Nonpolynomial Schrödinger equation for resonantly absorbing gratings

Lior Shabtay and Boris A. Malomed
Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel
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We derive a nonlinear Schrödinger equation with a radical term, $\sim \sqrt{1-|V|^2}$, as an asymptotic model of the resonantly absorbing Bragg reflector (RABR), i.e., a periodic set of thin layers of two-level atoms, resonantly interacting with the electromagnetic field and inducing the Bragg reflection. A family of bright solitons is found, which splits into stable and unstable parts, exactly obeying the Vakhitov-Kolokolov criterion. The soliton with the largest amplitude, $(|V|)_{\max} = 1$, is a "quasipeakon," i.e., a solution with a discontinuity of the third derivative at the center. Families of exact cnoidal waves, built as periodic chains of quasipeakons, are found too. The ultimate solution belonging to the family of dark solitons, with the background level $V = 1$, is a dark compacton. Those bright solitons that are unstable destroy themselves (if perturbed) attaining the critical amplitude, $|V| = 1$. The dynamics of the wave field around this critical point is studied analytically, revealing a switch of the system into an unstable phase, in terms of the RABR model. Collisions between bright solitons are investigated too. The collisions between fast solitons are quasielastic, while slowly moving ones merge into breathers, which may persist or perish (in the latter case, also by attaining $|V| = 1$).

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I. INTRODUCTION AND THE MODEL

The interplay between the resonant reflection of light on Bragg gratings (BGs) and resonant interaction of light with nanolayers of two-level atoms (of width $\lesssim$100 nm, which is much smaller than the wavelength of light) or with similar active elements, deposited at reflecting layers of which the BG is built, gives rise to artificial optical media in the form of *resonantly absorbing Bragg reflectors* (RABRs). They are promising for fundamental studies and applications [1–5], including the storage of slow-light pulses [6], negative reflection [7], and periodically amplifying BGs [8]. The currently available nanofabrication techniques make the creation of RABRs with required properties quite feasible [9]. In particular, the interplay of the resonant nonlinearity (which gives rise to the self-induced transparency in uniform media [10]) with the bandgap spectrum induced by the BG may give rise to peculiar species of temporal solitons in RABRs (see Ref. [11] for a review and more recent works [12–14]). These solitons belong to the class of *gap solitons*, whose propagation constant falls into the underlying bandgap. Gap solitons in fiber Bragg gratings with the uniform Kerr nonlinearity have been a subject of intensive theoretical studies [15] and have been created experimentally, in the form of moving *Bragg solitons* [16].

Using the rotating-wave approximation, the Maxwell-Bloch equations governing the transmission of light in the RABR can be reduced to the system of equations for the scaled slowly varying variables, *viz.*, the amplitude of the electromagnetic field Σ_+, polarization of the medium P, and population inversion w of two-level atoms [3,11]:

$$(\Sigma_+)_{\tau\tau} - (\Sigma_+)_{\zeta\zeta} = -\eta^2\Sigma_+ + 2i\eta P + 2P_\tau, \qquad (1)$$

$$P_\tau = -i\delta P + \Sigma_+ w, \quad w_\tau = -\mathrm{Re}(\Sigma_+ P^*), \qquad (2)$$

where τ and ζ are the scaled time and coordinate (another amplitude, Σ_-, is governed by a detached equation). Coefficient δ, which may be positive or negative, measures the detuning of the transition frequency of the two-level atoms from the carrier frequency of the electromagnetic field, and η, which is defined to be positive, is the scaled BG reflectivity.

A straightforward corollary of Eqs. (2) is $\partial(|P|^2 + w^2)/\partial\tau = 0$, i.e., $|P|^2 + w^2 = \text{const}$. The normalization may be set by fixing const = 1, hence w can be eliminated in favor of P [3]:

$$w = \pm\sqrt{1 - |P|^2}. \qquad (3)$$

The stable situation is determined by the condition that, in the absence of the polarization ($P = 0$), the atomic population must be uninverted ($w = -1$); hence the lower sign must be chosen in Eq. (3). The opposite situation is possible too, but it is unstable, corresponding to an inverted population in the absence of the field.

Thus, assuming that the stable branch of square root (3) is used, $w = -\sqrt{1 - |P|^2}$, one arrives at the system of two equations [3]:

$$(\Sigma_+)_{\tau\tau} - (\Sigma_+)_{\zeta\zeta} = -\eta^2\Sigma_+ + 2i(\eta - \delta)P - 2\sqrt{1 - |P|^2}\Sigma_+, \qquad (4)$$

$$P_\tau = -i\delta P - \sqrt{1 - |P|^2}\,\Sigma_+. \qquad (5)$$

Looking for bright-soliton solutions to Eqs. (4) and (5) as $\Sigma_+ = \exp(-i\omega\tau)\mathcal{S}(\zeta), P = i\exp(-i\omega\tau)\mathcal{P}(\zeta)$, solutions for real functions $\mathcal{S}(\zeta)$ and $\mathcal{P}(\zeta)$ were found in an implicit analytical form in Ref. [3]. In the general case, these solitons fall into two distinct bandgaps produced by the linearized version of the system. Dark-soliton solutions were also studied in Ref. [3].

The present paper is focused on subfamilies of solitons residing near the edge of one of the gaps, $\omega = \delta$. First, we aim to show that, in an asymptotic approximation valid in this case, Eqs. (4) and (5) reduce to a single nonpolynomial Schrödinger equation (NPSE), with the nonlinear term in the form of a radical. To this end, solutions to Eqs. (4) and (5) are looked for as

$$\Sigma_+ = e^{-i\delta\tau}Q^{-1}V(\zeta,\tau), \quad P = -ie^{-i\delta\tau}[V(\zeta,t) + R(\zeta,\tau)], \qquad (6)$$

where $Q \equiv (\eta + \delta)/2$, $V(\zeta,\tau)$ is assumed to be a slowly varying function of time, in comparison with $\exp(-i\delta\tau)$, and R is a small correction to V required by the self-consistent derivation. Actually, the slow time dependence in V accounts for a small deviation of the full frequency from $\omega = \delta$; in other words, for the purpose of the asymptotic analysis δ and η may be considered as large parameters, which corresponds to a far-detuned strongly reflecting RABR. Then it is a simple exercise to demonstrate that the self-consistent asymptotic approximation corresponds to

$$R = \delta^{-1}(iV_\tau - Q^{-1}\sqrt{1-|V|^2}V) \quad (7)$$

in Eq. (6), and the asymptotic NPSE takes the following form:

$$iV_t + V_{\zeta\zeta} - \epsilon\sqrt{1-|V|^2}V = 0, \quad (8)$$

where the rescaled time is $t \equiv \delta(\eta^2+\delta^2)^{-1}\tau$ and $\epsilon \equiv 2\eta/\delta$. An additional obvious rescaling of t and ζ allows one to fix $\epsilon \equiv \pm 1$ for $\delta \gtrless 0$, which is adopted below. It is easy to see that Eq. (8) gives rise to bright- and dark-soliton solutions for $\epsilon = +1$ and -1, respectively, i.e., for positive and negative values of the mismatch, $\delta > 0$ and $\delta < 0$. Equation (8) conserves three dynamical invariants, *viz*., the momentum, Hamiltonian, and norm:

$$N = \int_{-\infty}^{+\infty} |V(\zeta)|^2 d\zeta. \quad (9)$$

Stationary solutions to Eq. (8) (in particular, solitons) are looked for in the usual form:

$$V(\zeta,t) = e^{-i\chi t}W(\zeta), \quad (10)$$

where real function $W(\zeta)$ satisfies the equation

$$\frac{d^2W}{d\zeta^2} = -\chi W + \epsilon\sqrt{1-W^2}W \equiv -\frac{dU_{\mathrm{eff}}}{dW}, \quad (11)$$

with effective potential $U_{\mathrm{eff}} = (1/2)\chi W^2 + (\epsilon/3)(1-W^2)^{3/2}$. Substituting such stationary solutions back into Eqs. (6) and (7), one can reproduce the respective solutions in the framework of the underlying RABR model. In particular, it is worthy to note that real stationary solutions $W(\zeta)$ correspond, according to Eqs. (6) and (7), to *complex* stationary states of the material polarization, i.e., those with an intrinsic *chirp*.

Another noteworthy finding reported below, which directly pertains to the relation between solutions to the NPSE (8) and solutions of the underlying system of Eqs. (1) and (2), is that when, in the course of the dynamical evolution governed by Eq. (8), $|V(\zeta,t)|$ attains the critical value, $|V| = 1$, i.e., according to Eq. (6), the polarization attains its critical value, $|P| = 1$, the further evolution of the system leads to switching from the stable branch of relation (3) to the unstable one, i.e., from $w = -\sqrt{1-|P|^2}$ to $w = +\sqrt{1-|P|^2}$. If this happens, the subsequent use of Eq. (8) for the slowly varying field becomes irrelevant, because the decay of the unstable state may be fast, making it necessary to get back to the use of the full system (1), (2) (which is beyond the scope of this work).

Asymptotic equations for a single slowly varying amplitude can also be derived near other edges of the two gaps that may be occupied by solitons in the full system of Eqs. (1) and (2). However, in other cases the eventual equation reduces to the usual cubic nonlinear Schrödinger equation (CNLSE). On the other hand, it is relevant to compare the NPSE in the form of Eq. (8) with the equation which was first derived, under the same name (NPSE), as the one-dimensional asymptotic form of the Gross-Pitaevskii equation (GPE) for the wavefunction of a self-attractive Bose-Einstein condensate (BEC) trapped in a cigar-shaped potential [17]. In the notation similar to that adopted here, the BEC equation (a.k.a. the *Salasnich equation*) is

$$iV_t + V_{\zeta\zeta} + \frac{1-(3/2)|V|^2}{\sqrt{1-|V|^2}}V = 0. \quad (12)$$

Both equations (8) and (12), give rise to a singularity when the local amplitude attains the critical value, $|V| = 1$. In the case of Eq. (12), this singularity leads to the *collapse* of the wavefunction, which is a property inherited from the full GPE for the self-attractive BEC in the three-dimensional space [17]. The purport of the singularity in Eq. (8) is demonstrated below: Hitting the critical amplitude, the system switches into the unstable phase, which is represented, in terms of the underlying RABR model, by square root (3) with the upper sign.

Another type of the NPSE was derived in Ref. [18] as the effective one-dimensional reduction of the GPE for the *self-repulsive* BEC. It seems as Eq. (8) with $\epsilon = +1$ and $\sqrt{1-|V|^2}$ replaced by $\sqrt{1+|V|^2}$. Of course, such an equation does not give rise to bright solitons. Nevertheless, it can generate bright *gap solitons*, if this nonlinearity is combined with a periodic linear potential (the optical lattice) [19].

The rest of the paper is organized as follows. In Sec. II, we report analytical results for bright and dark solitons in Eq. (8) with $\epsilon = +1$ and -1, respectively, as well as for cnoidal waves in the former case and for the switch of the system into the unstable phase in both cases. In that section we also report results of simulations confirming the stability and instability of the bright solitons, as predicted in the analytical form by means of the Vakhitov-Kolokolov criterion. Simulations of two-soliton states and collisions between moving bright solitons are reported in Sec. III. The paper concludes in Sec. IV.

II. ANALYTICAL RESULTS

A. Bright solitons

Solutions to stationary equation (11) can be represented by means of the formal energy integral:

$$\left(\frac{dW}{d\zeta}\right)^2 + \chi W^2 + \frac{2\epsilon}{3}(1-W^2)^{3/2} = \mathrm{const}. \quad (13)$$

For bright solitons, with $W(|\zeta| = \infty) = 0$, which correspond to $\epsilon = +1$, as said above, and, accordingly, const $= 2/3$ in Eq. (13), $W(\zeta)$ attains its maximum value (A) at the center of the soliton, where $dW/d\zeta$ vanishes. Therefore, amplitude A can be found by setting $dW/d\zeta = 0$ in Eq. (13) with $\epsilon = +1$:

$$A^2 = \frac{3}{2}\left[1 - \frac{3}{4}\chi^2 - \sqrt{\left(1-\frac{3}{4}\chi^2\right)^2 - \frac{4}{3}(1-\chi)}\right]. \quad (14)$$

While the bandgap where bright solitons may reside is, formally, semi-infinite in the framework of Eq. (8): $\chi < 1$, the solitons actually exist in a finite interval, $2/3 < \chi < 1$,

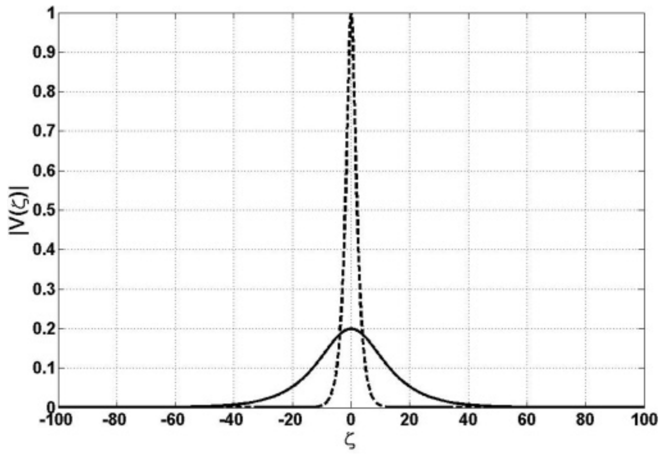

FIG. 1. The solitonic profiles corresponding to $\chi = 0.67$ and $\chi = 0.99$ (dashed and solid curves, respectively). The numerically found solutions are indistinguishable from, severally, the exact "quasipeakon" solution (15) found for $\chi = 2/3$ and the approximate solution (17), which is relevant for $1 - \chi \ll 1$.

in which squared amplitude (14) varies from 1 to 0. At the limit point of $\chi = 2/3$, the soliton solution can be found in an explicit form:

$$W_{\chi=2/3}(\zeta) = 2\tanh\left[\frac{|\zeta|}{\sqrt{3}} + \ln(\sqrt{2}+1)\right] \times \operatorname{sech}\left[\frac{|\zeta|}{\sqrt{3}} + \ln(\sqrt{2}+1)\right]. \quad (15)$$

The expansion of solution (15) around zero is

$$W_{\chi=2/3}(\zeta) = 1 - (1/3)\zeta^2 + (1/6)\sqrt{2/3}|\zeta|^3 + O(\zeta^4). \quad (16)$$

As seen from here, a peculiarity of this solution is that, while both $W(\zeta)$ and its first two derivatives are continuous at $\zeta = 0$, the third derivative, $d^3W/d\zeta^3$, suffers a discontinuity, jumping from $-\sqrt{2/3}$ to $+\sqrt{2/3}$ as ζ crosses zero. In this sense, this exact solution may be called a *quasipeakon*, usual peakons being solitons with a jump of the first derivative at the center [21].

In the limit of $1 - \chi \to 0$, the smallness of amplitude (14) suggests expanding the radical in Eq. (8), which reduces the equation to the usual CNLSE, and the soliton solutions, accordingly, take the following form:

$$V = 2\sqrt{1-\chi}\, e^{-i\chi t} \operatorname{sech}(\sqrt{1-\chi}\,\zeta). \quad (17)$$

Solitons close to those given by Eqs. (15) and (17) are displayed in Fig. 1.

The stability of the solitons can be predicted by means of the Vakhitov-Kolokolov (VK) criterion, $dN/d\chi < 0$, where N is the norm defined by Eq. (9) [22]. The numerically calculated curve $N(\chi)$ for the entire soliton family is displayed in Fig. 2, along with the amplitude $A(\chi)$, as given by analytical expression (14) (which completely coincides with its numerical counterpart), and the soliton's width $L(\chi)$, defined as $L^2 \equiv N^{-1}\int_{-\infty}^{+\infty}\zeta^2 W^2(\zeta)\, d\zeta$. It is seen in Fig. 2(a) that the VK criterion predicts the stability of the solitons in interval

$$\chi_{\rm cr} \approx 0.7120 < \chi < 1, \quad (18)$$

and instability in the remaining part of the existence region, $2/3 < \chi < \chi_{\rm cr}$.

Direct simulations of the evolution of perturbed bright solitons, performed in the framework of Eq. (8), corroborate this prediction: Strong perturbations added to VK-stable solitons gradually fade, leaving the soliton intact, as shown in Fig. 3(a). On the other hand, the evolution of perturbed VK-unstable solitons does not lead to their immediate destruction. Rather, as shown in Fig. 3(b), the amplitude of the soliton grows, crossing the critical level $|V| = 1$. Then formal continuation of the simulations shows a blowup [see, e.g., the panel corresponding to $t = 4$ in Fig. 3(b)], which is a manifestation of the fact that the model as a whole becomes unstable after hitting the critical level (see below).

B. Cnoidal waves

In addition to the bright solitons, a subfamily of exact periodic solutions in the form of *cnoidal waves* (expressed in terms of the Jacobi's elliptic functions) can also be found from energy equation (13) with $\epsilon = +1$, by setting const $= \chi$ on its right-hand side. First, in the case of

$$0 < \chi < 2/3, \quad (19)$$

the period of the cnoidal solution is defined as an interval of coordinate ζ in which a continuously varying elliptic function,

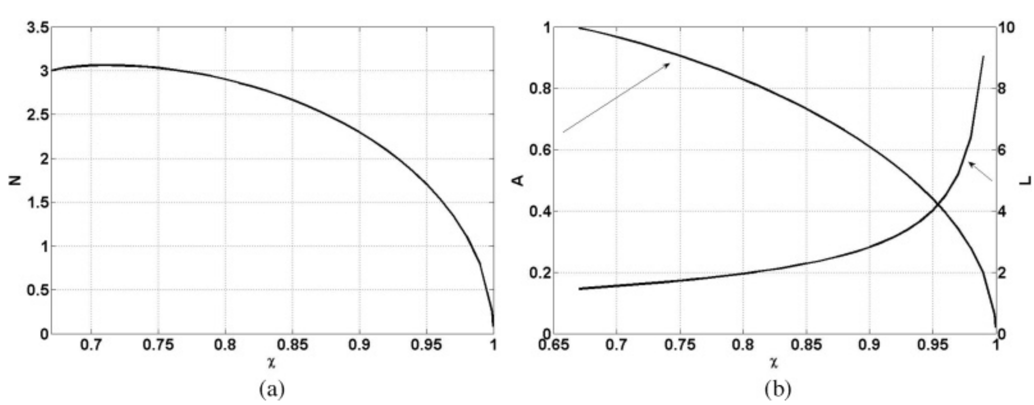

FIG. 2. The norm (a) and amplitude and width (b) of the soliton versus its intrinsic frequency χ. According to the VK criterion, the solitons are stable at $dN/d\chi < 0$.

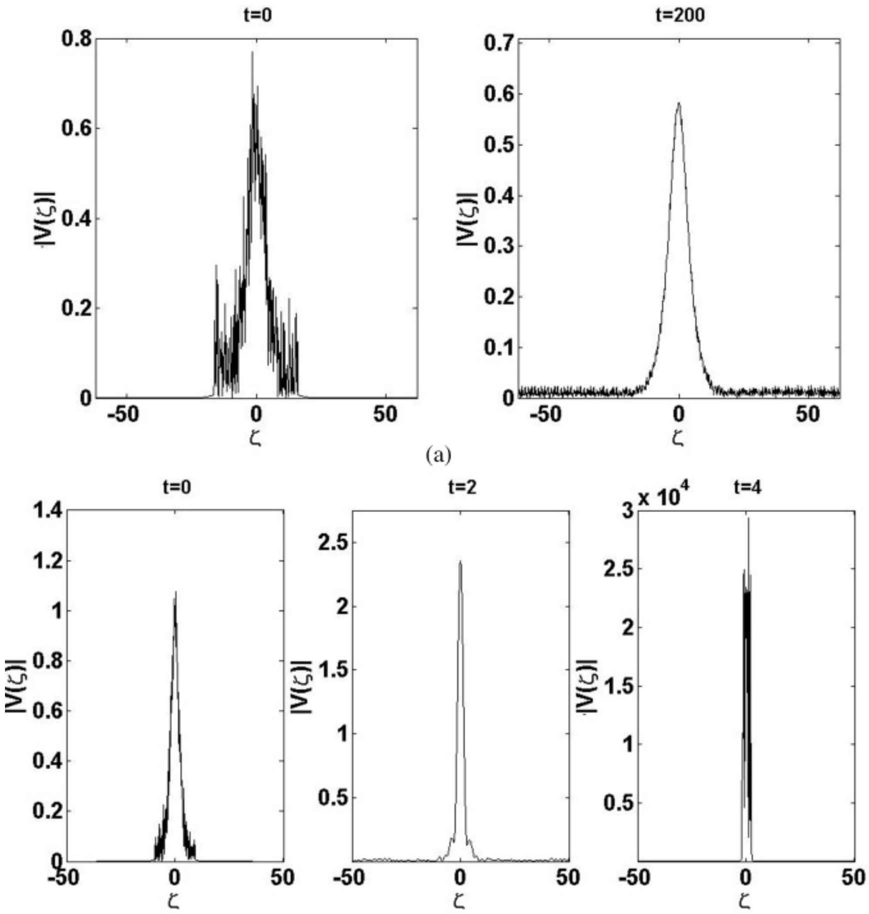

FIG. 3. (a) Self-cleaning of a stable soliton with $\chi = 0.9$, to which a strong random perturbation was added at $t = 0$. (b) The evolution of a slightly perturbed soliton with $\chi = 0.69$, which belongs to the VK-unstable subfamily. The crossing of the critical level, $|V| = 1$, implies the transition into the unstable phase. The subsequent blowup indicates the loss of the system's stability.

$\mathrm{sn}(\zeta/\sqrt{3},k)$ with modulus

$$k = \sqrt{(1/2)[1 + (3/2)\chi]}, \tag{20}$$

takes values that are not too small:

$$(\sqrt{2}k)^{-1} \equiv (1 + 3\chi/2)^{-1/2} \leqslant \mathrm{sn}(\zeta/\sqrt{3},k) \leqslant 1. \tag{21}$$

In this interval, the solution is

$$W(\zeta) = 2k\,\mathrm{sn}(\zeta/\sqrt{3},k)\mathrm{dn}(\zeta/\sqrt{3},k), \tag{22}$$

where dn is the other standard elliptic function, and the entire solution is built as a chain of the so defined periods. Note that k given by Eq. (20) takes values $1/\sqrt{2} < k < 1$ if χ belongs to region (19). As follows from Eqs. (22) and (21), at junction points between adjacent periods, where the left inequality in Eq. (21) turns into the equality, the solution attains the critical value, $W = 1$, and its derivative vanishes, $dW/d\zeta = 0$, hence the matching at the junctions is continuous for both $W(\zeta)$ and $dW/d\zeta$, as well as for $d^2W/d\zeta^2$ [according to Eq. (11), $d^2W/d\zeta^2 = -\chi$ at $W = 1$]. Only the third derivative is discontinuous at the junction, jumping between $-\sqrt{\chi}$ and $+\sqrt{\chi}$. In this sense, these cnoidal solutions are built as periodic chains of *quasipeakons*, cf. the exact soliton solution [21] of the same type. Note also that the bright solitons do not exist in the entire interval (19); i.e., the existence regions of this type of the cnoidal waves and solitons are separated.

Exact solutions for cnoidal waves take a different form at $\chi > 2/3$ (recall the solitons exist in the region of $2/3 \leqslant \chi < 1$, i.e., the cnoidal waves may coexist with the solitons in this case, although the cnoidal solutions exists also at $\chi > 1$, where the solitons cannot be found). In this case, the solution is built of elliptic functions sn and cn with modulus

$$k = \sqrt{2[1 + (3/2)\chi]^{-1}} \tag{23}$$

[cf. Eq. (20)]. Note that expression (23) is the inverse of (20), and it takes values $k < 1$ for $\chi > 2/3$. The period of the cnoidal solution is now defined by inequalities

$$1/\sqrt{2} \leqslant \mathrm{sn}[\sqrt{(1/6)[1 + (3/2)\chi]}\zeta,k] \leqslant 1 \tag{24}$$

[cf. Eq. (21)], and the solution is

$$W(\zeta) = 2\,\mathrm{sn}[\sqrt{(1/6)[1 + (3/2)\chi]}\zeta,k] \times \mathrm{cn}[\sqrt{(1/6)[1 + (3/2)\chi]}\zeta,k]. \tag{25}$$

At junction points between adjacent periods, W again attains the critical value, $W = 1$, the first derivative vanishes, $dW/d\zeta = 0$, the second derivative is continuous, taking value $d^2W/d\zeta^2 = -\chi$, while the third derivative jumps between values $\pm\sqrt{\chi}$ [cf. the jump of the third derivative in expansion (16) for the "quasipeakon"].

Actually, the quasipeakon solution (15) corresponds to the limit form of both cnoidal families at the border between them, $\chi = 2/3$. We stress that the above exact cnoidal-wave solutions, which depend on the single parameter χ, represent only particular cases of a general family of periodic solutions, which depend on two parameters, χ and const in Eq. (13) (and cannot be expressed in terms of the Jacobi's elliptic functions).

C. Dark solitons

An obvious condition necessary for the stability of dark solitons is the modulational stability of the continuous wave (CW) states,

$$V_{\rm CW} = e^{-i\chi t}\sqrt{1-\chi^2}, \tag{26}$$

with χ taking values $-1 < \chi < 0$. A straightforward analysis demonstrates that all CWs (26) are indeed stable, if $\epsilon = -1$ in Eq. (8).

Solutions for dark solitons, $V(\zeta,t) = e^{-i\chi t}W(\zeta)$, which approach CW (26) with frequency χ at $\zeta \to \pm\infty$, can be found as solutions to Eq. (13) with $\epsilon = -1$ and const $= \chi - (1/3)\chi^3$. A straightforward analysis demonstrates that, in the limit case of $\chi = 0$, the solution takes a peculiar form of a *compacton-shaped* dark soliton:

$$W(\zeta) = \begin{cases} \mathrm{sn}(\zeta/\sqrt{3},1/\sqrt{2})\sqrt{2-\mathrm{sn}^2(\zeta/\sqrt{3},1/\sqrt{2})}, & \text{at } |\zeta| < \sqrt{3}K(1/\sqrt{2}), \\ 1, & \text{at } |\zeta| \geqslant \sqrt{3}K(1/\sqrt{2}), \end{cases} \tag{27}$$

where the modulus of sn is $1/\sqrt{2}$, and $K(1/\sqrt{2})$ is the corresponding value of the complete elliptic integral of the first kind. Previously, dark compactons were reported in several discrete models [20], but we are not aware of solutions similar to that given by Eq. (27) in continual models.

D. The switch into the unstable phase

The crossing of the critical amplitude level, $|V| = 1$, by an evolving solution can be studied in an analytical form too. To this end, a near-critical solution is looked for as

$$V(\zeta,t) = 1 - [v_1(\zeta,t) + i v_2(\zeta,t)], \tag{28}$$

where functions v_1 and v_2 are real and imaginary parts of small perturbations around $V = 1$, i.e., $v_{1,2}^2 \ll 1$. The substitution of this expression into Eq. (8) and a straightforward asymptotic expansion yields the following equations:

$$(v_2)_t - (v_1)_{\zeta\zeta} - \epsilon\sqrt{2v_1} = 0, \tag{29}$$

$$(v_1)_t + (v_2)_{\zeta\zeta} = 0. \tag{30}$$

Solutions to Eq. (29) make sense if $v_1(\zeta,t)$ does not become negative.

1. *The case of* $\epsilon = -1$

First, we consider this issue for Eq. (8) with $\epsilon = -1$, which admits stable CW states. In this case, a family of *exact* solutions to asymptotic equations (29) and (30) can be looked for in the following form:

$$v_1(\zeta,t) = (1/2)\,(\beta\zeta^2 - at)^2,$$
$$v_2(\zeta,t) = (1/2)[bt^2 + ct\zeta^2 + (\gamma/2)\zeta^4], \tag{31}$$

with constants a,β,b,c,γ. Here $t = 0$ is defined as the moment of time at which v_1 vanishes for the first time, i.e., the critical value $|V| = 1$ is attained, in the framework of the asymptotic approximation. Note that the choice of the ansatz for v_1 in the form of the full square in expression (31) guarantees that $v_1(t,\zeta)$ remains positive, as it must be. The substitution of the ansatz into Eqs. (29) and (30) gives rise to the following exact relations:

$$b = (1-2\beta)a, \quad c = -a^2, \quad \gamma = (1/3)a\beta,$$
$$a = \sqrt{2\beta(1-6\beta)}, \tag{32}$$

where β remains an arbitrary parameter, taking values $0 < \beta < 1/6$.

Ansatz (31), subject to conditions (32), yields an exact solution to Eqs. (29) and (30) with $\epsilon = -1$, provided that $\sqrt{2v_1(\zeta,t)}$ is realized, when the ansatz is substituted into Eq. (29), as $\beta\zeta^2 - at$, but *not* as $|at - \beta\zeta^2|$ (the latter expression cannot provide for a solution). An explicit form of $\sqrt{1-|V|^2}$, as given by Eqs. (28), (31), and (32), with regard to the above-mentioned realization of $\sqrt{2v_1(\zeta,t)}$, is

$$\sqrt{1-|V|^2} \approx \sqrt{2v_1(\zeta,t)} = \beta\zeta^2 - \sqrt{2\beta(1-6\beta)}t. \tag{33}$$

As follows from Eq. (33), $\sqrt{1-|V|^2}$ does not vanish (i.e., $|V| < 1$ holds) at $t < 0$, which is the *precritical* stage of the evolution. At the *critical moment* of time, $t = 0$, $\sqrt{1-|V|^2}$ vanishes at point $\zeta = 0$. Then, as seen from Eq. (29), at $t > 0$ (at the *postcritical* stage) $\sqrt{1-|V|^2}$ vanishes at two points, $\zeta_0(t) = \pm\sqrt{2(\beta^{-1}-6)t}$. In the *instability domain* between these points,

$$-\sqrt{2(\beta^{-1}-6)t} < \zeta < +\sqrt{2(\beta^{-1}-6)t}, \tag{34}$$

which emerges at $t = 0$ and expands with time as $\sqrt{t}$, expression (33), i.e., eventually, square root (3) [see Eq. (6)], switches from the stable (lower) branch to the unstable (upper) one. Further evolution of the system is expected to be strongly affected by the presence of the instability domain and should be studied by means of direct simulations of the full system of Eqs. (1) and (2), which is is beyond the scope of this work.

2. *The case of* $\epsilon = +1$

In the case when the bright solitons exist, i.e., $\epsilon = +1$, the critical level, $|V| = 1$, is attained at the center of *quasipeakon* (15). In this case, it is not possible to find an exact solution to Eqs. (29) and (30) describing the crossing of $|V| = 1$ by a perturbed peakon, unlike the above solution given by Eqs. (31) and (32). However, taking into regard expansion (16) of the quasipeakon around its center, and the fact that its frequency is $\chi = 2/3$, an approximate nonstationary solution can be sought for, at small $|\zeta|$ and $|t|$, as

$$v_1 = [(1/\sqrt{3})|\zeta| - at]^2 + O(\zeta^4),$$
$$v_2 = (2/3)t + \sqrt{2/3}t|\zeta| - (1/\sqrt{2})at^2 + O(|\zeta|^3, t\zeta^2), \tag{35}$$

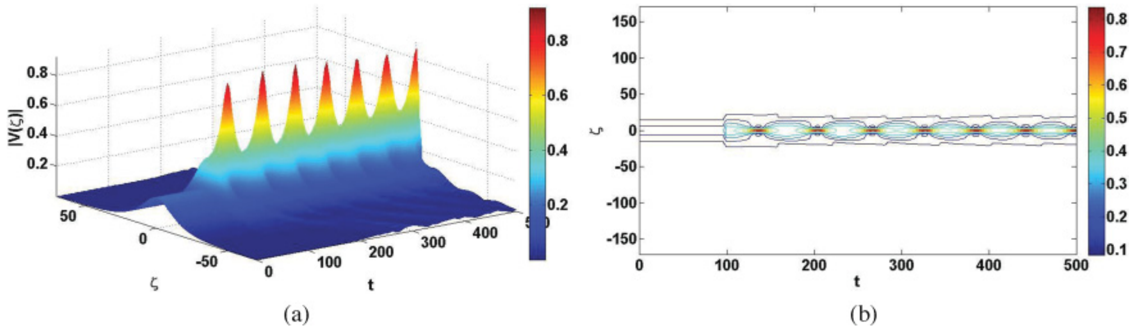

FIG. 4. (Color online) An example of the breather generated by the two-soliton initial condition in the case of $\chi = 0.99$ (the fundamental-soliton solution is suddenly multiplied by 2 at $t = 100$): (a) the three-dimensional image; (b) the contour plot in the (t,ζ) plane.

where $a > 0$ is an arbitrary constant, and the square root is realized as follows:

$$\sqrt{1-|V|^2} \approx \sqrt{2v_1(\zeta,t)} \approx (\sqrt{2/3})|\zeta| - \sqrt{2}at \qquad (36)$$

[cf. Eq. (33)]. As seen from Eq. (35), at $t \to -0$ this approximate solution describes a small perturbation in the form of a *cuspon* introduced at the central point, $\zeta = 0$. At $t > 0$, Eq. (36) demonstrates that the square root switches into the *unstable branch* in the respective instability domain, $|\zeta| < \sqrt{3}at$, which expands linearly with t [cf. Eq. (34)]. As said above, Eq. (8) cannot be used after the switching into the unstable phase.

III. DYNAMICS OF BRIGHT SOLITONS

It was mentioned above that, in the limit of $1 - \chi \to 0$, Eq. (8) goes over into the CNLSE. A well-known peculiarity of the latter (integrable) equation is the existence of higher-order

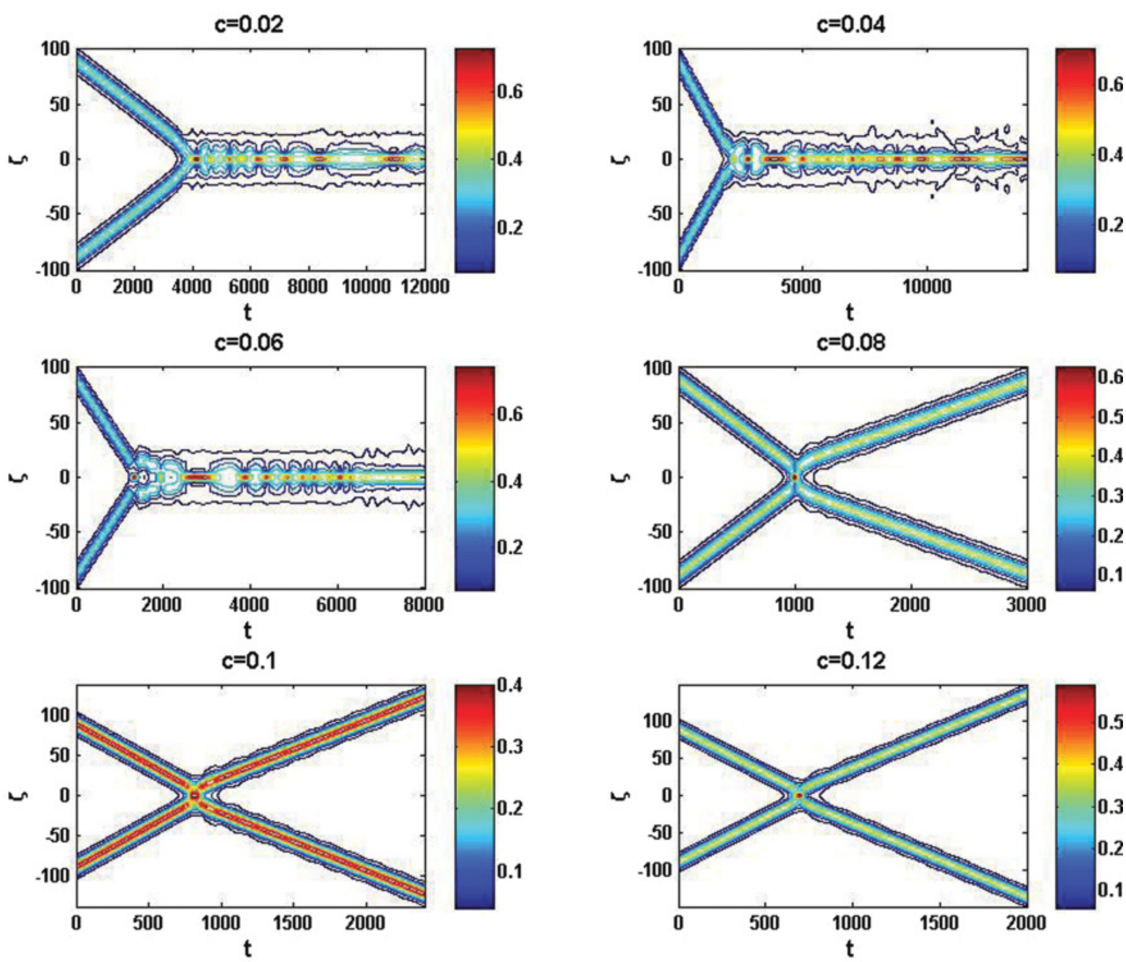

FIG. 5. (Color online) Outcomes of collisions between identical stable solitons with $\chi = 0.96$, observed with a gradual increase of the collision velocity c. A transition from the merger to quasielastic collisions is observed at $c = c_{\min} \approx 0.07$.

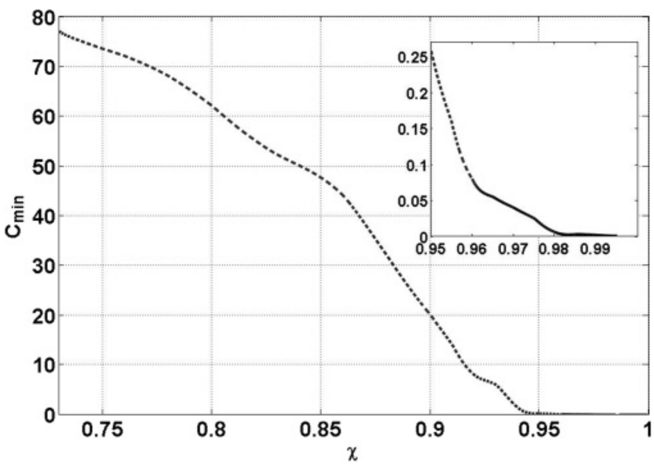

FIG. 6. The velocity separating quasielastic collisions and the merger of identical in-phase solitons, versus the intrinsic frequency of the soliton χ. The inset zooms the region of small $1-\chi$. The solid and dashed portions of the curve correspond, respectively, to the merger of the colliding solitons into breathers remaining in the stability domain (at $|V| < 1$) and to the case when the pulses produced by the merger (at $c < c_{\min}$) attain values $|V| = 1$ and thus switch into the unstable phase.

soliton solutions (breathers), the simplest one, *two-soliton*, being obtained by the multiplication of a fundamental soliton by 2 at the initial moment [23]. This circumstance suggests to simulate the evolution of so produced double pulses in the framework of Eq. (8), if the initial fundamental soliton is taken with $1-\chi$ small enough. A typical example is displayed in Fig. 4, which demonstrates that, in this case, Eq. (8) indeed supports robust breathers of the two-soliton type, although some emission of radiation is observed too.

Collisions between moving solitons are another natural dynamical problem [24]. The corresponding simulations were carried out, as usual, by taking a pair of far separated identical stable solitons [those with $\chi > \chi_{\rm cr}$; see Eq. (18)] and setting them in motion with velocities $\pm c$ by the application of the "kick", i.e., multiplying each soliton by $\exp(\pm ic\zeta/2)$. Figure 5 demonstrates that, as might be expected, the collisions are quasielastic if c exceeds a certain minimum velocity, $c_{\min}$, while, at $c < c_{\min}$, the colliding solitons merge into a single pulse that features irregular intrinsic oscillations (as seen from the figure, the quasielastic collisions may result in a change of the velocities).

In fact, the latter outcome is observed only for $\chi \geqslant 0.96$, i.e., close enough to the CNLSE limit. At $\chi < 0.96$, the merger produces pulses whose amplitude exceeds the critical level, $|V| = 1$ (not shown here in detail), which is followed by the switch of the pulse into the unstable phase, as outlined above.

The velocity separating quasielastic and inelastic collisions $c_{\min}$ is shown as a function of χ in Fig. 6. Naturally, $c_{\min}$ increases with the decrease of χ, as this implies the growth of the amplitudes of the colliding solitons, and thus moving farther from the integrable CNLSE limit, where collisions between solitons are completely elastic.

We also simulated collisions between solitons with a phase shift of π between them. In that case, as might be expected, the colliding solitons always demonstrate a quasielastic rebound, irrespective of the values of χ and c (not shown here).

Finally, three-soliton collisions (between two moving solitons and a central quiescent one) were considered too. The results for them (not shown here) are similar to those presented above for collisions between two solitons: quasielastic passage of fast solitons and the merger of slowly moving ones.

IV. CONCLUSION

This work aimed to present a physically relevant variant of the NPSE (nonpolynomial Schrödinger equation), which naturally appears in the model of the RABR (resonantly absorbing Bragg reflector), as the equation for gap solitons residing near the edge of the bandgap. The equation features the nonlinearity in the form of the radical term. Previously, different forms of the NPSE were derived as models of the BEC dynamics in Refs. [17] and [18]. A full family of bright-soliton solutions to the present NPSE was obtained in an implicit form, the ultimate solution being the explicitly found "quasipeakon." Cnoidal waves in the form of a chain of quasipeakons were found too. The stability of the bright solitons is correctly predicted by the VK criterion, which separates the family into stable and unstable parts. The ultimate form of the dark-soliton solution, shaped as a "dark compacton," was also obtained.

The instability has a peculiar form in the present model: It occurs when the amplitude of the wave field attains the critical value ($|V| = 1$), which is followed by the switch of the system into an unstable phase (it corresponds to the inverted atomic population unsupported by the external field), in terms of the underlying RABR model. The passage of the system through the instability threshold was investigated analytically.

The dynamics of the stable bright solitons was further explored by means of direct simulations. In particular, two-soliton breathers were found, and the border between the merger and mutual passage of colliding solitons was identified. The merger may lead either to the formation of robust irregularly oscillating pulses or to the switch into the unstable phase.

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