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Family of continuous-variable entanglement criteria using general entropy functions

A. Saboia,* F. Toscano, and S. P. Walborn

Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Rio de Janeiro, RJ 21941-972, Brazil

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We derive a family of entanglement criteria for continuous-variable systems based on the Rényi entropy of complementary distributions. We show that these entanglement witnesses can be more sensitive than those based on second-order moments, as well as previous tests involving the Shannon entropy [Phys. Rev. Lett. **103**, 160505 (2009)]. We extend our results to include the case of discrete sampling. We provide several numerical results which show that our criteria can be used to identify entanglement in a number of experimentally relevant quantum states.

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I. INTRODUCTION

Quantum entanglement is a fundamental property of quantum systems that can be exploited for quantum computation, quantum teleportation, and quantum cryptography [1]. As such, its detection is an essential task in an experimental setting. Many techniques exist for detecting entanglement in discrete systems (see [2,3] for review). In continuous-variable systems, its identification can be more complicated due to the large Hilbert space structure. However, there is a considerable amount of work concerning entanglement detection and characterization of Gaussian states [4,5], where tests involving only the second-order moments [6–12] are adequate. However, there is a large interest in non-Gaussian states, since non-Gaussianity is necessary for some quantum information tasks, such as quantum computation [13–15] and entanglement distillation [16,17]. Second-order criteria are sufficient but not necessary for entanglement in non-Gaussian states. As such, there has been some work dedicated toward entanglement detection in non-Gaussian states [18–29]. The set of criteria derived by Shchukin and Vogel (SV) [19], for instance, is very powerful and general, but may require a large number of measurements [30]. We note that the SV criteria have been applied for the experimental detection of non-Gaussian entanglement [31].

It has been shown that classical entropy functions can be used to formulate Bell's inequalities [32] and entanglement witnesses for bipartite $d \times d$ level systems [33]. These are examples of nonlinear entanglement witnesses, which provide improvements in sensibility at little to no extra experimental effort [34,35]. In Ref. [24], the Shannon entropy of complementary distributions was used to derive a set of entanglement witnesses for bipartite continuous-variable quantum systems. This approach is especially useful in the experimental characterization of entanglement, since it considers only a pair of joint quadrature measurements. At the same time, these entropic witnesses are more sensitive than second-order tests (i.e., those based solely on the elements of the covariance matrix) [6–10]. In the present work, we extend this approach by deriving entanglement criteria using more general entropy functions. For example, we use the classical Rényi entropy, characterized by the continuous parameter α, to derive a family of entropic entanglement witnesses which provides a more powerful tool for identification of entanglement. We note that the Wehrl entropy [36], and also quantum versions of the Shannon [37] and Rényi entropies [38,39], have been used to identify quantum entanglement. In general, these criteria require complete knowledge of the density matrix or more complicated measurement schemes [40].

This paper is organized as follows. In Sec. II we define our notation and briefly review the criteria of Ref. [24]. In Sec. III we develop a family of entanglement witnesses for continuous variables using the classical Rényi entropy. We then extend these results to include the case of discrete sampling. We tested the continuous variable Rényi criteria on several experimentally relevant states. Section V provides numerical results which show that the generalized Rényi witnesses detect entanglement in a wider variety of quantum states than second-order tests or witnesses based solely on Shannon entropy [24]. In Sec. VI we provide concluding remarks.

II. ENTANGLEMENT CRITERIA WITH SHANNON ENTROPY

First, we review two sets of inequalities which were developed in Ref. [24]. These inequalities are satisfied for all separable states, so that the violation of either one indicates that the bipartite state is entangled.

We will take into account a rotation of the usual canonical operators $\mathbf{x}$ and $\mathbf{p}$, and define a pair of general complementary operators for systems 1 and 2 as

$$\mathbf{r}_j = \cos\theta_j \mathbf{x}_j + \sin\theta_j \mathbf{p}_j, \tag{1a}$$

$$\mathbf{s}_j = \cos\theta_j \mathbf{p}_j - \sin\theta_j \mathbf{x}_j, \tag{1b}$$

where $j = 1,2$ refers to each subsystem of the bipartite state. The commutation relation $[\mathbf{x}_j,\mathbf{p}_k] = i\delta_{j,k}$ for canonical operators $\mathbf{x}_j$ and $\mathbf{p}_k$ implies $[\mathbf{r}_j,\mathbf{s}_k] = i\delta_{j,k}$, $j,k = 1,2$. Here x and p are dimensionless continuous variables, such as quadratures of electromagnetic field modes or dimensionless position and momentum of a point particle, for example. Let us define the global operators $\mathbf{r}_\pm$ and $\mathbf{s}_\pm$ as

$$\mathbf{r}_\pm = \mathbf{r}_1 \pm \mathbf{r}_2, \tag{2a}$$

*saboia@if.ufrj.br

and

$$\mathbf{s}_{\pm} = \mathbf{s}_1 \pm \mathbf{s}_2. \tag{2b}$$

Since $[\mathbf{r}_j,\mathbf{s}_k] = i\delta_{j,k}$, $j,k = 1,2$, it is easy to see that $[\mathbf{r}_\mu,\mathbf{s}_\nu] = 2i\delta_{\mu,\nu}$ with $\mu,\nu = \pm$.

The inequalities in Ref. [24] were developed initially for a separable pure state $|\psi_1\rangle \otimes |\psi_2\rangle$, corresponding to the wave function $\Psi(r_1,r_2) = \psi_1(r_1)\psi_2(r_2)$, which can also be written as

$$\Psi(r_+,r_-) = \frac{1}{\sqrt{2}}\psi_1\left(\frac{r_+ + r_-}{2}\right)\psi_2\left(\frac{r_+ - r_-}{2}\right). \tag{3}$$

For simplicity, we denote the probability distributions associated to measurement of $\mathbf{r}_\pm$ as simply $R_\pm$. They are given by

$$R_\pm = \frac{1}{2}\int dr_\mp R_1\left(\frac{r_+ + r_-}{2}\right)R_2\left(\frac{r_+ - r_-}{2}\right), \tag{4}$$

which is equivalent to the convolution

$$R_\pm = R_1 * R_2^{(\pm)}, \tag{5}$$

where $R_i(r_i) = |\psi_i(r_i)|^2$, $R_2^+ \equiv R_2(r)$, and $R_2^- \equiv R_2(-r)$. The Shannon entropy for continuous variables is defined by

$$H[R] = -\int dr\, R(r) \ln R(r), \tag{6}$$

where $R(r)$ is the probability distribution associated to the measurement of an arbitrary continuous variable r. Similar expressions are obtained for the probability distribution S of the complementary variable s.

Two inequalities were introduced in Ref. [24]. Their violation indicates the presence of entanglement. Using the probability distributions $R_\pm$ and $S_\pm$ defined above and applying the entropy power inequality [41], the following criteria were obtained:

$$H[R_\pm] + H[S_\mp] \geqslant \frac{1}{2}\ln\left[\sum_{i,j} e^{(2H[R_i]+2H[S_j])}\right]. \tag{7}$$

These criteria are particularly useful only in the case of pure states. They can be extended to include mixed states as well, but numerical optimization procedures are required [24]. By further applying an entropic uncertainty relation for the distributions R_j and S_j [42], a second set of entropic witnesses were derived:

$$H[R_\pm] + H[S_\mp] \geqslant \ln(2\pi e). \tag{8}$$

Although inequality (8) is weaker than inequality (7), it has the advantage that it is directly applicable to mixed states. It was shown that these criteria are more sensitive than second-order tests involving the same operators.

III. GENERALIZATION OF ENTROPIC CRITERIA

A natural attempt to improve the entropic entanglement witnesses described in Sec. II is the application of a more general function of information entropy. For this purpose, we employ the Rényi entropy for continuous variables, defined by [41,43]

$$H_\alpha[R] = \frac{1}{1-\alpha}\ln\left[\int dr\, R^\alpha(r)\right] = \frac{\alpha}{1-\alpha}\ln\|R\|_\alpha, \tag{9}$$

where $\|R\|_\alpha$ is the $\mathcal{L}_\alpha$ norm of the distribution R (see Ref. [41]):

$$\|R\|_\alpha = \left[\int dr\, R^\alpha(r)\right]^{1/\alpha}. \tag{10}$$

As in Sec. II, let us first consider only pure states of the form $|\psi_1\rangle \otimes |\psi_2\rangle$. In the Appendix, we show that any separable pure state of this form will satisfy

$$\begin{aligned}\left(\frac{\alpha-1}{\alpha}\right)&H_\alpha[R_\pm] + \left(\frac{1-\beta}{\beta}\right)H_\beta[S_\mp]\\ &\geqslant \left(\frac{\alpha_1-1}{\alpha_1}\right)H_{\alpha_1}[R_1] + \left(\frac{1-\beta_1}{\beta_1}\right)H_{\beta_1}[S_1] + \left(\frac{\alpha_2-1}{\alpha_2}\right)\\ &\quad\times H_{\alpha_2}[R_2] + \left(\frac{1-\beta_2}{\beta_2}\right)H_{\beta_2}[S_2] + \ln\left[\frac{C(\beta_1,\beta_2)}{C(\alpha_1,\alpha_2)}\right],\end{aligned} \tag{11}$$

where $C(\alpha_1,\alpha_2)$ is defined in Eq. (A4) and α,α_1,α_2 (β,β_1,β_2) are related through Eq. (A7). Inequality (11) is a generalization of criteria (7). In order to recover (7) from (11) we first consider the case $\alpha = \beta$ and then take the limit $\alpha \to 1$. Violation of inequality (11) implies that the pure state considered is entangled. Extension of (11) to include mixed states is possible, although one must calculate the supremum of the right-hand side over all possible decompositions of the mixed state. As in Sec. II, it is possible to arrive at an inequality which is directly applicable to mixed states and requires no numerical maximization.

To derive a second inequality that does not depend on the entropy functions $H_{\alpha_j}[R_j]$ and $H_{\beta_j}[S_j]$, we employ the entropic uncertainty relation for Rényi entropy given by Ref. [44]:

$$H_{\alpha_j}[R_j] + H_{\beta_j}[S_j] \geqslant -\frac{1}{2(1-\alpha_j)}\ln\frac{\alpha_j}{\pi} - \frac{1}{2(1-\beta_j)}\ln\frac{\beta_j}{\pi}, \tag{12}$$

where it is necessary to include the restriction [44]

$$\frac{1}{\alpha_j} + \frac{1}{\beta_j} = 2, \quad j = 1,2. \tag{13}$$

Equation (13), along with Eq. (A7), lead to

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2. \tag{14}$$

Applying the uncertainty relation (12) to inequality (11) and performing some algebra we obtain the inequality

$$\begin{aligned}H_\alpha[R_\pm] + H_\beta[S_\mp] &\geqslant -\frac{1}{2(1-\alpha)}\ln\frac{\alpha}{\pi} - \frac{1}{2(1-\beta)}\ln\frac{\beta}{\pi}\\ &\quad + \frac{\alpha}{\alpha-1}\sum_{j=1,2}\frac{\alpha_j-1}{\alpha_j}\ln\left|\frac{\alpha_j}{\alpha_j-1}\right|\\ &\quad - \ln\left|\frac{\alpha}{\alpha-1}\right|.\end{aligned} \tag{15}$$

The sum of terms in the last two terms of Eq. (15) is always nonnegative. α_1 and α_2 are arbitrary parameters within the restrictions imposed by Eqs. (A7a) and (14), which guarantee that $1 \leqslant 1/\alpha_1 + 1/\alpha_2 \leqslant 2$. Within this domain we can maximize the last two terms on the right-hand side of inequality (15), which reach a maximum value of ln 2 when $\alpha_1 = \alpha_2$. This leads directly to the inequality

$$H_\alpha[R_\pm] + H_\beta[S_\mp] \geqslant -\frac{1}{2(1-\alpha)}\ln\frac{\alpha}{2\pi} - \frac{1}{2(1-\beta)}\ln\frac{\beta}{2\pi}. \tag{16}$$

Note that our choice $\alpha \geqslant 1$ and $1/2 \leqslant \beta \leqslant 1$ is arbitrary, and that these restrictions can be switched with no alteration in the derivation. Inequality (16) reduces to (8) when $\alpha \longrightarrow 1$.

We will now show that inequality (16) is also valid for mixed states. Noting that $[\mathbf{r}_\mu, \mathbf{s}_\nu] = 2i\delta_{\mu,\nu}$, $(\mu,\nu = \pm)$, then the uncertainty relation for the Rényi entropy of complementary distributions $R_\pm$ and $S_\pm$ is

$$H_\alpha[R_\pm] + H_\beta[S_\pm] \geqslant -\frac{1}{2(1-\alpha)}\ln\frac{\alpha}{2\pi} - \frac{1}{2(1-\beta)}\ln\frac{\beta}{2\pi}, \tag{17}$$

where again $1/\alpha + 1/\beta = 2$. Bialynicki-Birula has shown that this uncertainty relation is also valid for mixed states [44], in which case $R_\pm$ and $S_\pm$ are complementary marginal distributions obtained from the Wigner function associated to the mixed quantum state. We can now make use of an alternative way of deriving inequality (16) by means of the positive partial transpose (PPT) criterion [6,45,46]. For any continuous-variable quantum state, the transpose operation is equivalent to a mirror reflection in phase space, taking $(r_j, s_j) \longrightarrow (r_j, -s_j)$ [6]. Thus, the partial transpose of a bipartite state ϱ_{12} thus takes the global variables $r_\pm \longrightarrow r_\pm$ and $s_\pm \longrightarrow s_\mp$, where we take the transpose of subsystem 2. The marginal probability distributions under partial transposition T transform as

$$R_\pm^T = R_\pm, \tag{18a}$$

$$S_\pm^T = S_\mp, \tag{18b}$$

and we have

$$H_\alpha[R_\pm^T] + H_\beta[S_\pm^T] = H_\alpha[R_\pm] + H_\beta[S_\mp]. \tag{19}$$

The partial transpose of a separable density operator is a positive operator, and thus it is still a physical state [6,45,46] and will satisfy the uncertainty relation (17). Substituting Eq. (19) into inequality (17) leads directly to inequality (16), where we have made no assumptions about the purity of the bipartite state ϱ_{12}. Thus, criteria (16) is also valid for bipartite mixed states.

The above argument illustrates that the family of entropic entanglement witnesses (16) are in fact PPT criteria. This illustrates a general method for developing new PPT criteria: apply *any* quantum mechanical uncertainty relation to distributions $R_\pm$ and $S_\pm$ and use Eqs. (18). We note that this was the general spirit of the procedure used by Simon to develop a criterion based on second-order moments [6] and has also been used in Ref. [47].

A. Relationship with second-order criteria

The second-order Mancini-Giovannetti-Vitali-Tombesi (MGVT) criteria is [8]

$$\Delta_{r_\pm}\Delta_{s_\mp} \geqslant 1, \tag{20}$$

where Δ_q^2 is the variance in variable q. Inequality (20) is verified by any separable state. In Ref. [24], it was shown that the MGVT criteria can be derived directly from the Shannon criteria (8) by maximizing the sum $H[R_\pm] + H[S_\mp]$. This leads to the inequalities

$$\ln(2\pi e\Delta_{r_\pm}\Delta_{s_\mp}) \geqslant H[R_\pm] + H[S_\mp] \geqslant \ln(2\pi e). \tag{21}$$

Note that the MGVT criteria (20) derives from the two extremes of (21). This upper bound is saturated for Gaussian probability distributions [48]. Since $R_\pm$ and $S_\pm$ are arbitrary (although complementary) marginal distributions in phase space, this implies that the bound is saturated for Gaussian states. Nevertheless, within the class of non-Gaussian states, inequalities (21) show that the criteria given in (8) may detect entanglement in states that the MGVT criteria might not (20). Several examples were provided in [24].

A natural question to ask is whether we can derive new entanglement witnesses by maximizing the sum of Rényi entropies $H_\alpha[R_\pm] + H_\beta[S_\mp]$ in criteria (16). Doing so leads to an inequality also involving second-order moments, due to the fact that the Rényi entropy is maximized for the Student-t and Student-r distributions [49,50], which (for zero mean) are completely characterized by the variance. More specifically, we arrive at

$$\Delta_{r_\pm}\Delta_{s_\mp} \geqslant f(\alpha,\beta), \tag{22}$$

where $f(\alpha,\beta) \leqslant 1$ for all allowed values of α and β. In the limiting case $\alpha,\beta \longrightarrow 1$, $f(\alpha,\beta) = 1$ and we recover the MGVT criteria (20). Thus, inequality (22) is not an improvement over the already-established MGVT criteria.

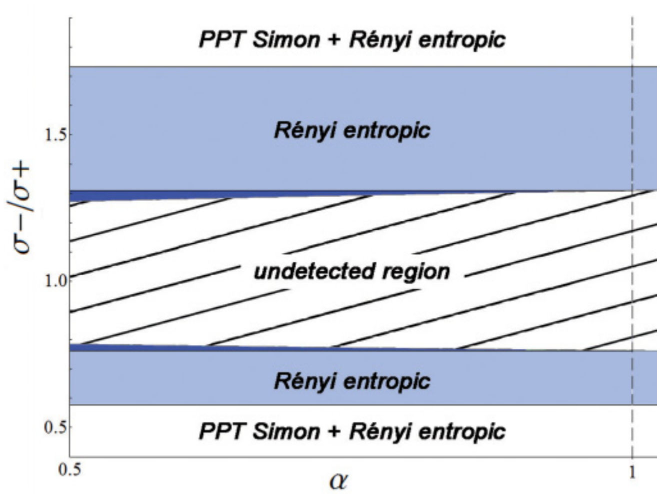

FIG. 1. (Color online) Entanglement detection of state (27) for $n = 1$. The light-blue shaded region is where the Rényi entropic criteria in (16) identifies entanglement, while the Simon second-order PPT criterion does not. The dark-blue shaded region shows its improvement from Shannon entropic criteria. The uppermost and lowermost areas designate the regions in which the Simon PPT and Rényi criteria detect entanglement in state (27). In the center hatched region neither test detect entanglement.

IV. DISCRETE DISTRIBUTIONS

Inequalities (11) and (16) derived in the above section were developed for continuous distributions $R_\pm$ and $S_\pm$. However, in an experimental setting one typically measures discrete distributions due to the finite resolution of the measurement apparatus. Here, we show how to deal with discrete resolution and we derive an entanglement witness equivalent to (16), but for discrete distributions. The same procedure can be adopted for a derivation of inequalities equivalent to (11). Let us call these discrete distributions $R^\delta_\pm$ and $S^\Delta_\pm$, and suppose that their elements are

$$\rho^\delta_{k\pm} = \int_{(k-1/2)\delta}^{(k+1/2)\delta} R_\pm(r)\,dr, \tag{23a}$$

and

$$\sigma^\Delta_{k\pm} = \int_{(k-1/2)\Delta}^{(k+1/2)\Delta} S_\pm(s)\,ds, \tag{23b}$$

respectively. Here we assume that r measurements have resolution δ and s measurements are performed with resolution Δ. To apply these inequalities to discrete distributions, one can write the entropy of the continuous distribution in terms of the discrete distribution as [41]

$$H_\alpha[R_\pm] = H_\alpha[R^\delta_\pm] + \ln\delta, \tag{24a}$$

$$H_\beta[S_\pm] = H_\beta[S^\Delta_\pm] + \ln\Delta, \tag{24b}$$

provided that δ and Δ are sufficiently small. Here the discrete Rényi entropy is

$$H_\alpha[R^\delta_\pm] = \frac{1}{1-\alpha}\ln\left(\sum_k \left(\rho^\delta_{k\pm}\right)^\alpha\right), \tag{25}$$

and similarly for $H_\beta[S^\Delta_\pm]$. Inequality (16) can then be written in terms of the discrete distributions:

$$H_\alpha[R^\delta_\pm] + H_\beta[S^\Delta_\mp] \geqslant -\frac{1}{2}\left(\frac{\ln\alpha}{1-\alpha} + \frac{\ln\beta}{1-\beta}\right) + \ln\left(\frac{2\pi}{\delta\Delta}\right). \tag{26}$$

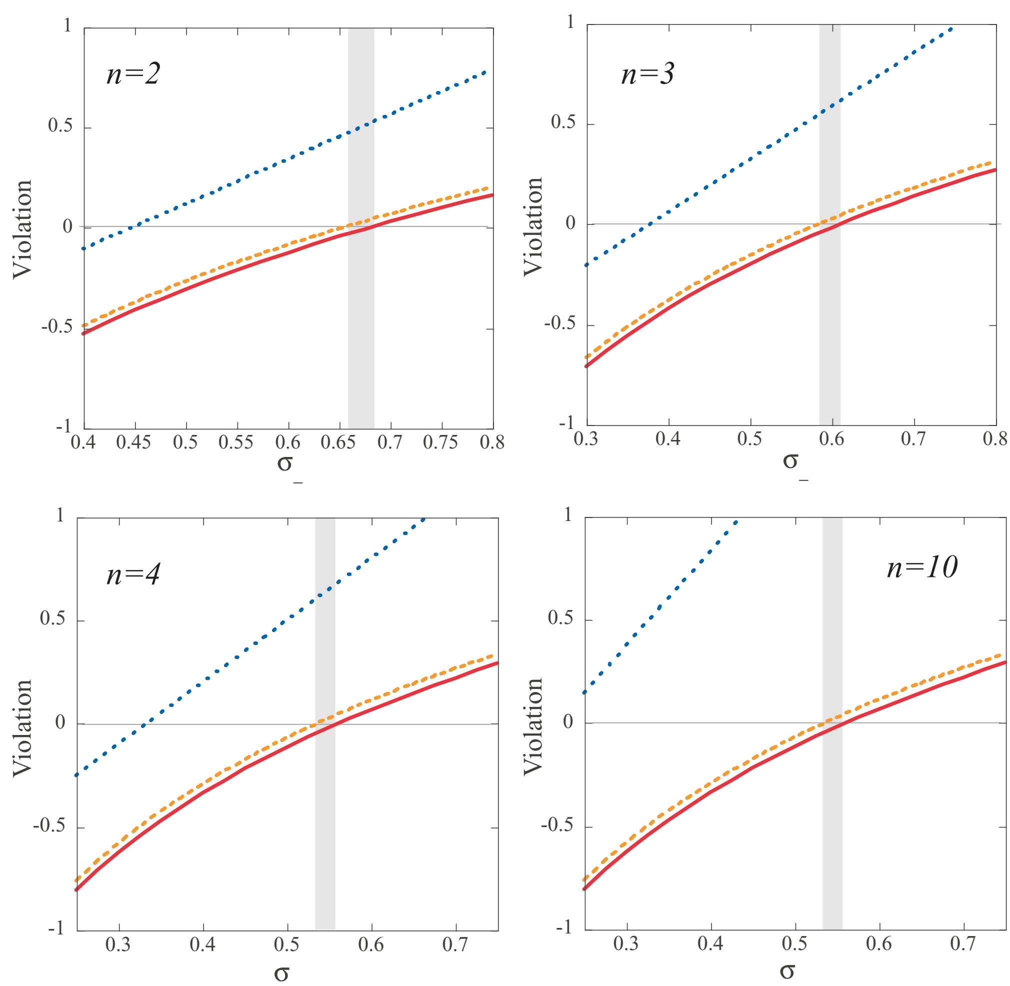

FIG. 2. (Color online) Violation of entanglement criteria as a function of σ_- for Hermite-Gaussian states (27) with $n = 2,3,4,10$ and $\sigma_+ = 1$. Negative values correspond to the detection of entanglement. In all plots the solid red line corresponds to the violation of inequality (16) with $\alpha = 1/2$, the orange dashed line is the Shannon criteria (8) and the blue dotted line the MGVT criteria (20). The gray shaded region shows improvement gained from using the Rényi entropy.

It is also possible to obtain these inequalities by direct application of the uncertainty relation for the discrete Rényi entropies, as developed by Bialynicki-Birula [44,51].

V. EXAMPLES

Here we provide some examples which show the utility of the Rényi entropic criteria presented in Sec. III. We focus on several examples of continuous-variable states which are currently of experimental interest. We leave further numerical investigation to future work.

A. Hermite-Gaussian states

Entropy functions are better quantifiers of information uncertainty, in particular when the probability distributions are not Gaussian functions. In light of this, we consider the general family of states given by

$$\Phi_n(r_1,r_2) = \frac{A_n}{\sqrt{\sigma_+\sigma_-}}\mathcal{H}_n\left(\frac{r_1+r_2}{\sqrt{2}\sigma_+}\right)e^{\frac{-(r_1+r_2)^2}{4\sigma_+^2}}\, e^{\frac{-(r_1-r_2)^2}{4\sigma_-^2}}, \qquad (27)$$

where $\mathcal{H}(x)$ is the nth-order Hermite polynomial. The index n and the widths σ_+ and σ_- characterize the state. For $n \neq 0$, state (27) is nonseparable for any value of parameters σ_+ and σ_-. For $n = 1$ this state has been experimentally produced using spontaneous parametric down-conversion [31] and been shown to have several interesting properties [52–54]. We note that it is equivalent to the single-photon entangled state considered in Ref. [18], when $\sigma_+ = \sigma_- = 1$.

Let us consider first the case $n = 1$. The application of the witness (16), after a lengthy but straightforward calculation, leads to

$$\frac{\sigma_-}{\sigma_+} < \left[\frac{\pi^{\frac{1}{2}}}{\Gamma\left(\alpha+\frac{1}{2}\right)}\left(\frac{\alpha}{2}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}}, \qquad (28a)$$

$$\frac{\sigma_-}{\sigma_+} > \left[\frac{\pi^{\frac{1}{2}}}{\Gamma\left(\alpha+\frac{1}{2}\right)}\left(\frac{\alpha}{2}\right)^{\alpha}\right]^{-\frac{1}{1-\alpha}}, \qquad (28b)$$

where we have included both cases: $\alpha \geqslant 1$ and $1/2 \leqslant \alpha \leqslant 1$. Thus, only entangled states of the form (27) with $n = 1$ that violate one of these inequalities are detected by our entropic Rényi criteria (16). For $\alpha = 1$ the limits $\sigma_-/\sigma_+ < \frac{e^{1-\gamma}}{2}$ and $\sigma_-/\sigma_+ > \frac{2}{e^{1-\gamma}}$ (γ is the Euler's constant) obtained in [24] are recovered. Figure 1 shows the limits of entanglement detection as a function of α. The graph shows that we improve sensibility

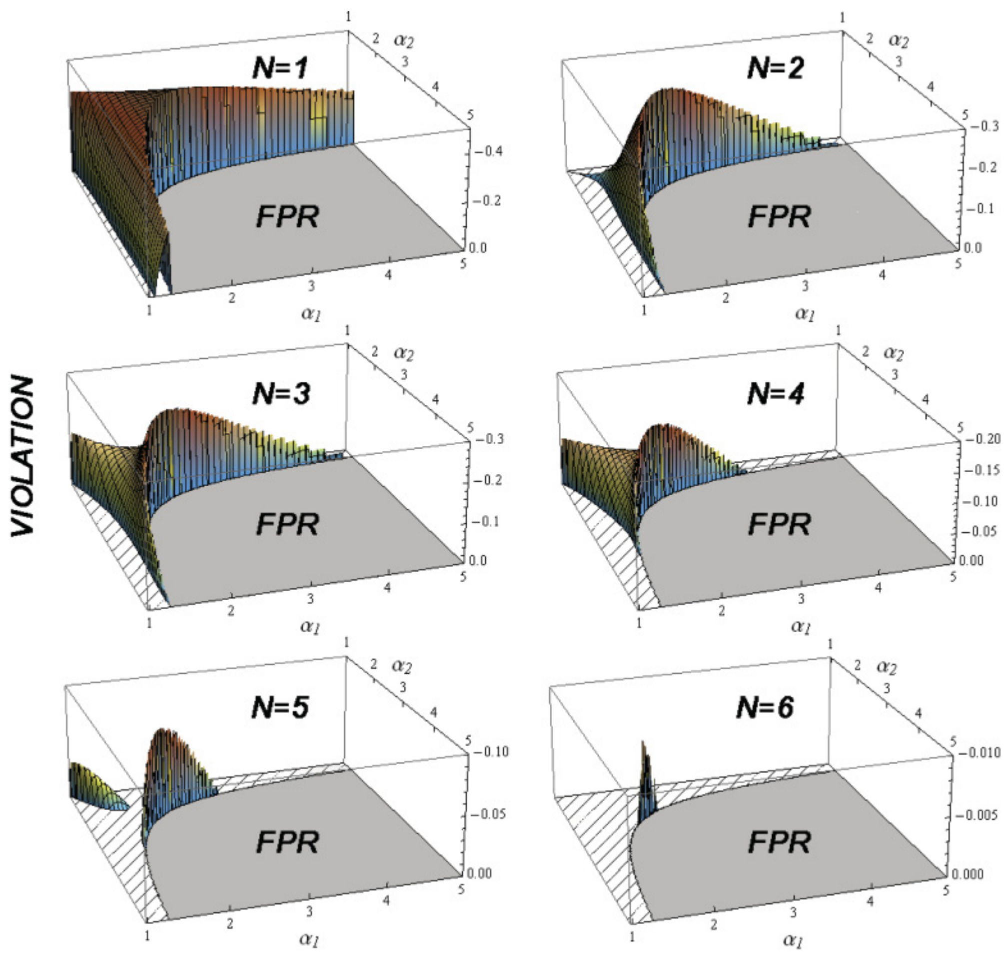

FIG. 3. (Color online) Entanglement detection for NOON state for $N = 1$ to 6. The surfaces represents the regions where the strong Rényi entropic criteria (11) detects entanglement as a function of α_1 and α_2. The criteria were tested for $\theta_j = 0$. FPR designates the "forbidden parameter region," as determined by Eqs. (A7a) and (A7b).

using the Rényi entropic inequality (16) when $\alpha \longrightarrow 1/2$. For example, in the particular case of $\sigma_-/\sigma_+ = 1.3$, entanglement is not detected by the Shannon entropy criteria of (8), but it is detected by the more general Rényi entropy criteria (16). At the same time, there is a large region $(1/\sqrt{3} < \sigma_-/\sigma_+ < \sqrt{3})$ where the second-order Simon criterion [6] does not detect entanglement. The Simon criterion is a necessary and sufficient condition for entanglement in bipartite Gaussian states. So, in the case where the Simon criterion fails to detect entanglement, the covariance matrix of the state is "separable," or in other words, the bipartite Gaussian state with the same covariance matrix is separable. Thus, we can guarantee that any second-order entanglement criterion also fails to detect entanglement in this region.

Inspired by the results shown in Fig. 1 for $n = 1$, we tested the Rényi entanglement criteria (16) with $\alpha = 1/2$, the Shannon criteria (8), and the MGVT criteria (20) for $n = 2,3,4,10$. Figure 2 shows the numerical results for $\sigma_+ = 1$ and $\sigma_- < 1$. For the family of states in (27), the Simon and MGVT criteria detect entanglement in the same region of values of σ_-, so we include only the MGVT criteria for comparison. An additional motivation to compare these three criteria is due to the fact that they can be determined by knowledge of the marginal probability distributions R_+ and S_- (R_- and S_+) alone, while evaluation of the Simon criterion is more costly in terms of measurements since it requires complete reconstruction of the covariance matrix. In all cases the Rényi criteria with $\alpha = 1/2$ outperforms both the Shannon criteria and the second-order MGVT criteria. The gray shaded regions show the interval where only the Rényi criteria with $\alpha = 1/2$ detects entanglement.

B. NOON states

There is a lot of interest in generating entangled "NOON" states of the form

$$|\psi\rangle_{\mathrm{NOON}} = \frac{1}{\sqrt{2}}(|N\rangle_1|0\rangle_2 + |0\rangle_1|N\rangle_2), \tag{29}$$

where $|n\rangle$ is an n-photon Fock state. NOON states are particularly useful for quantum metrology [55]. Here we consider detection of entanglement using continuous-variable quadrature measurements. For NOON states the inequality (16) does not detect entanglement for any value of α (tested for $N \leqslant 10$). However, we have investigated their entanglement detection with the stronger Rényi criteria (11). The results are shown in Figure 3. We have studied the violation of inequality (11) as a function of parameters α_1, α_2, β_1, β_2. In order to simplify the calculations, we have constrained β_1 and β_2 as functions of α_1 and α_2, according to restriction (13) (see Figure 3). The best violations were found for $\alpha_1 = \alpha_2 = 2$. In all cases, we chose quadrature operators (1) with $\theta = 0$. With this choice of parameters we were able to detect entanglement up to $N = 6$, which is an improvement over the Shannon criteria (7) [24]. Numerical results show that entanglement in the NOON states goes undetected under any second-order criterion (tested for $N \leqslant 10$).

C. Dephased cat state

Entangled Schrödinger cat states have been produced experimentally in quadrature variables of two single-mode fields using optical parametric amplification [56]. Due to experimental imperfections these states are mixed. Here we consider mixed states given by the dephased entangled cat states,

$$\rho = N(\nu,p)\{|\nu,\nu\rangle\langle\nu,\nu| + |-\nu,-\nu\rangle\langle-\nu,-\nu| \\ - (1-p)(|\nu,\nu\rangle\langle-\nu,-\nu| + |-\nu,-\nu\rangle\langle\nu,\nu|)\}, \tag{30}$$

where $N(\nu,p)$ is a normalization constant. Parameter p characterizes the dephasing [1], and ν is the complex amplitude of the coherent state $|\nu\rangle$. Entanglement in this state goes undetected under any second-order criterion for all values of ν or p. On the other hand, the Shannon criteria (8) identifies

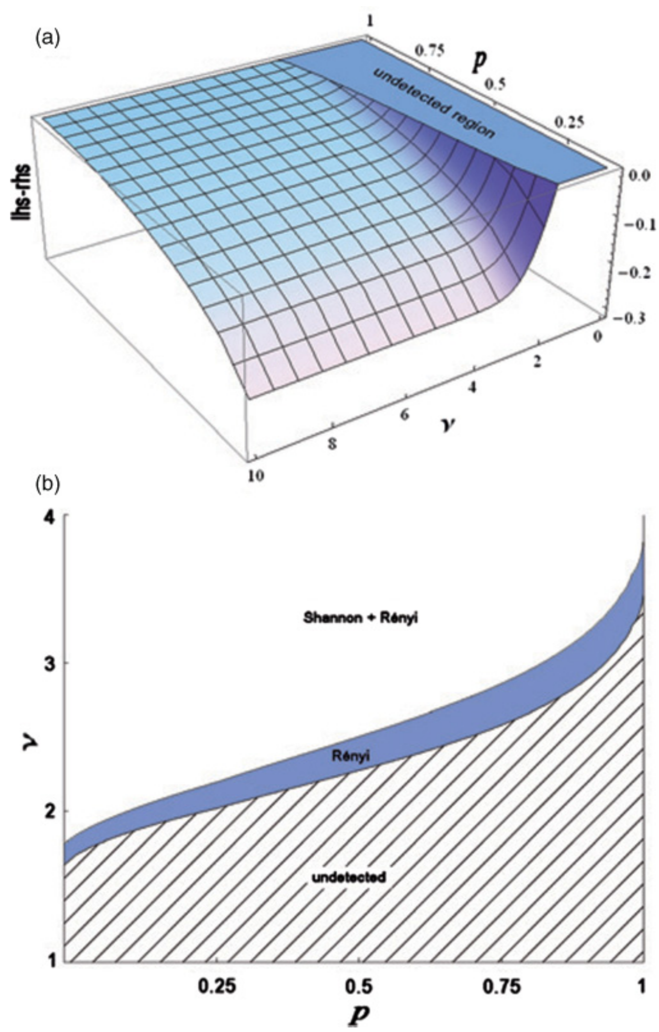

FIG. 4. (Color online) (a) Violation of Rényi entanglement criteria (α very close to 1/2), given by the difference of the left-hand side (lhs) and right-hand side (rhs) of (16) for the dephased cat state (30). (b) Comparison of Shannon criteria and Rényi criteria. The white region is detected by Shannon and Rényi entropic criteria, the blue one is detected only by Rényi entropic criteria ($\alpha \longrightarrow 1/2$) and the hatched area represents the region which remains undetected as function of ν and p.

entanglement for a broad range of values of parameters p and ν [24]. The Rényi entropic criteria (16) with α very close to $1/2$ extends entanglement detection in this state, as we can see in Figure 4(a). Figure 4(b) compares these two results. Presumably, for small ν there is very little correlation present in the nonlocal sum or difference variables, as both criteria fail for small values of ν.

VI. CONCLUSIONS

We have presented a family of entanglement witnesses using generalized classical entropy functions applied to marginal probability distributions $R_\pm$ and $S_\pm$ associated with the measurement of global canonical operators $\mathbf{r}_\pm$ and $\mathbf{s}_\pm$ in continuous-variable systems. First, we employed the Rényi entropy (parameterized by α) for continuous distributions to arrive at a set of inequalities [see Eq. (11)] which are satisfied for all pure bipartite separable states. Second, we introduced a set of inequalities in Eq. (16), using also the Rényi entropy of continuous distributions, which are satisfied for all bipartite states (pure or mixed). We have demonstrated that these criteria offer a greater sensitivity to detection of entanglement. We illustrated this point with several examples where the Rényi entropic criteria identify entanglement, while the Shannon entropic criteria [24] and second-order criteria, such as the Duan-Giedke-Cirac-Zoller (DGCZ) [7], Simon [6], and MGVT [8] tests, do not. We also showed that the entropic criteria given in Eq. (16) are in fact PPT criteria and gave a general recipe to obtain new PPT criteria based on marginal probability distributions $R_\pm$ and $S_\pm$ in continuous-variable systems.

The entanglement witnesses presented here should be very convenient in an experimental setting, as they involve a relatively small number of measurements. In particular, fixing the local rotations involved in the definition of the global operators $\mathbf{r}_\pm$ and $\mathbf{s}_\pm$, it is necessary to determine only the probability distributions $R_\pm$ and $S_\pm$. This can be done directly via measurement of $\mathbf{r}_\pm$ and $\mathbf{s}_\pm$ or from local measurements of the joint probability distributions $R(r_1,r_2)$ and $S(s_1,s_2)$. In order to take into account the precision of the measurement apparatus we extended our Rényi entropy criteria (16) to include discrete distributions [see Eq. (26)].

In addition to practical relevance, the improvement offered by the entropic entanglement criteria is interesting from a theoretical point of view, since there is an entire family of entropic inequalities parametrized by the order of the Rényi entropy (a continuous quantity) that could be explored. Moreover, these results encourage the use of other types of entropy functionals and/or uncertainty relations for entanglement characterization.

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APPENDIX

Here we derive inequality (11) for separable pure states. Using the probability distributions (5), the Rényi entropy for global distributions is

$$H_\alpha[R_\pm] = \frac{\alpha}{1-\alpha}\ln \|R_1 * R_2^{(\pm)}\|_\alpha. \tag{A1}$$

To derive an inequality, we employ Young's inequality, which is valid for convolutions of distributions [57]. For $1/\alpha = 1/\alpha_1 + 1/\alpha_2 - 1$, Young's inequality is

$$\|R_1 * R_2^{(\pm)}\|_\alpha \leqslant C(\alpha_1,\alpha_2)\|R_1\|_{\alpha_1}\|R_2\|_{\alpha_2}, \tag{A2}$$

with $\alpha,\alpha_1,\alpha_2 \geqslant 1$ or

$$\|R_1 * R_2^{(\pm)}\|_\alpha \geqslant C(\alpha_1,\alpha_2)\|R_1\|_{\alpha_1}\|R_2\|_{\alpha_2}, \tag{A3}$$

for $\alpha,\alpha_1,\alpha_2 \leqslant 1$. The coefficient $C(\alpha_1,\alpha_2)$ is given by

$$C(\alpha_1,\alpha_2) = \frac{C_{\alpha_1}C_{\alpha_2}}{C_\alpha}, \tag{A4}$$

where

$$C_t = \sqrt{\frac{t^{\frac{1}{t}}}{|t'|^{\frac{1}{t'}}}}, \tag{A5}$$

with $t' \equiv t/(t-1)$. Without loss of generality, we choose variables such that $\alpha, \alpha_1, \alpha_2 \geqslant 1$ and $0 \leqslant \beta, \beta_1, \beta_2 \leqslant 1$. Then, from inequalities (A2) and (A3) we can write:

$$\|R_\pm\|_\alpha \leqslant C(\alpha_1,\alpha_2)\|R_1\|_{\alpha_1}\|R_2\|_{\alpha_2}, \tag{A6a}$$

and

$$\|S_\mp\|_\beta \geqslant C(\beta_1,\beta_2)\|S_1\|_{\beta_1}\|S_2\|_{\beta_2}, \tag{A6b}$$

where we remember that

$$\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} - 1, \tag{A7a}$$

and

$$\frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2} - 1. \tag{A7b}$$

Dividing inequality (A6a) by inequality (A6b), we can set up a new inequality

$$\frac{\|R_\pm\|_\alpha}{\|S_\mp\|_\beta} \leqslant \frac{C(\alpha_1,\alpha_2)}{C(\beta_1,\beta_2)}\frac{\|R_1\|_{\alpha_1}}{\|S_1\|_{\beta_1}}\frac{\|R_2\|_{\alpha_2}}{\|S_2\|_{\beta_2}}, \tag{A8}$$

which will be verified when the pure state is separable, since the distributions $R_\pm$ and $S_\mp$ can be expressed in terms of convolutions of the probability distributions of the two subsystems.

We can write the norm in terms of the Rényi entropy. Taking the logarithm of inequality (A8) and using Eq. (9) results in an inequality in terms of Rényi entropies:

$$\begin{aligned}
&\left(\frac{\alpha-1}{\alpha}\right)H_\alpha[R_\pm] + \left(\frac{1-\beta}{\beta}\right)H_\beta[S_\mp] \\
&\geqslant \left(\frac{\alpha_1-1}{\alpha_1}\right)H_{\alpha_1}[R_1] + \left(\frac{1-\beta_1}{\beta_1}\right)H_{\beta_1}[S_1] + \left(\frac{\alpha_2-1}{\alpha_2}\right) \\
&\quad\times H_{\alpha_2}[R_2] + \left(\frac{1-\beta_2}{\beta_2}\right)H_{\beta_2}[S_2] + \ln\left[\frac{C(\beta_1,\beta_2)}{C(\alpha_1,\alpha_2)}\right].
\end{aligned} \tag{A9}$$

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