Information, fidelity, and reversibility in single-qubit measurements

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We explicitly calculate information, fidelity, and reversibility of an arbitrary single-qubit measurement on a completely unknown state. These quantities are expressed as functions of a single parameter, which is the ratio of the two singular values of the measurement operator corresponding to the obtained outcome. Thus, our results give information tradeoff relations to the fidelity and to the reversibility at the level of a single outcome rather than that of an overall outcome average.

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I. INTRODUCTION

Quantum measurement provides information on a physical system, while it inevitably changes the state of the system depending on the obtained outcome. This property is of great interest in the foundations of quantum mechanics and is of practical importance in quantum information processing and communication [1] such as quantum cryptography [2–5]. Therefore, numerous studies [6–15] have discussed tradeoff relations between the information gain and the state change in quantum measurement by quantifying them in various ways. For example, Banaszek [7] has shown an inequality between two fidelities quantifying the information gain and the state change.

Interestingly, in connection with such a state change, quantum measurement was widely believed to have intrinsic irreversibility [16] because of nonunitary state reduction. However, it has been shown that quantum measurement is not necessarily irreversible [17,18] if all the information on the system is preserved during the measurement process. In particular, a quantum measurement is said to be physically reversible [18,19] if the pre-measurement state can be recovered from the post-measurement state with a nonzero probability of success by means of a second measurement, known as a reversing measurement. Several physically reversible measurements have been proposed with various systems [20–26] and have been experimentally demonstrated using various qubits [27,28]. Thus, it would be interesting to involve physical reversibility while discussing information tradeoff relations. In fact, in a recent discussion on photodetection processes [29] the existence of a tradeoff relation between the information gain and the physical reversibility has been suggested. Such a tradeoff relation is also expected in view of a different type of reversible measurement, known as a unitarily reversible measurement [30,31], in which the pre-measurement state can be recovered with unit probability by means of a unitary operation, whereas the measurement provides no information about the measured system.

Moreover, physically reversible measurements naturally prompt investigation of the information tradeoff relation at the level of a single outcome [10] rather than that of an overall outcome average because the state recovery by a reversing measurement relies on the postselection of outcomes. That is, the reversing measurement can recover the state of the system changed by a physically reversible measurement only when it yields a preferred outcome. Unfortunately, this state recovery is always accompanied by the erasure of information obtained by the physically reversible measurement (see the Erratum of [21]), implying a tradeoff relation between the information gain and the state change at the single outcome level. However, an approximate recovery by the Hermitian conjugate measurement [32] does not necessarily decrease the information gain.

In this paper, we derive general formulas for the information gain, the state change, and the physical reversibility in quantum measurements, in which the system to be measured is a two-level system or qubit in a completely unknown state. We evaluate the amount of information gain by using a decrease in Shannon entropy [10,32], the degree of state change by using fidelity [33], and the degree of physical reversibility by using the maximal successful probability of a reversing measurement [34]. Because the formulas are written as functions of a single parameter, they lead to information tradeoff relations to the state change and the physical reversibility at a single outcome level. We also consider two efficiencies of the measurement with respect to the state change and the physical reversibility, and we show their different behaviors as functions of a single parameter.

This paper is organized as follows: Section II explains the procedure to quantify the information gain, the state change, and the physical reversibility, and it shows their explicitly calculated formulas in the case of an arbitrary single-qubit measurement. Section III discusses information tradeoff relations to the state change and the physical reversibility, and it defines two efficiencies of the measurement with respect to the state change and the physical reversibility. Section IV summarizes our results.

II. FORMULATION

To evaluate the amount of information provided by a single-qubit measurement, we assume that the pre-measurement state of the qubit is known to be one of the predefined pure states $\{|\psi(a)\rangle\}$ with equal probability, $p(a) = 1/N$, where $a = 1,\ldots,N$, although the index a of the pre-measurement state is unknown to us. Since the pre-measurement state is usually an arbitrary unknown state in quantum measurement, the set $\{|\psi(a)\rangle\}$ actually consists of all possible pure states of the qubit with $N \to \infty$. The lack of information on the state of the qubit can initially be evaluated by the Shannon entropy as

$$H_0 = -\sum_a p(a)\log_2 p(a) = \log_2 N. \tag{1}$$

Next, we measure the qubit to obtain information on its state. In a more general formulation of quantum measurement [1,35], a quantum measurement is described by a set of measurement operators $\{\hat{M}_m\}$ that satisfies

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}, \tag{2}$$

where $\hat{I}$ is the identity operator. That is, if the system to be measured is in a state $|\psi\rangle$, the measurement yields an outcome m with probability

$$p_m = \langle\psi|\hat{M}_m^\dagger \hat{M}_m|\psi\rangle, \tag{3}$$

causing a state reduction of the measured system to

$$|\psi_m\rangle = \frac{1}{\sqrt{p_m}}\hat{M}_m|\psi\rangle. \tag{4}$$

Here, we have assumed that the quantum measurement is efficient [8] or ideal [31] to ignore classical noise that yields a mixed post-measurement state, because we are interested in the quantum nature of measurement. From now on, we focus on a single measurement process with outcome m described by a measurement operator $\hat{M}_m$. The measurement operator $\hat{M}_m$ can always be written by singular-value decomposition as

$$\hat{M}_m = \kappa_m \hat{U}_m \hat{D}_m \hat{V}_m, \tag{5}$$

where κ_m is a real number, $\hat{U}_m$ and $\hat{V}_m$ are unitary operators, and $\hat{D}_m$ is a non-negative operator with diagonal matrix representation in an orthonormal basis $\{|0\rangle,|1\rangle\}$,

$$\hat{D}_m = |0\rangle\langle 0| + \lambda_m|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_m \end{pmatrix}, \tag{6}$$

with $0 \leqslant \lambda_m \leqslant 1$, for the single-qubit measurement. Note that the diagonal element λ_m is the ratio of the two singular values of $\hat{M}_m$. Without loss of generality, we can omit the unitary operator $\hat{V}_m$ as

$$\hat{M}_m = \kappa_m \hat{U}_m \hat{D}_m, \tag{7}$$

by relabeling the index a as $|\psi'(a)\rangle = \hat{V}_m|\psi(a)\rangle$.

If the pre-measurement state is $|\psi(a)\rangle$, measurement (7) yields the outcome m with probability

$$p(m|a) = \kappa_m^2\langle\psi(a)|\hat{D}_m^2|\psi(a)\rangle \equiv \kappa_m^2 q_m(a) \tag{8}$$

as given in Eq. (3). Since the probability for $|\psi(a)\rangle$ is $p(a) = 1/N$, the total probability for the outcome m is given by

$$p(m) = \sum_a p(m|a)p(a) = \frac{1}{N}\sum_a \kappa_m^2 q_m(a) = \kappa_m^2\overline{q_m}, \tag{9}$$

where the overline denotes the average over a,

$$\overline{f} \equiv \frac{1}{N}\sum_a f(a). \tag{10}$$

On the contrary, given the outcome m, we can find the probability for the pre-measurement state $|\psi(a)\rangle$ as

$$p(a|m) = \frac{p(m|a)p(a)}{p(m)} = \frac{q_m(a)}{N\overline{q_m}} \tag{11}$$

from Bayes's rule, which means that the lack of information on the pre-measurement state becomes the Shannon entropy

$$H(m) = -\sum_a p(a|m)\log_2 p(a|m) \tag{12}$$

after the measurement. Therefore, the information gain by the measurement with the *single* outcome m can be defined by the decrease in Shannon entropy as [10,32]

$$I(m) \equiv H_0 - H(m) = \frac{\overline{q_m\log_2 q_m} - \overline{q_m}\log_2\overline{q_m}}{\overline{q_m}}. \tag{13}$$

Note that this information gain is positive and is free from the divergent term $\log_2 N \to \infty$ in Eq. (1). These results essentially arise from the assumption that the probability distribution $p(a)$ is uniform. If averaged over all the outcomes, the information gain reduces to the mutual information [1] of the random variables $\{a\}$ and $\{m\}$, namely,

$$I \equiv \sum_m p(m)I(m) = \sum_{m,a} p(a|m)p(m)\log_2\frac{p(a|m)}{p(a)}. \tag{14}$$

To explicitly calculate the information gain (13), we parametrize the state of the qubit by two continuous angles (θ,ϕ) as

$$|\psi(a)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{15}$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$. Thus, the summation over a is replaced with an integral over (θ,ϕ) as

$$\frac{1}{N}\sum_a \longrightarrow \frac{1}{4\pi}\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta. \tag{16}$$

Since

$$q_m(a) = \cos^2\frac{\theta}{2} + \lambda_m^2\sin^2\frac{\theta}{2} \tag{17}$$

from Eq. (8), the information gain (13) is calculated to be

$$I(m) = 1 - \frac{1}{2\ln 2} - \frac{\lambda_m^4}{1-\lambda_m^4}\log_2\lambda_m^2 - \log_2\left(1+\lambda_m^2\right), \tag{18}$$

which depends only on λ_m. Figure 1 shows the information gain $I(m)$ as a function of λ_m. The information gain $I(m)$ has a maximal value $1 - 1/(2\ln 2)$ at $\lambda_m = 0$ and a minimal value 0 at $\lambda_m = 1$, while monotonically decreasing as λ_m increases. In fact, measurement (7) is a projective measurement when $\lambda_m = 0$ and is the identity operation when $\lambda_m = 1$, except for the unitary operation $\hat{U}_m$.

Unfortunately, the measurement changes the state of the qubit. When the pre-measurement state is $|\psi(a)\rangle$ and the measurement outcome is m, the post-measurement state is given by

$$|\psi(m,a)\rangle = \frac{1}{\sqrt{p(m|a)}}\kappa_m\hat{U}_m\hat{D}_m|\psi(a)\rangle \tag{19}$$

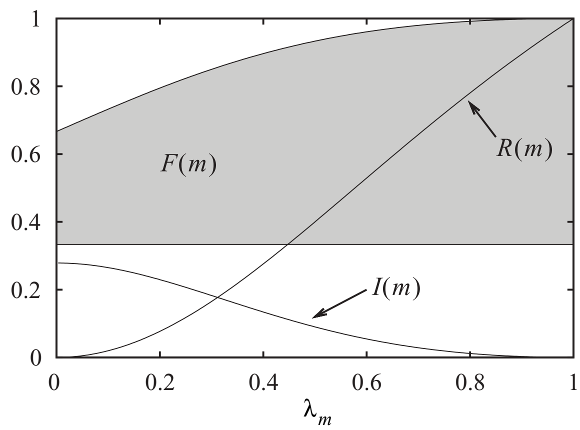

FIG. 1. Information gain $I(m)$, fidelity $F(m)$, and reversibility $R(m)$ when the measurement yields a single outcome m, as functions of λ_m. The parameter $\lambda_m = 0$ corresponds to a projective measurement, and $\lambda_m = 1$ corresponds to the identity operation except for a unitary operation.

from Eqs. (4) and (7). This state change can be quantified by the fidelity [1,33] between the pre-measurement and post-measurement states as

$$F(m,a) = \left|\langle\psi(a)|\psi(m,a)\rangle\right|. \tag{20}$$

As the process of measurement changes the state of qubit to a greater extent, the fidelity becomes smaller. Averaged over a with the probability (11), the fidelity after the measurement with the single outcome m is evaluated as

$$F(m) = \sum_a p(a|m)[F(m,a)]^2 = \frac{1}{q_m}\overline{|\langle\psi|\hat{U}_m\hat{D}_m|\psi\rangle|^2}. \tag{21}$$

Here, we have averaged the squared fidelity rather than the fidelity for simplicity; this choice does not qualitatively affect our results. If the fidelity $F(m)$ is averaged over all the outcomes, it reduces to the mean operation fidelity [7],

$$F \equiv \sum_m p(m)F(m) = \sum_m \overline{|\langle\psi|\hat{M}_m|\psi\rangle|^2}. \tag{22}$$

To explicitly calculate the fidelity (21), we must specify the unitary operator $\hat{U}_m$, in sharp contrast with the case of the information gain (13). We parametrize it in the matrix representation as

$$\hat{U}_m = e^{i\alpha_m}\begin{pmatrix} e^{i\beta_m}\cos\gamma_m & -e^{i\delta_m}\sin\gamma_m \\ e^{-i\delta_m}\sin\gamma_m & e^{-i\beta_m}\cos\gamma_m \end{pmatrix}, \tag{23}$$

where α_m, β_m, γ_m, and δ_m are real. Therefore, the fidelity is calculated to be

$$F(m) = \frac{1}{3} + \frac{1}{3}\left[1 + \frac{2\lambda_m}{1+\lambda_m^2}\cos 2\beta_m\right]\cos^2\gamma_m. \tag{24}$$

For a given λ_m, the lower and upper bounds on the fidelity are given by

$$\frac{1}{3} \leqslant F(m) \leqslant \frac{2}{3}\left[1 + \frac{\lambda_m}{1+\lambda_m^2}\right]. \tag{25}$$

The lower bound does not depend on λ_m and is achieved, e.g., if $\hat{U}_m = |0\rangle\langle 1| + |1\rangle\langle 0|$, whereas the upper bound depends on λ_m and is achieved, e.g., if $\hat{U}_m = \hat{I}$. Because the unitary operator $\hat{U}_m$ causes the state change irrelevant to the information gain $I(m)$, the upper bound

$$F_{\rm opt}(m) \equiv \frac{2}{3}\left[1 + \frac{\lambda_m}{1+\lambda_m^2}\right], \tag{26}$$

which we refer to as optimal fidelity, can be regarded as a measure of the inevitable state change by the extraction of information through the measurement operator $\hat{M}_m$. The fidelity $F(m)$ is also shown in Fig. 1 as a function of λ_m. In particular, the optimal fidelity $F_{\rm opt}(m)$ has a minimal value $2/3$ at $\lambda_m = 0$ and a maximal value 1 at $\lambda_m = 1$, while monotonically increasing as λ_m increases.

Although the measurement changes the state of the qubit as mentioned above, if the measurement is physically reversible [18,19], we can reverse this state change by a reversing measurement. The reversing measurement is constructed so that when it yields a preferred outcome (e.g., 0), it applies a measurement operator

$$\hat{R}_0^{(m)} = \eta_m\hat{M}_m^{-1} = \frac{\eta_m}{\kappa_m}\hat{D}_m^{-1}\hat{U}_m^{\dagger} \tag{27}$$

with a complex number η_m to the post-measurement state $|\psi(m,a)\rangle$ of the qubit, thereby canceling the effect of $\hat{M}_m$ owing to

$$\hat{R}_0^{(m)}\hat{M}_m = \eta_m\hat{I}. \tag{28}$$

That is, when the reversing measurement on $|\psi(m,a)\rangle$ yields the preferred outcome 0, the state of the qubit reverts to the pre-measurement state $|\psi(a)\rangle$ except for an overall phase factor via the state reduction (4),

$$|\psi_{\rm rev}(m,a)\rangle = \frac{1}{\sqrt{p_{\rm rev}(m,a)}}\hat{R}_0^{(m)}|\psi(m,a)\rangle \propto |\psi(a)\rangle, \tag{29}$$

where $p_{\rm rev}(m,a)$ is the successful probability of the reversing measurement defined by

$$p_{\rm rev}(m,a) = \langle\psi(m,a)|\hat{R}_0^{(m)\dagger}\hat{R}_0^{(m)}|\psi(m,a)\rangle = \frac{|\eta_m|^2}{p(m|a)} \tag{30}$$

as given in Eq. (3). Here, we define the physical reversibility by the maximal successful probability of the reversing measurement [23,34,36]. Since the upper bound on $|\eta_m|^2$ is given by [34]

$$|\eta_m|^2 \leqslant \inf_{|\psi\rangle}\langle\psi|\hat{M}_m^{\dagger}\hat{M}_m|\psi\rangle = \kappa_m^2\lambda_m^2 \tag{31}$$

to satisfy $\langle\psi|\hat{R}_0^{(m)\dagger}\hat{R}_0^{(m)}|\psi\rangle \leqslant 1$ for any $|\psi\rangle$, the physical reversibility becomes

$$R(m,a) \equiv \max_{\eta_m} p_{\rm rev}(m,a) = \frac{\kappa_m^2\lambda_m^2}{p(m|a)} = \frac{\lambda_m^2}{q_m(a)}. \tag{32}$$

Averaged over a with the probability (11), the reversibility of the measurement with the single outcome m is evaluated as

$$R(m) = \sum_a p(a|m)R(m,a) = \frac{\lambda_m^2}{q_m} = \frac{2\lambda_m^2}{1+\lambda_m^2}, \tag{33}$$

which depends only on λ_m. The reversibility $R(m)$ is also shown in Fig. 1 as a function of λ_m. It has a minimal value 0 at $\lambda_m = 0$ and a maximal value 1 at $\lambda_m = 1$, while monotonically increasing as λ_m increases. Clearly, measurement (7) is physically reversible unless $\lambda_m = 0$. If the reversibility $R(m)$ is averaged over all the outcomes, it reduces to the degree of physical reversibility of measurement discussed by Koashi and Ueda [34],

$$R \equiv \sum_m p(m)R(m) = \sum_m \inf_{|\psi\rangle} \langle\psi|\hat{M}_m^\dagger \hat{M}_m|\psi\rangle. \tag{34}$$

III. TRADEOFF RELATIONS

Since we have written the information gain $I(m)$, the optimal fidelity $F_{\text{opt}}(m)$, and the reversibility $R(m)$ as functions of the same single parameter λ_m, as given in Eqs. (18), (26), and (33), respectively, it is easy to find relations among them. In fact, we can plot $F_{\text{opt}}(m)$ and $R(m)$ as functions of $I(m)$, as in Fig. 2, to show tradeoff relations *at a single outcome level*. That is, as the measurement provides more information about the state of the qubit, the process of measurement changes the state to a greater extent and makes it even less reversible. These tradeoff relations derive two types of measurement efficiencies: the ratio of the information gain to the optimal fidelity loss,

$$E_F(m) \equiv \frac{I(m)}{1 - F_{\text{opt}}(m)}, \tag{35}$$

and the ratio of the information gain to the reversibility loss,

$$E_R(m) \equiv \frac{I(m)}{1 - R(m)}. \tag{36}$$

Figure 3 shows the efficiencies $E_F(m)$ and $E_R(m)$ as functions of λ_m. Note that $E_F(m)$ is a monotonically increasing function, whereas $E_R(m)$ is a monotonically decreasing function. Therefore, at $\lambda_m = 0$, $E_F(m)$ has a minimal value $3[1 - 1/(2\ln 2)]$ and $E_R(m)$ has a maximal value $1 - 1/(2\ln 2)$. This means that the projective measurement, which provides the most information and causes the largest state change with no reversibility, is the most efficient with respect to the reversibility but is the least efficient with respect to the fidelity. In the limit of $\lambda_m \to 1$, we obtain $E_F(m) \to 1/\ln 2$ and $E_R(m) \to 0$.

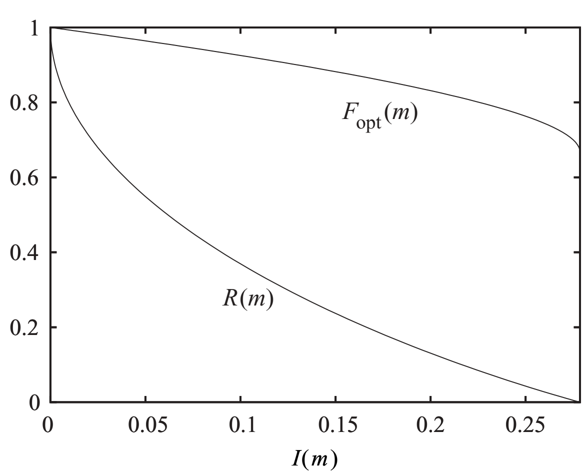

FIG. 2. Optimal fidelity $F_{\text{opt}}(m)$ and reversibility $R(m)$ as functions of the information gain $I(m)$.

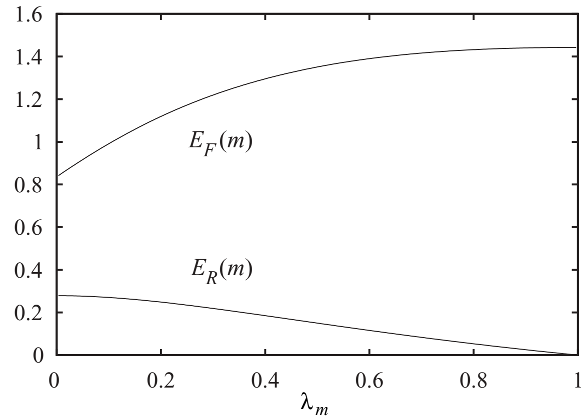

FIG. 3. Efficiencies $E_F(m)$ and $E_R(m)$ of measurement as functions of λ_m.

IV. CONCLUSION

In conclusion, we calculated the information gain, fidelity, and physical reversibility of an arbitrary single-qubit measurement, assuming that the qubit to be measured was in a completely unknown state. These quantities are expressed as functions of the same single parameter λ_m, which is the ratio of the two singular values of the measurement operator corresponding to the outcome, as shown in Eqs. (18), (26), and (33). Our results gave information tradeoff relations to the fidelity and reversibility at the level of a single outcome without averaging all outcomes. Moreover, two efficiencies of the measurement were discussed to show their different behaviors: the ratio of the information gain to the optimal fidelity loss and the ratio of the information gain to the reversibility loss. As the information gain decreases by increasing the parameter λ_m, the former ratio increases whereas the latter decreases.

Our tradeoff relations are applicable to any efficient measurement on a qubit or two-level system with postselection. A characteristic feature of our tradeoff relations is that the information gain is directly related to the fidelity and reversibility for a given measurement $\hat{M}_m$, because all the quantities are functions of the single parameter λ_m. By only eliminating the parameter λ_m, we can obtain the tradeoff curves, as shown in Fig. 2, without optimization problems [7,9,13]. Unfortunately, this does not apply to more general situations. For example, in measurements with an overall outcome average, the information gain (14), fidelity (22), and reversibility (34) are functions of all $\{\lambda_m\}$ and $\{\kappa_m\}$ corresponding to possible outcomes because m is summed over by using the value of the total probability (9),

$$p(m) = \tfrac{1}{2}\kappa_m^2\left(1 + \lambda_m^2\right). \tag{37}$$

In measurements on d-level systems such as qudit or multiple qubits, all quantities are functions of $d-1$ parameters $\{\lambda_m^{(1)}, \ldots, \lambda_m^{(d-1)}\}$ as will be shown elsewhere because the measurement operator is represented by a $d \times d$ matrix in an orthonormal basis. Moreover, in measurements with classical noise, they are functions of multiple parameters because a single measurement process is described by a set of measurement operators. To find tradeoff curves in such situations, we must optimize measurements by maximizing the fidelity or the reversibility with a fixed value of the information gain by using numerical calculations. Our simple and direct tradeoff relations are free from such optimization problems; therefore, they can be regarded as highly fundamental in quantum measurement.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] C. H. Bennett and G. Brassard, in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984), pp. 175–179.

[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[4] C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).

[5] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. **68**, 557 (1992).

[6] C. A. Fuchs and A. Peres, Phys. Rev. A **53**, 2038 (1996).

[7] K. Banaszek, Phys. Rev. Lett. **86**, 1366 (2001).

[8] C. A. Fuchs and K. Jacobs, Phys. Rev. A **63**, 062305 (2001).

[9] K. Banaszek and I. Devetak, Phys. Rev. A **64**, 052307 (2001).

[10] G. M. D'Ariano, Fortschr. Phys. **51**, 318 (2003).

[11] M. Ozawa, Ann. Phys. **311**, 350 (2004).

[12] L. Maccone, Phys. Rev. A **73**, 042307 (2006).

[13] M. F. Sacchi, Phys. Rev. Lett. **96**, 220502 (2006).

[14] F. Buscemi and M. F. Sacchi, Phys. Rev. A **74**, 052320 (2006).

[15] F. Buscemi, M. Hayashi, and M. Horodecki, Phys. Rev. Lett. **100**, 210504 (2008).

[16] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (*Non-Relativistic Theory*), 3rd ed. (Butterworth-Heinemann, Oxford, 1977).

[17] M. Ueda and M. Kitagawa, Phys. Rev. Lett. **68**, 3424 (1992).

[18] M. Ueda, N. Imoto, and H. Nagaoka, Phys. Rev. A **53**, 3808 (1996).

[19] M. Ueda, in *Frontiers in Quantum Physics: Proceedings of the International Conference on Frontiers in Quantum Physics, Kuala Lumpur, Malaysia, 1997*, edited by S. C. Lim, R. Abd-Shukor, and K. H. Kwek (Springer-Verlag, Singapore, 1999), pp. 136–144.

[20] A. Imamoglu, Phys. Rev. A **47**, R4577 (1993).

[21] A. Royer, Phys. Rev. Lett. **73**, 913 (1994); **74**, 1040(E) (1995).

[22] H. Terashima and M. Ueda, Phys. Rev. A **74**, 012102 (2006).

[23] A. N. Korotkov and A. N. Jordan, Phys. Rev. Lett. **97**, 166805 (2006).

[24] H. Terashima and M. Ueda, Phys. Rev. A **75**, 052323 (2007).

[25] Q. Sun, M. Al Amri, and M. S. Zubairy, Phys. Rev. A **80**, 033838 (2009).

[26] Y.-Y. Xu and F. Zhou, Commun. Theor. Phys. **53**, 469 (2010).

[27] N. Katz, M. Neeley, M. Ansmann, R. C. Bialczak, M. Hofheinz, E. Lucero, A. O'Connell, H. Wang, A. N. Cleland, J. M. Martinis, and A. N. Korotkov, Phys. Rev. Lett. **101**, 200401 (2008).

[28] Y.-S. Kim, Y.-W. Cho, Y.-S. Ra, and Y.-H. Kim, Opt. Express **17**, 11978 (2009).

[29] H. Terashima, e-print arXiv:1010.5075.

[30] H. Mabuchi and P. Zoller, Phys. Rev. Lett. **76**, 3108 (1996).

[31] M. A. Nielsen and C. M. Caves, Phys. Rev. A **55**, 2547 (1997).

[32] H. Terashima and M. Ueda, Phys. Rev. A **81**, 012110 (2010).

[33] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976).

[34] M. Koashi and M. Ueda, Phys. Rev. Lett. **82**, 2598 (1999).

[35] E. B. Davies and J. T. Lewis, Commun. Math. Phys. **17**, 239 (1970).

[36] M. Ban, J. Phys. A: Math. Gen. **34**, 9669 (2001).