$ acquires the value $1/2$ and 1 at $t = \tau_q$ and $t = 0$, respectively, i.e., we have $\phi_{k_0}(\tau_q) = \frac{1}{2}$, $\phi_{k_0}(0) = 1$. This suggests that the random variable $2\phi_{k_0}(\tau_q)\phi_{k_0}(0)$ attains the value 1 with probability p_{k_0} . Thus using Eq. (7) the excitation probability p_{k_0} can be written as

$$p_{k_0} = 2\langle \phi_{k_0}(\tau_q) \phi_{k_0}(0) \rangle = e^{-\omega_{k_0} \tau_q}. \quad (8)$$

The energy ϵ_{k_0} near the critical point is given by [12]

$$\epsilon_{k_0} = 2(1 - \cos k_0) = 4 \sin^2 \frac{k_0}{2} \sim k_0^2, \quad (9)$$

for small k_0 . This gives

$$p_{k_0} = e^{-2\pi k_0^2 \tau_q}. \quad (10)$$

The number density of defects is given by

$$n_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_{k_0} dk_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2\pi \tau_q k_0^2} dk_0 = \frac{1}{2\pi} \frac{1}{\sqrt{2\tau_q}}. \quad (11)$$

This gives rise to the domain size corresponding to the Kibble-Zurek (KZ) correlation length $\hat{\xi} \sim \sqrt{\tau_q}$. This result is identical with that obtained by other authors [3–5].

At the critical point the ground states are characterized by the fact that the entanglement entropy S_L of a block of L spins with the rest of the system diverges like $\log L$ with a prefactor determined by the central charge c of the relevant conformal field theory [10]. In fact,

$$S_L \approx \frac{c}{3} \log_2 L. \quad (12)$$

For the critical Ising model $c = 1/2$,

$$S_L \approx \frac{1}{6} \log_2 L, \quad (13)$$

and the system falls into the free fermion universality class. It has been shown that the central charge c is related to the Berry phase factor ϕ , the phase being $e^{i2\pi\phi}$ and just like the central charge c , the Berry phase factor ϕ also undergoes a renormalization group (RG) flow [18] such that $L \frac{\partial \phi}{\partial L} \leq 0$ where L is a length scale. From this we have

$$|\phi|_L = a \ln L = \bar{a} \log_2 L, \quad (14)$$

where $a(\leq 0)$ is a parameter.

For a pair of nearest neighbor spins ($L = 2$), $\bar{a} = |\phi|$ which corresponds to the entanglement of formation given by concurrence in a mixed state [7,8]. So for the entropy of a block of L spins due to entanglement with the rest of the system in the pure state we can write

$$S_L \approx |\phi| \log_2 L. \quad (15)$$

At criticality the concurrence for the entanglement of a pair of nearest neighbor spins in transverse Ising model is given by $|\phi| = 0.18$ [7] and so we find

$$S_L \approx 0.18 \log_2 L. \quad (16)$$

It is observed that the prefactor in Eq. (13) is very close to the value 0.18. We may mention here that the entanglement of a block of L spins with the rest of the system can be considered to be equivalent to the entanglement between a single spin representing the block spin with another spin represented by the rest of the system in block variable RG scheme. In view of this, S_L in Eq. (13) can be considered as the concurrence C for the entanglement between the pair of this two block variable renormalized spins in a mixed state. The slight departure of the prefactor $\frac{1}{6}$ in Eq. (13) from the value 0.18 in Eq. (16) may be associated with the block variable renormalization of the spin system which induces a change in the coupling constant. The factor $\frac{1/6}{0.18} = 0.926$ is considered to be the correction factor associated with the block spin variable.

The introduction of a quench incorporates a new length scale given by the KZ correlation length $\hat{\xi} \sim \sqrt{\tau_q}$. So to evaluate the entanglement entropy immediately after the dynamical phase transition we have to rescale the Berry phase factor $|\phi|$ in expression (15) corresponding to the entanglement of a pair of nearest-neighbor spins in terms of that of the block spins in the domain $\hat{\xi}$ with the rest of the system. This follows from the fact that after the formation of defects the system in the final state represents a kink-antikink chain with lattice space approximately given by $\hat{\xi}$. So we write [13]

$$S(L, \tau_q) = 2 \frac{|\phi| \log_2 L}{|\phi| \log_2 \hat{\xi}} \times 0.926 \approx 3.7 \frac{\ln L}{\ln \tau_q}. \quad (17)$$

The factor 0.926 has been introduced as the correction factor of the block variable renormalization as discussed above. Equation (17) shows that in the scaling law of the entanglement entropy there is a prefactor depending on τ_q . This imposes a restriction on the maximum value of L, L_{\max} which is allowed in the system. As in the final state the system represents a kink-antikink chain with lattice constant $\hat{\xi}$, the maximum value of the entanglement entropy attained by the system corresponds to the entanglement of a block $\hat{\xi}$ spins with the rest of the system along with the entropy associated with the entanglement of kink-antikink pair. As we know for a bipartite spin 1/2 system the maximum value of entanglement entropy is 1, which at criticality is distributed over the whole chain, we can write for the entire chain [13]

$$S_{\max} = 2(0.18 \log_2 \hat{\xi} + 1) \times 0.926 = 0.12 \ln \tau_q + 1.85. \quad (18)$$

This suggests the relation $\frac{S(L, \tau_q)}{S_{\max}} \leq 1$ which implies [19]

$$\ln L \leq 0.03(\ln \tau_q)^2 + 0.5 \ln \tau_q. \quad (19)$$

This restricts the block size L when QPT is induced by a quench and suggests that for sufficiently small τ_q the block size $L \leq \hat{\xi}$. Thus we find that during critical slowing down due to the formation of defects, the scaling law for the entanglement entropy is valid only in a restrictive sense.

We have compared our result with that obtained by Cincio *et al.* [5] using the standard technique of density matrix

analysis in [13]. Within the limited range of the block size L determined by Eq. (19) the result was found to be in good agreement with that of Cincio *et al.*

III. THE XX MODEL

The XX model given by the Hamiltonian

$$H = - \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \lambda \sum_i \sigma_i^z \quad (20)$$

has two-limit behavior. At $\lambda = 2$ the system corresponds to a ferromagnetic state while at $\lambda = 0$ the system falls into the free boson universality class. The interval between these two points corresponds to the critical region. As the system transits through the point $\lambda = 2$ in the vicinity of the critical point we introduce a quench so that the time dependent magnetic field is given by

$$\lambda(t < 0) = -2t/\tau_q, \quad (21)$$

τ_q being the quench time. At $t = 0$ ($\lambda = 0$) the system settles down in a free bosonic state when the ground state is ordered in the xy plane and we have pairs of spins with opposite orientations. However, due to nonadiabaticity in the transition, this simple bosonic picture is disturbed and the system is characterized by domains having spins with same orientations, i.e., defects (kinks) are formed. Following the analysis in the previous section, in the critical region we consider the fluctuation of the Berry phase acquired by a spin state when the ground state evolves in a closed path. Let the system initially be in the region $\lambda > 2$. As the system transits through the point $\lambda = 2$, the ferromagnetic order is destroyed when certain spins are flipped. From Eq. (4) we can easily show that the Berry phase factor $\phi_{k_0}(\tau_q)$ takes the value 1 corresponding to the value $\theta_{k_0}(\tau_q) = \pi$ where k_0 is the momentum mode of the quasiparticle associated with the flipped spin.

As time evolves, in the vicinity of $\lambda = 0$ due to nonadiabaticity certain spins get excited with probability p_{k_0} and their orientations are reversed. Finally at $\lambda = 0$, the system settles down with domains characterized by spins having this reversed orientation destroying the free bosonic property. This suggests that the random variable $\phi_{k_0}(0)$ attains the value 1 corresponding to $\theta_{k_0} = \pi$ as follows from Eq. (4). From this we find that the random variable $\phi_{k_0}(\tau_q)\phi_{k_0}(0)$ acquires the value 1 with probability p_{k_0} . Thus we have

$$p_{k_0} = \langle \phi_{k_0}(\tau_q) \phi_{k_0}(0) \rangle, \quad (22)$$

and from Eq. (7) we find

$$p_{k_0} = \frac{1}{2} e^{-\omega_{k_0} \tau_q}, \quad (23)$$

ω_{k_0} being the frequency associated with the energy in the critical region.

Now we note that the energy of the quasiparticle associated with a spin state having the momentum mode k_0 is given by [20]

$$\epsilon_{k_0} = \lambda - 2 \cos \frac{2\pi k_0}{N}, \quad (24)$$

N being the number of sites. So taking the lattice constant $a = \frac{2\pi}{N} = 1$, we find at $\lambda = 2$, $\epsilon_{k_0} = 2(1 - \cos k_0) = 4 \sin^2 \frac{k_0}{2} \sim k_0^2$

for small k_0 . This suggests that for p_{k_0} we can write

$$p_{k_0} = \frac{1}{2} e^{-2\pi\tau_q k_0^2}. \quad (25)$$

The number density of defects (kinks) formed when the system finally settles down at $\lambda = 0$ is given by

$$n_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_{k_0} dk_0 = \frac{1}{2\pi} \frac{1}{2} \int_{-\pi}^{\pi} e^{-2\pi\tau_q k_0^2} dk_0 = \frac{1}{4\pi} \frac{1}{\sqrt{2\tau_q}}. \quad (26)$$

Incidentally it has the same scaling behavior with τ_q as in the transverse Ising model and the KZ correlation length is $\hat{\xi} \sim \sqrt{\tau_q}$.

To estimate the entanglement entropy at criticality we note that at $\lambda = 0$ the system belongs to the free boson universality class and the central charge of the relevant conformal field theory corresponds to $c = 1$. This gives the scaling law of the entanglement entropy at $\lambda = 0$ of a block of L spins with the rest of the system as

$$S_L \sim \frac{1}{3} \log_2 L. \quad (27)$$

As mentioned earlier the entanglement entropy of a block of L spins with the rest in a pure state is equivalent to the entanglement of formation of a pair of nearest neighbor spins in a mixed state given by concurrence C . As the concurrence C is associated with the Berry phase factor acquired by a spin state while evolving in a closed path we write $C = |\phi|$ where the phase is $e^{i2\pi\phi}$. Thus as in Sec. II using a RG flow equation we can write for the entanglement entropy of a block of L spins with the rest of the system

$$S_L \sim |\phi| \log_2 L. \quad (28)$$

At the critical point $\lambda = 0$, the system belongs to the free boson universality class, and the Berry phase factor $|\phi|$ here is identical to the concurrence of a pair of nearest-neighbor spins in an antiferromagnetic system which is given by $C = |\phi| = 0.386$ [7]. Now we take into account the correction factor 0.926 due to the block spin renormalization. In fact the von Neumann entropy of a block of L spins with the rest of the system in pure state is here transcribed into the concurrence associated with the entanglement of a renormalized block spin with another spin corresponding to the rest of the system in a mixed state.

This gives the effective value of $|\phi| = 0.386 \times 0.926 = 0.35$ which is close to the value $1/3$ derived from conformal field theory. Utilizing this result we now estimate the entanglement entropy of a block of L spins with the rest of the system at criticality near $\lambda = 0$ induced by a quench. Indeed in analogy to Eq. (17) we write

$$S(L, \tau_q) = 4 \frac{|\phi|}{|\phi| \ln \tau_q} \ln L \times 0.926 = 3.7 \frac{\ln L}{\ln \tau_q}. \quad (29)$$

This result is independent of the prefactor in Eq. (28) and thus is universal for all one-dimensional spin systems. However, the maximum value of L , L_{\max} allowed in the system is different. Following Eq. (18), the maximum value of the entanglement entropy at $\lambda = 0$ is derived as

$$\begin{aligned} S_{\max} &= 2(|\phi| \log_2 \hat{\xi} + 1) \times 0.926, \\ &= 0.25 \ln \tau_q + 1.85. \end{aligned} \quad (30)$$

Now from the constraint $S(L, \tau_q)/S_{\max} \leq 1$ we obtain

$$\ln L \leq 0.07(\ln \tau_q)^2 + 0.5 \ln \tau_q. \quad (31)$$

Thus we find that due to generation of defects, the scaling law for the entanglement entropy is restricted by the maximum value of the block size allowed in the system. This picture is in the vicinity of the point $\lambda = 0$. However, in the region $2 > \lambda > 0$, the entropy of entanglement gradually decreases with the increase in the magnetic field following the same scaling law until at $\lambda = 2$, the entanglement vanishes.

It is pointed out by Latorre, Rico, and Vidal [21] that the maximum entropy is reached at $\lambda = 0$. The entropy is fitted by a logarithmic scaling law as given by Eq. (27) with a constant term which has been determined analytically by Jin and Korepin [22]. These results have been obtained without critical slowing down when nonadiabaticity comes into play. It is noted from Eq. (29) that when nonadiabaticity near criticality is introduced the entropy decreases with quench time. In Fig. 1 (left) we have compared our result with the exact value of the entropy at $\lambda = 0$ as reported in [20] (where no quench was introduced) for $\tau_q = 200, 400$, and 800 . As expected we have found that the entropy increases with decreasing quench time. At $\lambda = 0$ the entropy is maximum and we find that our result is in reasonable good agreement with the exact value for small

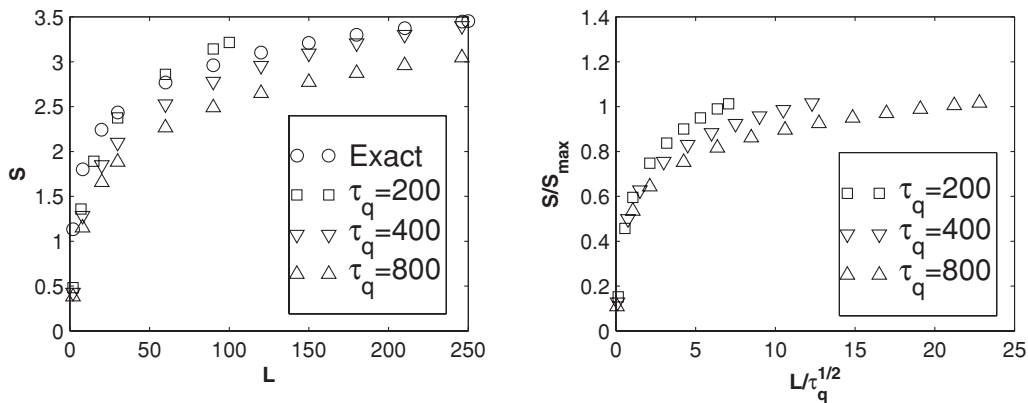


FIG. 1. (Left) the entanglement entropy of the XX model for various quench time (τ_q) is compared with the exact value plotted for $\lambda = 0$ from the result as reported in [20]. (Right) shows the variation of $S(L, \tau_q)/S_{\max}$ with $L/\sqrt{\tau_q}$.

value of τ_q . Indeed, we find that for $\tau_q = 200$ it is close to the exact value and for higher values of τ_q the entropy decreases. In Fig. 1 (right) we have plotted $S(L, \tau_q)/S_{\max}$ vs. $L/\sqrt{\tau_q}$.

IV. THE XXX MODEL

The Hamiltonian for the XXX model is given by

$$H_{XXX} = \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z) + \sum_i \lambda \sigma_i^z. \quad (32)$$

The critical behavior of this model is analogous to that of the XX model. It has two-limit behavior. At $\lambda = 2$ the system represents the ferromagnetic state and at $\lambda = 0$ the system corresponds to the antiferromagnetic state. The interval $2 > \lambda > 0$ is gapless and hence critical. Let us denote the time evolution of the magnetic field by the relation

$$\lambda(t < 0) = -2t/\tau_q, \quad (33)$$

τ_q being the quench time and analyze the dynamics of the system when it transits through the point $\lambda = 2$. In the critical region, there is random fluctuation in the Berry phase and from the two-point correlation of the Berry phase factor we derive the excitation probability. Actually, this is given by the same relation as in Eq. (25) and we write

$$p_{k_0} = \frac{1}{2} e^{-\omega_{k_0} \tau_q}. \quad (34)$$

In this case the energy of the quasiparticle in the vicinity of the critical point is different from that of the XX model.

A generalization of the XXX model is the anisotropic Heisenberg model which is known as the XXZ model given by the Hamiltonian

$$H_{XXZ} = \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z) + \lambda \sum_i \sigma_i^z. \quad (35)$$

It may be mentioned that for $\lambda = 0$ the anisotropic system shows a gapless phase in the interval $-1 \leq \Delta \leq 1$. For $\Delta = 1$ the system corresponds to the isotropic Heisenberg model. For $\lambda > 2$ the isotropic Heisenberg model corresponds to a ferromagnetic state. As the magnetic field decreases and passes through the critical point $\lambda = 2$ some spins get flipped. For an excited state of the Heisenberg model belonging to the subspace of states where some spins are flipped the energy of the quasiparticle with momentum mode k_0 is given by [23]

$$\epsilon_{k_0} = 4(1 - \cos k_0) = 8 \sin^2 \frac{k_0}{2} \sim 2k_0^2, \quad (36)$$

for small k_0 . Thus from the relation (34) we can write

$$p_{k_0} = \frac{1}{2} e^{-2\pi \epsilon_{k_0} \tau_q} = \frac{1}{2} e^{-4\pi \tau_q k_0^2}, \quad (37)$$

and the number density of defects is given by

$$n_3 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_{k_0} dk_0 = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{-4\pi \tau_q k_0^2} dk_0 = \frac{1}{8\pi} \frac{1}{\sqrt{\tau_q}}. \quad (38)$$

Thus the number density of kinks scales like $\tau_q^{-1/2}$ and the KZ correlation length is given by $\hat{\xi} \sim \sqrt{\tau_q}$.

As the XXX chain at $\lambda = 0$ corresponds to the antiferromagnetic state, the behavior of the entanglement entropy for a

block of L spins with the rest of the system is identical with that of the XX model. This is valid for $S(L, \tau_q)$ and S_{\max} also. Just as in the XX model, in this model also, the entanglement entropy is maximum for $\lambda = 0$ which gradually diminishes as the magnetic field increases, and finally it vanishes at $\lambda = 2$.

V. THE LIPKIN-MESHKOV-GLICK MODEL

The Lipkin-Meshkov-Glick (LMG) model demonstrates the mechanism of a phase transition for a many-body system and was introduced almost 45 years ago [24]. In contrast to the conventional spin models in the LMG model each spin interacts with all the spins of the system with the same coupling strength. This highly symmetric interaction pattern introduces the loss of the notion of geometry as there is no distance between the spins. This implies that we cannot consider the notion of a block of spins as a set of contiguous spins here. The symmetry of the Hamiltonian suggests that the ground state belongs to a symmetric subspace where all the spins are indistinguishable and this subspace restricts the entanglement entropy of a block of L spins with the remaining spins. Interestingly the scaling behavior shows a similar pattern with that of the XX model where conformal symmetry plays a significant role at the critical region. But in the LMG model the scaling law seems to have nothing to do with any underlying conformal symmetry.

In this note, we show that in case we introduce point-splitting regularization the regularized LMG Hamiltonian in the isotropic case ($\gamma = 1$) appears as a combination of Heisenberg (XXX) model and Ising model with magnetic field along the negative z axis. In view of this, the regularized Hamiltonian follows the pattern of the usual spin systems where we can conceive of the notions of local interactions and translational invariance. The underlying conformal symmetry at the critical region of these systems is manifested in the LMG model and the similarity of the pattern of the scaling behavior of the entanglement entropy of this system can be understood. Thus the point-splitting regularization of the LMG model uncovers the hidden conformal symmetry at criticality which is lost when we take the sharp point limit.

The Hamiltonian for the LMG model is given by

$$H = \frac{1}{N} \sum_{i < j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) + \lambda \sum_i \sigma_i^z. \quad (39)$$

N being the total number of spins.

In what follows we shall consider the isotropic case with $\gamma = 1$. The Hamiltonian (39) can be written in terms of the total spin operator $S^\alpha = \frac{1}{2} \sum_i \sigma_i^\alpha$ as

$$H = \frac{2}{N} \left(\vec{S}^2 - (S^z)^2 - \frac{N}{2} \right) + 2\lambda S^z. \quad (40)$$

Let us consider that this total spin operator is located at a spatial point k . Now we introduce the point-splitting regularization so that we write

$$\vec{S}^2 = \vec{S}_k \cdot \vec{S}_{k'} \cdot \delta_{kk'}, \quad (41)$$

where k and k' are two adjacent sites with an infinitesimal distance $k' - k = \epsilon$. Thus \vec{S}^2 corresponds to the product $\vec{S}_k \cdot \vec{S}_{k'}$ in the limit $\epsilon \rightarrow 0$. We take

$$S_k^\alpha = \frac{1}{2} \sum_i \sigma_i^\alpha, \quad S_{k'}^\alpha = \frac{1}{2} \sum_j \sigma_j^\alpha, \quad (42)$$

where i and j are two adjacent sites with an infinitesimal distance. Considering only nearest-neighbor interactions we can write the Hamiltonian (40) in the regularized form

$$\begin{aligned} H_{reg} &= \frac{2}{N} \left[\frac{1}{4} \sum_{i,j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) \right. \\ &\quad \left. - \frac{1}{4} \sum_{i,j} \sigma_i^z \sigma_j^z - \frac{N}{2} \right] + \lambda \sum_i \sigma_i^z, \\ &= \frac{1}{2N} \sum_{i,j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) \\ &\quad - \frac{1}{2N} \sum_{i,j} \sigma_i^z \sigma_j^z - 1 + \lambda \sum_i \sigma_i^z, \\ &= \frac{1}{2N} \left[\sum_{i,j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + N\lambda \sum_i \sigma_i^z \right] \\ &\quad - \frac{1}{2N} \left(\sum_{i,j} \sigma_i^z \sigma_j^z - N\lambda \sum_i \sigma_i^z \right) - 1. \end{aligned} \quad (43)$$

In deriving Eq. (43) we have considered nearest-neighbor interaction only because of the fact that the introduction of next nearest-neighbor interaction gives rise to frustration in the antiferromagnetic phase of the XXX model which is unsolicited in the LMG model. In fact, this happens when the spin at the initial site interacts with all odd sites. To avoid this unwanted feature only the nearest neighbor interaction has been taken into account.

Now considering $\frac{1}{2N}$ as a dimensionless coupling constant J we write $H_{reg} = H_1 + H_2 - 1$ with

$$H_1 = J \left[\sum_{i,j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + \frac{\lambda}{2J} \sum_i \sigma_i^z \right], \quad (44)$$

$$H_2 = -J \left(\sum_{i,j} \sigma_i^z \sigma_j^z - \frac{\lambda}{2J} \sum_i \sigma_i^z \right). \quad (45)$$

The Hamiltonian given by Eq. (43) can be split into various other H_1 and H_2 such that both of them describe critical systems. The motivation behind this particular choice follows from the fact that cluster of even (odd) number of interacting spins represent bosonic (fermionic) systems. As the LMG model is characterized by the fact that each spin interacts with every other spin, in the regularized Hamiltonian we have chosen the critical systems such that one of these represents bosonic and the other fermionic features. In view of this we have taken H_1 and H_2 such that one represents the XXX model and the other Ising model.

The criticality of this system can be studied by noting that this is a combination of the XXX model given by H_1 and the Ising model with magnetic field along the negative z direction

given by H_2 . It is observed that $\frac{\lambda}{2J}$ denotes the intensity of the magnetic field. We know that the XXX model has two limit behavior. When $\frac{|\lambda|}{2J} > 2$, the XXX system is gapped and for $\lambda = 0$, the magnetization is zero and the system is in an entangled (antiferromagnetic) state. In the interval $2 > \frac{|\lambda|}{2J} > 0$ the system is gapless and this denotes the critical region. Noting that the dimensionless parameter J corresponds to $\frac{1}{2N}$, N being an integer we find that $\frac{|\lambda|}{2J} = 2$ corresponds to the value $|\lambda| = \frac{2}{N}$. Since in an entangled state the minimum number of spins must be 2, i.e., $N_{\min} = 2$, at criticality $|\lambda|$ lies in the interval $0 < |\lambda| < 1$. The value of λ is independent of the *real* number of sites as we have treated the parameter $\frac{1}{2N}$ as the coupling constant. Now it is observed that in the Ising chain given by H_2 for $|\lambda| > 1$ the system is in the ferromagnetic state when all spins are oriented along the negative z axis. So in the interval $0 < |\lambda| < 1$ as $|\lambda|$ is tuned from 1 to 0 the spin system will undergo a transition when down spins will be excited and at $\lambda = 0$ all spins will settle down with opposite orientations. Evidently, with the evolution of $|\lambda|$ the spins cross a point when these are oriented along the x axis. Hence, this corresponds to the double ferromagnet transition and in the interval $0 < |\lambda| < 1$ spins evolve through a paramagnetic state. Thus during this transition in the interval $0 < |\lambda| < 1$ the entanglement entropy of the spins evolves through a nonzero value which is similar to that of the transverse Ising model. In the regularized Hamiltonian we can consider the time dependent magnetic field corresponding to the summation of that given by Eqs. (3) and (33) and so we write

$$\lambda(t < 0) = -\frac{2t}{\tau_q} + \frac{t}{\tau_q} = -\frac{t}{\tau_q}, \quad (46)$$

τ_q being the quench time. We have taken $\frac{t}{\tau_q}$ for the Ising system with positive sign as the magnetic field here is in the negative z direction. Finally, when the system settles down at $t = 0$ ($\lambda = 0$), the number density of defects is estimated and is given by $n_4 = n_1 + n_3$ where $n_1(n_3)$ is given by Eq. (11) [Eq. (38)] in the thermodynamic limit. From this we obtain

$$n_4 = n_1 + n_3 = \frac{1}{2\pi} \frac{1}{\sqrt{2\tau_q}} + \frac{1}{8\pi} \frac{1}{\sqrt{\tau_q}} = \frac{2\sqrt{2} + 1}{8\pi \sqrt{\tau_q}}. \quad (47)$$

Thus we note that in LMG model also the number density of kinks formed scales as $\tau_q^{-\frac{1}{2}}$. However, in this model a correlation length characterizing the typical distance between defects cannot be introduced though one can estimate the fraction of flipped spins after the quench.

The entanglement entropy in the LMG model of a block of L spins with respect to the rest of the spins can be derived in an analogous way. The entanglement entropy in the critical region $0 < |\lambda| < 1$ is given by the summation of the entanglement entropy of the XXX model and the transverse Ising model at criticality. As the entanglement entropy for a block of L spins with the rest of the system is given by $S_1(L) \sim \frac{1}{3} \log_2 L$ for the XXX model and that for the transverse Ising model is given by $S_2(L) \sim \frac{1}{6} \log_2 L$, the entanglement entropy in the critical region $0 < |\lambda| < 1$ for the LMG model corresponds to the scaling law in the thermodynamical limit as

$$S(L) \sim \frac{1}{3} \log_2 L + \frac{1}{6} \log_2 L \sim \frac{1}{2} \log_2 L, \quad (48)$$

which is identical with that obtained by Latorre, Orus, Rico, and Vidal [25]. One should note that for $\lambda \geq 1$, the system represents a polarized product state. Thus when the LMG model is recast in the regularized form the critical region manifests the same logarithmic scaling law as observed in XXX model and the transverse Ising model which are characterized by conformal symmetry at the critical region in these systems. Though the LMG model appears to have nothing to do with conformal symmetry at the critical region, the point-splitting regularization reveals an underlying conformal symmetry at criticality in this system which is lost in the sharp point limit.

Now we consider that the criticality is induced by a linear quench (46). As the regularized Hamiltonian suggests that at the critical region this corresponds to a combination of the XXX model and the transverse Ising model so following Eqs. (17) and (29), we write

$$S(L, \tau_q) = \frac{2|\tilde{\phi}| \log_2 L}{|\tilde{\phi}| \log_2 \hat{\xi}} \times 0.926, \quad (49)$$

where $|\tilde{\phi}| = |\phi|_{XXX} + |\phi|_{\text{Ising}} = 0.386 + 0.18 = 0.566$ and $\hat{\xi} (\sim \sqrt{\tau_q})$ is the associated KZ correlation length. The final result is independent of $|\tilde{\phi}|$ and we obtain the universal value

$$S(L, \tau_q) = 3.7 \frac{\ln L}{\ln \tau_q}. \quad (50)$$

Here the maximum value of L , L_{\max} is different from that of the XXX and the transverse Ising model. In analogy to Eqs. (18) and (30) we find that the maximum value of the entanglement entropy

$$S_{\max|LMG} = 2(|\tilde{\phi}| \log_2 \hat{\xi} + 1) \times 0.926 \approx 0.36 \ln \tau_q + 1.85. \quad (51)$$

So from the constraint $S(L, \tau_q)/S_{\max} \leq 1$ we find

$$\ln L|_{LMG} \leq 0.097(\ln \tau_q)^2 + 0.5 \ln \tau_q. \quad (52)$$

In the sharp point limit the LMG model does not allow any correlation length. However, we can consider the coherence number [26] which can be identified with the KZ correlation length $\hat{\xi}$ in the regularized formalism. It is noted that in

the critical region $0 < \lambda < 1$ the entanglement entropy is maximum at $\lambda = 0$ and decreases with the increase in the magnetic field until at $\lambda = 1$ it vanishes when the system represents a product state. However, within this region the scaling behavior remains the same.

It may be pointed out here that recently the LMG model has been studied in which the system is dragged adiabatically through the critical point [27]. Using the Landau-Zener formula [28] the fraction of the flipped spins has been found to scale like $\tau_q^{-\frac{3}{2}}$, τ_q being the quench time. The failure of obtaining the scaling behavior $\tau_q^{-\frac{1}{2}}$ as derived here may be related to the definition of the defect density in this formalism. Indeed, here the degree of adiabaticity is estimated through the residual energy E_{res} given by $E_{res} = E_{fin} - E_{gs}$ where E_{fin} is the average energy in the final time-evolved state and E_{gs} is the ground-state energy. It is found that the system has three regions: (i) for fast quenches E_{res} persists independent of τ_q , (ii) for slower quenches E_{res} persists $\sim \tau_q^{-\frac{3}{2}}$, and (iii) for further slowing E_{res} persists $\sim \tau_q^{-2}$. It is noted that for fast quench, the transition is nonadiabatic. However, for slow transitions the adiabatic dynamics is quite different from our present formalism. In fact, the result we have obtained here is fully determined by the nonadiabaticity condition in the vicinity of the critical point.

In Fig. 2 (left) we have compared our result for the entanglement entropy for the LMG model with the value for $\lambda = 0$ derived in [20] extrapolating it to the thermodynamic limit. It is noted that when we introduce a quench the entropy decreases with the increase in the quench time τ_q . So it is expected that the value reported in [20] will be in agreement with our result for small τ_q . However for very small τ_q , the maximum value of the block size L will be very small which follows from Eq. (52). In view of this we have taken moderate values of $\tau_q = 200, 400$, and 800 for comparison. It is noted that for $\tau_q = 200$ the result is very close to the value derived from [20].

Caneva *et al.* [27] have computed the entanglement entropy at $\lambda = 0$ for various values of τ_q when the system is adiabatically dragged through the critical point for finite system size. According to their result, for fast quench $\tau_q \rightarrow 0$

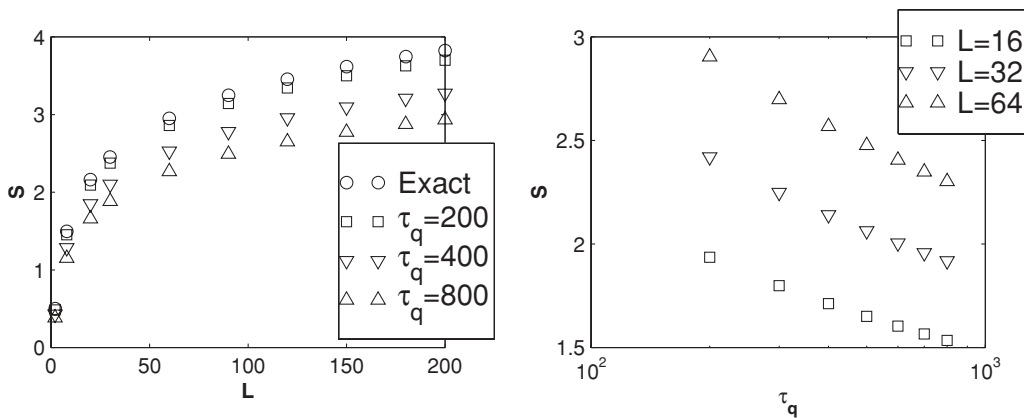


FIG. 2. (Left) we have compared our result for $S(L, \tau_q)$ for $\tau_q = 200, 400$, and 800 with the values derived from that reported in [20] at $\lambda = 0$ extrapolating it to the thermodynamic limit. (Right) we have plotted our results for $L = 16, 32$, and 64 in the region $\tau_q > 200$.

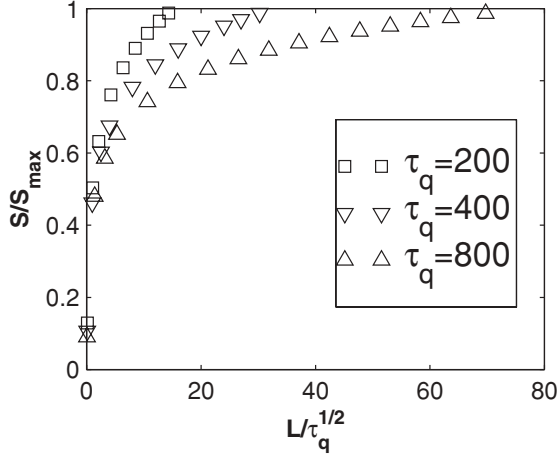


FIG. 3. The entanglement entropy $S(L, \tau_q)$ vs. $L/\sqrt{\tau_q}$.

the entanglement entropy tends to be vanishing and for slow dynamics $\tau_q \rightarrow \infty$ the entropy picks up the value it assumes in the final ground state $S_{gs}(\lambda = 0) = 1$ independent of the system size. Between this two limiting behaviors the entropy reaches a size dependent maximum at τ_q nearing 10 and then decreases with the increase in τ_q . In our formalism we find that for τ_q nearing 10, the maximum value of the block size L will be very small as follows from Eq. (52). In view of this we have taken moderately large $\tau_q > 200$ for comparison with their result. Caneva *et al.* [27] have computed their result for block size $L = N/2$, N being the system size. However our result has been computed in the thermodynamic limit. In Fig. 2 (right) we have plotted our results for $L = 16, 32$, and 64 and $\tau_q > 200$. We find that our result is in reasonable good agreement with that of Caneva *et al.* [27]. In Fig. 3 we have plotted $S(L, \tau_q)/S_{\max}$ vs. $L/\sqrt{\tau_q}$.

VI. DEFECT FORMATION AND MAXIMAL ENTANGLEMENT ENTROPY IN DIFFERENT ONE DIMENSIONAL SYSTEMS: A COMPARATIVE STUDY

We have studied here defect formation and the scaling of entanglement entropy in QPT in different one-dimensional

systems induced by a quench. It is found that they all show a universal behavior. The number density of defects in all these systems scales like $\tau_q^{-1/2}$ with difference only in the prefactors. Also the entanglement entropy at criticality takes a universal value having the same scaling pattern with block spin size L . However, there is a restriction on the maximum value of L allowed in different systems. The maximum value of the entropy in all these systems follows the same pattern with difference only in the coefficients. In all these systems the constraint on the maximum value of the block size L restricts the validity of the scaling law for any arbitrary L . In fact, the main implication of the scaling law for the entanglement entropy is that with the addition of every spin, entanglement increases. However, in QPT induced by a quench this is not valid beyond the upper limit of the value of L allowed in various systems. Indeed, the nonadiabatic transition due to quench makes the scaling law a restrictive one and cannot be treated in the conventional sense.

In Fig. 4 (left) we have plotted the number density of defects formed during QPT induced by a quench in different spin systems for different values of τ_q . The density of defects for the isotropic XY , XX , XXX , LMG models are denoted by n_1, n_2, n_3, n_4 , respectively. In Fig. 4 (right) we have plotted the maximum value of entanglement entropy S_{\max} for different values of τ_q in isotropic XY , XX , and LMG models. S_{\max} is identical for XX and XXX systems.

VII. DISCUSSION

We have analyzed here the defect formation and the scaling law of the entanglement entropy in QPT in several one-dimensional systems at critical slowing down when nonadiabaticity plays a dominant role. We have argued that in the vicinity of the critical point nonadiabaticity in QPT causes spin fluctuation which in turn makes the Berry phase factor a random one. From the two-point correlation of the Berry phase factor we have estimated the probability of the generation of defects. Indeed, in a recent paper [12] we estimated the number density of defects as well as the spin-spin correlation at criticality of the transverse Ising

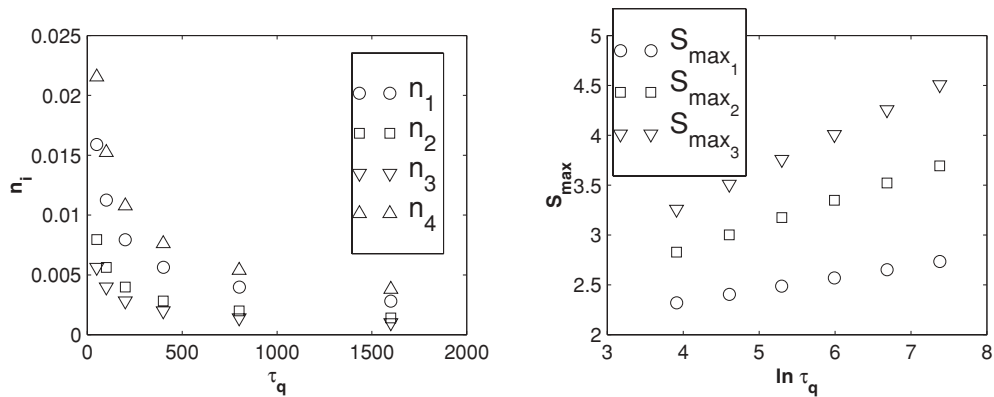


FIG. 4. (Left) shows the variation of the number density of defects n_1, n_2, n_3, n_4 with quench time τ_q in isotropic XY , XX , XXX and LMG models respectively for different values of τ_q . (Right) shows the maximum value of the entanglement entropy S_{\max} for different values of τ_q in isotropic XY (S_{\max_1}), XX (XXX) (S_{\max_2}) and LMG (S_{\max_3}) models. It is noted that S_{\max} is identical for XX and XXX systems.

model from this dynamical mechanism of QPT induced by a quench. The results are found to be identical with those derived from the standard Landau-Zener transition probability studied by other authors [3–5]. Here we have analyzed

several other one-dimensional spin systems using this dynamical mechanism. Our analysis suggests that this formalism represents the universal dynamics of QPT induced by a quench.

-
- [1] T. W. B. Kibble, *J. Phys. A: Math. Gen.* **9**, 1387 (1976); *Phys. Rep.* **67**, 183 (1980).
 - [2] W. H. Zurek, *Nature (London)* **317**, 505 (1985); *Phys. Rep.* **276**, 477 (1996).
 - [3] W. H. Zurek, U. Dorner, and P. Zoller, *Phys. Rev. Lett.* **95**, 105701 (2005).
 - [4] J. Dziarmaga, *Phys. Rev. Lett.* **95**, 245701 (2005).
 - [5] L. Cincio, J. Dziarmaga, M. M. Rams, and W. H. Zurek, *Phys. Rev. A* **75**, 052321 (2007).
 - [6] B. Basu, *Europhys. Lett.* **76**, 833 (2006).
 - [7] B. Basu and P. Bandyopadhyay, *Int. J. Geo. Meth. Mod. Phys.* **4**, 707 (2007).
 - [8] B. Basu and P. Bandyopadhyay, *J. Phys. A: Math. Theor.* **41**, 055301 (2008).
 - [9] T. J. Osborne and M. A. Nielsen, *Phys. Rev. A* **66**, 032110 (2002).
 - [10] C. Holzhey, F. Larsen, and F. Wilczek, *Nucl. Phys. B* **424**, 443 (1994).
 - [11] B. Basu, *Phys. Lett. A* **374**, 1205 (2010).
 - [12] B. Basu and P. Bandyopadhyay, *J. Phys. A: Math. Theor.* **43**, 354023 (2010).
 - [13] P. Majumdar and P. Bandyopadhyay, *Phys. Rev. A* **81**, 012311 (2010).
 - [14] R. A. Bertlmann, K. Durstberger, Y. Hasegawa, and B. C. Hiesmayr, *Phys. Rev. A* **69**, 032112 (2004).
 - [15] E. Sjöqvist, *Phys. Rev. A* **62**, 022109 (2000).
 - [16] D. M. Tong, L. C. Kurek, and C. H. Oh, *J. Phys. A: Math. Gen.* **36**, 1149 (2003).
 - [17] X. X. Yi, L. C. Wang, and T. Y. Zheng, *Phys. Rev. Lett.* **92**, 150406 (2004).
 - [18] P. Bandyopadhyay, *Proc. Roy. Soc. (London) A* **467**, 427 (2011); *Int. J. Mod. Phys. A* **15**, 1415 (2002).
 - [19] There is a typographical error in Eq. (26) in [13]. The correct prefactor will be 0.03 and not 0.3 as printed.
 - [20] J. I. Latorre and A. Riera, *J. Phys. A: Math. Theor.* **42**, 054002 (2009).
 - [21] J. I. Latorre, E. Rico, and G. Vidal, *Quantum Inf. Comput.* **4**, 48 (2004).
 - [22] B. Q. Jin and V. E. Korepin, *J. Stat. Phys.* **116**, 79 (2004).
 - [23] J. Gonzalez, M. A. Martin-Delgado, G. Sierra, and A. H. Vozmediano, *Quantum Electron Liquids and High- T_c Superconductivity* (Springer, Berlin, 1990).
 - [24] H. J. Lipkin, N. Meshkov, and A. J. Glick, *Nucl. Phys.* **62**, 188 (1965).
 - [25] J. I. Latorre, R. Orus, E. Rico, and J. Vidal, *Phys. Rev. A* **71**, 064101 (2005).
 - [26] R. Botet, R. Jullien, and P. Pfeuty, *Phys. Rev. Lett.* **49**, 478 (1982).
 - [27] T. Caneva, R. Fazio, and G. E. Santoro, *Phys. Rev. B* **78**, 104426 (2008).
 - [28] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958); C. Zener, *Proc. R. Soc. London A* **137**, 696 (1932).