Transverse multimode effects on the performance of photon-photon gates

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The multimode character of quantum fields imposes constraints on the implementation of high-fidelity quantum gates between individual photons. So far this has only been studied for the longitudinal degree of freedom. Here we show that effects due to the transverse degrees of freedom significantly affect quantum gate performance. We also discuss potential solutions, in particular separating the two photons in the transverse direction.

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I. INTRODUCTION

Photons are attractive as carriers of quantum information because they propagate quickly over long distances and interact weakly with their environment. Their utility for quantum information processing applications, such as quantum repeaters [1] or quantum computing [2], would be further enhanced if it were possible to efficiently implement two-qubit gates between individual photons. Such two-qubit gates can be implemented probabilistically using just linear optics and photon detection [3], but strong photon-photon interactions would allow much more direct and deterministic implementations. Several approaches to the implementation of interaction-based photon-photon gates have been proposed, including those based on Kerr nonlinearities in fibers or crystals [4], on electromagnetically induced transparency (EIT) in atomic ensembles [5], and on the interaction of both photons with an individual quantum system [6]. See Refs. [7–11] for recent experimental progress.

In the simplest case, an ideal controlled-phase gate performs the transformations $|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle, |0\rangle|1\rangle \rightarrow |0\rangle|1\rangle, |1\rangle|0\rangle \rightarrow |1\rangle|0\rangle, |1\rangle|1\rangle \rightarrow e^{i\phi}|1\rangle|1\rangle$, where $|0\rangle$ and $|1\rangle$ are zero- and one-photon states, respectively. In the present context, the phase ϕ acquired by the state $|1\rangle|1\rangle$ is due to the interaction of the two input photons. For quantum information processing applications it is desirable to achieve $\phi = \pi$. In early theoretical papers the main focus was determining how to achieve interactions that are sufficiently strong to allow large phase shifts. The photonic pulses were typically idealized as single mode. Later it was realized that the (longitudinal) multimode character of the pulses impose important constraints on the implementation of high-fidelity quantum gates [12–19]. The phase shifts due to the interaction depend on the relative position of the two photons, and take different values over the pulses because the photons have to be described as extended wave packets rather than point particles [20]. As a consequence, an initial product state of the two photons is mapped by the interaction onto an output state that exhibits unwanted entanglement in the photons' external degrees of freedom. For large phase shifts this leads to low fidelities for the simplest quantum gate proposals [18].

It has been suggested that this difficulty can be overcome by more sophisticated quantum gate designs where the two photons pass through each other. This can be achieved by trapping one [8,13] or both [14] of the two photons, by having a counterpropagation geometry [15,16], or by considering two photons with different group velocities [19].

Here we show that having the photons pass through each other is not sufficient on its own, due to the presence of the transverse degrees of freedom. In short, the interaction-induced phases also depend on the relative transverse position of the two photons, which leads to fidelity limitations that cannot be mitigated by counterpropagation. In the following we describe these limitations in detail. We also discuss potential solutions, in particular separating the two wave packets in the transverse direction, which is possible for nonlinearities based on long-range interactions.

II. INTERACTING PULSE EVOLUTION

The general picture for the photon-photon gate is the interaction between two fields, $\hat{\Psi}_1(\mathbf{x},t)$ and $\hat{\Psi}_2(\mathbf{x},t)$, in three spatial dimensions. The field operators at $t = 0$ are defined as $\hat{\Psi}_i(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \hat{a}_{i,\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, where V is the quantization volume. They might describe photons in a Kerr medium or polaritons in an EIT medium. The field operators satisfy the equal-time commutation relation $[\hat{\Psi}_i(\mathbf{x},t), \hat{\Psi}_j^\dagger(\mathbf{x}',t)] = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}')$. This means that photon absorption is negligible (e.g., in the EIT case the spectra of the pulses are well inside the transparency window), so the evolution of the interacting fields can be regarded as unitary. Making the standard slowly varying envelope and paraxial approximations, we have the effective Hamiltonian [21] (where $\hbar \equiv 1$ is adopted hereafter)

$$\hat{K} = \sum_j \int d^3x\, \hat{\Psi}_j^\dagger(\mathbf{x},t) \left(v\frac{1}{i}\nabla_z - \frac{v\nabla_T^2}{2k} \right) \hat{\Psi}_j(\mathbf{x},t), \quad (1)$$

to describe their free evolution. Here, v is the group velocity in the positive or negative z direction, $k = \frac{2\pi}{\lambda}$ is the carrier wave vector, and ∇_T^2 is the transverse Laplace operator. The interaction for the two fields is [22]

$$\hat{V} = \sum_{i,j} \frac{1}{2} \int d^3x_1 \int d^3x_2 [\hat{\Psi}_i^\dagger(\mathbf{x}_1,t)\hat{\Psi}_j^\dagger(\mathbf{x}_2,t)\Delta(\mathbf{x}_1-\mathbf{x}_2) \times \hat{\Psi}_j(\mathbf{x}_2,t)\hat{\Psi}_i(\mathbf{x}_1,t)], \quad (2)$$

where the terms $i = j$ and $i \neq j$ in the sum correspond to self-phase and cross-phase modulation effects, respectively. Given the total Hamiltonian $\hat{H} = \hat{K} + \hat{V}$, the equations of motion $i\partial_t\hat{\Psi}_i(\mathbf{x},t) = [\hat{\Psi}_i(\mathbf{x},t),\hat{H}]$ for the counterpropagating fields read as

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z_1} - iv\frac{\nabla^2_{T,1}}{2k}\right)\hat{\Psi}_1(\mathbf{x}_1,t) = -i\hat{\alpha}(\mathbf{x}_1,t)\hat{\Psi}_1(\mathbf{x}_1,t),$$
$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial z_2} - iv\frac{\nabla^2_{T,2}}{2k}\right)\hat{\Psi}_2(\mathbf{x}_2,t) = -i\hat{\alpha}(\mathbf{x}_2,t)\hat{\Psi}_2(\mathbf{x}_2,t), \tag{3}$$

with the interaction potential

$$\hat{\alpha}(\mathbf{x}_i,t) = \int d^3x'\Delta(\mathbf{x}_i - \mathbf{x}')[\hat{I}_1(\mathbf{x}',t) + \hat{I}_2(\mathbf{x}',t)], \tag{4}$$

where $\hat{I}_n(\mathbf{x}) = \hat{\Psi}^\dagger_n(\mathbf{x})\hat{\Psi}_n(\mathbf{x})$.

A photon-photon gate is implemented by evolving an input biphoton state $|\Phi\rangle = |1\rangle_1|1\rangle_2 = \int d^3x_1 f_1(\mathbf{x}_1)\hat{\Psi}^\dagger_1(\mathbf{x}_1)\int d^3x_2 f_2(\mathbf{x}_2)\hat{\Psi}^\dagger_2(\mathbf{x}_2)|0\rangle$, where $f_i(\mathbf{x}) = \langle 0|\hat{\Psi}_i(\mathbf{x})|1\rangle$ are the pulse profiles, under unitary time evolution

$$\hat{U}(t) = \mathbb{T}e^{-i\int_0^t dt'\hat{H}(t')} = e^{-i\int_0^t dt'\hat{K}}e^{-i\int_0^t dt'\hat{V}}e^{-i\hat{C}}, \tag{5}$$

where $\mathbb{T}$ denotes the time-ordering operation, and the operator is factorized using the Baker-Campbell-Hausdorff (BCH) formula. The first factor in Eq. (5) describes the free evolution, including pulse propagation and pulse diffraction. The second factor is from the interaction between pulses. The third factor contains all commutators between the exponents of the first two terms; they reflect the interplay between pulse motion and pulse interaction, which generally changes the pulse profiles.

The ideal output state under a gate operation would be $e^{i\phi}e^{-i\int_0^t dt'\hat{K}}|\Phi\rangle$, with ϕ being a homogeneous controlled phase, where we take into account the free evolution of the photons. The actual output state, however, will be $\hat{U}(t)|\Phi\rangle = \int d^3x_1\int d^3x_2\psi(\mathbf{x}_1,\mathbf{x}_2,t)\hat{\Psi}^\dagger_1(\mathbf{x}_1)\hat{\Psi}^\dagger_2(\mathbf{x}_2)|0\rangle$ [23], giving the two-particle wave function $\psi(\mathbf{x}_1,\mathbf{x}_2,t) \equiv \langle 0|\hat{\Psi}_1(\mathbf{x}_1,t)\hat{\Psi}_2(\mathbf{x}_2,t)|\Phi\rangle$, which is generally nonfactorizable with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$. The fidelity F and the controlled phase ϕ of a gate operation are determined via the overlap between the actual output state and the freely evolved state $|\Phi_R\rangle = e^{-i\int_0^t dt'\hat{K}}|\Phi\rangle$:

$$\sqrt{F}e^{i\phi} = \langle\Phi_R|\hat{U}(t)|\Phi\rangle = \langle\Phi|e^{-i\int_0^t dt'\hat{V}}e^{-i\hat{C}}|\Phi\rangle$$
$$= \int d^3x_1d^3x_2\psi_0^*(\mathbf{x}_1,\mathbf{x}_2,t)\psi(\mathbf{x}_1,\mathbf{x}_2,t), \tag{6}$$

where $\psi_0(\mathbf{x}_1,\mathbf{x}_2,t)$ is the corresponding two-particle function for the freely evolved state $|\Phi_R\rangle$.

The field equations (3) allow one to obtain the evolution of $\psi(\mathbf{x}_1,\mathbf{x}_2,t)$ by multiplying $\hat{\Psi}_2(\mathbf{x}_2,t)$ to the right of the first equation of (3) and $\hat{\Psi}_1(\mathbf{x}_1,t)$ to the left of the second. Then, the product with $\langle 0|$ and $|\Phi\rangle$ takes on the addition of the equations, yielding the following linear equation for the two-particle function $\langle 0|\hat{\Psi}_1(\mathbf{x}_1,t)\hat{\Psi}_2(\mathbf{x}_2,t)|\Phi\rangle$:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z_1}\right)\langle 0|\hat{\Psi}_1(\mathbf{x}_1,t)\hat{\Psi}_2(\mathbf{x}_2,t)|\Phi\rangle$$
$$+\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial z_2}\right)\langle 0|\hat{\Psi}_1(\mathbf{x}_1,t)\hat{\Psi}_2(\mathbf{x}_2,t)|\Phi\rangle$$
$$-\left(iv\frac{\nabla^2_{T,1}}{2k_0} + iv\frac{\nabla^2_{T,2}}{2k_0}\right)\langle 0|\hat{\Psi}_1(\mathbf{x}_1,t)\hat{\Psi}_2(\mathbf{x}_2,t)|\Phi\rangle$$
$$= -i\Delta(\mathbf{x}_2 - \mathbf{x}_1)\langle 0|\hat{\Psi}_1(\mathbf{x}_1,t)\hat{\Psi}_2(\mathbf{x}_2,t)|\Phi\rangle. \tag{7}$$

Here we have used $\Psi_i(\mathbf{x})|0\rangle = 0$, $\Psi_i(\mathbf{x},t)\Psi^\dagger_i(\mathbf{x}',t)|0\rangle = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}|0\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{x}')|0\rangle$. With the center-of-mass coordinate $\mathbf{X} = \frac{\mathbf{x}_1+\mathbf{x}_2}{2}$ and $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, the derived equation for $\psi(\mathbf{x}_1,\mathbf{x}_2,t) = R(\mathbf{X},t)\xi(\mathbf{x},t)$ can be reduced to

$$\left(\frac{\partial}{\partial t} + 2v\frac{\partial}{\partial z} + iv\frac{\nabla^2_{T,x}}{2k}\right)\xi(\mathbf{x},t) = -i\Delta(\mathbf{x})\xi(\mathbf{x},t), \tag{8}$$

with $R(\mathbf{X},t)$ being trivially evolved under the diffraction term $e^{-i\frac{vt}{k}\nabla^2_{T,X}}$.

The linear equation noted above greatly simplifies the determination of the evolution of interacting pulses. This simplification is possible because we consider the interaction between two single photons (as opposed to multiphoton pulses). In the comoving coordinate, which eliminates the term $2v\partial_z\xi(\mathbf{x},t)$ in (8), the three factors in Eq. (5) are translated into $e^{-i\frac{vt}{2k}\nabla^2_{T,x}}$, $e^{-i\varphi(\mathbf{x},t)}$, and $e^{-i\hat{C}'}$, respectively, to evolve $\xi(\mathbf{x},0)$. Now, the third factor contains the exponentials of the commutators between $\frac{vt}{2k}\nabla^2_{T,x}$ and $\varphi(\mathbf{x},t)$ as follows:

$$e^{-i\hat{C}'} = \exp\left\{-\frac{1}{2}\left[\varphi,\frac{vt}{2k}\nabla^2_{T,x}\right]\right\}$$
$$\times\exp\left\{\frac{i}{3}\left[\varphi,\left[\varphi,\frac{vt}{2k}\nabla^2_{T,x}\right]\right]\right.$$
$$\left.+\frac{i}{6}\left[\frac{vt}{2k}\nabla^2_{T,x},\left[\varphi,\frac{vt}{2k}\nabla^2_{T,x}\right]\right]\right\}\cdots. \tag{9}$$

The commutators in the exponentials are of order l/r, where $l = vt$ is the medium length and $r = k\sigma^2$ is the Rayleigh length, with σ the transverse size of the pulses at $t = 0$. It is not difficult to achieve $l/r \ll 1$ in practice, making the effects from the third factor insignificant. For simplicity we will not include the first and third factors in the two-particle functions derived in the following, but the first-order contribution from the third factor will be considered in our numerical calculations. The second factor, $e^{-i\varphi(\mathbf{x},t)}$, where $\varphi(\mathbf{x},t) = \int_0^t dt'\Delta[\mathbf{x}_T, z - 2v(t-t')]$, is the main concern of the present paper. We will explain its effects with two examples.

III. CONTACT POTENTIAL

Our first example is a highly local interaction described by a Δ-function potential, $\Delta(\mathbf{x}_1 - \mathbf{x}_2) = V_0\delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_2)$. This is a good model for nonlinearities that are due to the interaction of both photons with the same atom in an atomic ensemble or crystal, or to short-range atomic collisions [4,5,13–15]. This interaction gives an output two-particle function of

$$\psi(\mathbf{x}_1,\mathbf{x}_2,t) = f_1(z_1 - vt,\mathbf{x}_{T,1})f_2(z_2 + vt,\mathbf{x}_{T,2})$$
$$\times\exp\left(i\frac{V_0}{2v}[H(z-2vt)-H(z)]\delta^{(2)}(\mathbf{x}_T)\right), \tag{10}$$

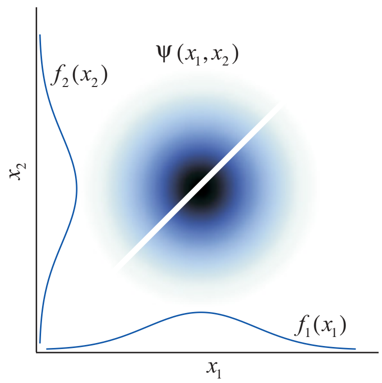

FIG. 1. (Color online) Qualitative plot showing the two-particle wave function as a function of the transverse coordinates x_1 and x_2 of the two photons. The photons are distributed over a multitude of transverse positions. For a contact interaction the interaction induces a nonzero phase only if the two photons happen to be at exactly the same position (see the diagonal line $x_1 = x_2$); see Eq. (10). The probability of this occurring is infinitesimal. As a consequence, the effective output phase defined by Eq. (6) is always zero, independent of the strength of the interaction.

where $H(z)$ is the Heaviside step function. From Eq. (10) one sees that the interaction-induced phase is nonzero only if the transverse coordinates of the two photons coincide, i.e., on a subset of configuration space, $\mathbf{x}_T = 0$, which has a measure of zero (see Fig. 1). As a consequence, one has $F = 1$ and $\phi = 0$. In the case of an ideal three-dimensional Δ-function potential the effective output phase is exactly zero, no matter how strong the interaction between the two photons. This is closely related to the results of Refs. [12,17] for the one-dimensional, but copropagating, case. One essentially finds equivalent results for any interaction whose range is much shorter than the transverse size of the wave packets. Note that this result is consistent with the nonzero conditional phase for photon-photon interactions obtained in Refs. [24,25], where the evolution is nonunitary, as manifested by a different field operator commutator.

IV. DIPOLE-DIPOLE INTERACTION

Related difficulties also arise for more long-range interactions. This can be seen from our second example, which is motivated by Refs. [11,16]. It concerns the interaction between polaritons whose atomic component is in a highly excited Rydberg state in an external electric field (cf. Fig. 2). This induces a dipole-dipole interaction between the polaritons,

$$\Delta(\mathbf{x}_1 - \mathbf{x}_2) = C(1 - 3\cos^2\vartheta)/|\mathbf{x}_1 - \mathbf{x}_2|^3, \tag{11}$$

where C depends on the specific Rydberg states used and ϑ is the angle between $\mathbf{x}_1 - \mathbf{x}_2$ and the external field (along which the electric dipoles of the Rydberg states are aligned). This is an attractive system because Rydberg states have large dipole moments, leading to potentially very strong interactions between the polaritons [26–28].

We consider the situation where the external field is perpendicular to the direction of motion. We assume the initial pulse profiles to be $f_1(\mathbf{x}_1) = \psi_0(x_1)\psi_0(y_1)\psi_0(z_1)$ and $f_2(\mathbf{x}_2) = \psi_0(x_2)\psi_0(y_2)\psi_0(z_2 - l)$,

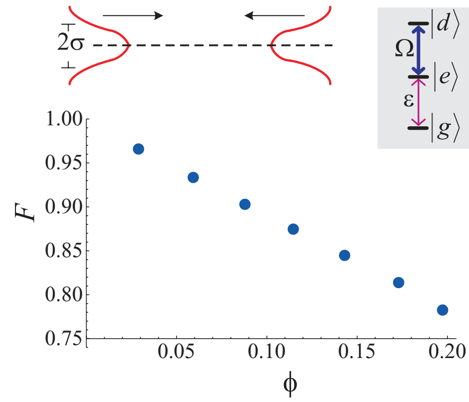

FIG. 2. (Color online) Schematic setup for a photon-photon gate working with counterpropagating single photon pulses in a medium under EIT conditions. The pulses interact with each other through the dipole-dipole force between the Rydberg states $|d_i\rangle$. The inset shows the relevant energy levels of the atoms. The pulses collide head-on, but they have a transverse extent σ. This leads to a dependence of the interaction-induced phase for the two-particle wave function on the relative transverse position, resulting in a trade-off between the effective phase ϕ of the two-photon operation and its fidelity F. Only very small phases are compatible with high fidelities.

where $\psi_n(x) = [\frac{1}{\sigma\sqrt{\pi}2^n n!}]^{\frac{1}{2}} H_n(\frac{x}{\sigma})e^{-\frac{1}{2}(\frac{x}{\sigma})^2}$ with $H_n(\frac{x}{\sigma})$ being the Hermite polynomials. The evolution according to Eq. (8) gives the output two-particle wave function

$$f_1(z_1 - l, \mathbf{x}_{T,1}) f_2(z_2 + l, \mathbf{x}_{T,2}) e^{-i\varphi(\mathbf{x}_1,\mathbf{x}_2,l/v)}. \tag{12}$$

The interaction-induced phase in the above is given by

$$\varphi(z,\mathbf{x}_T,l/v) = \frac{C}{2v}\frac{1}{\mathbf{x}_T^2}\left\{\frac{z^3 + 2z\mathbf{x}_T^2}{\left(z^2 + \mathbf{x}_T^2\right)^{\frac{3}{2}}} - \frac{(z-2l)^3 + 2(z-2l)\mathbf{x}_T^2}{\left[(z-2l)^2 + \mathbf{x}_T^2\right]^{\frac{3}{2}}}\right\}. \tag{13}$$

By Eq. (6), the conditional phase ϕ and the fidelity F in this case are determined as follows:

$$\sqrt{F}e^{i\phi} = \int d^3x_1' d^3x_2' f_1^2(\mathbf{x}_1') f_2^2(\mathbf{x}_2') \exp \times \left\{ -i\frac{C}{2v}\frac{1}{\mathbf{x}_T'^2}\left[\frac{(z'+l)^3 + 2(z'+l)\mathbf{x}_T'^2}{\left[(z'+l)^2 + \mathbf{x}_T'^2\right]^{\frac{3}{2}}} - \frac{(z'-l)^3 + 2(z'-l)\mathbf{x}_T'^2}{\left[(z'-l)^2 + \mathbf{x}_T'^2\right]^{\frac{3}{2}}}\right]\right\}, \tag{14}$$

where $\mathbf{x}_1' = \mathbf{x}_1 - l\hat{e}_z$, $\mathbf{x}_2' = \mathbf{x}_2$, and $\mathbf{x}' = \mathbf{x}_1' - \mathbf{x}_2'$. In this calculation we have chosen $l = 4\pi\sigma$ and $\sigma = 10\lambda$. The results are shown in Fig. 2 . Increasing the parameter $C/(2v\sigma^2)$, which indicates the interaction strength, increases the effective output phase ϕ but diminishes the output fidelity F. As a consequence, significant phase shifts are completely out of reach if one wants to achieve high fidelities. In the numerical calculations, we have included the first-order correction due to the third factor of Eq. (5). Such a correction comes from the commutators $[\frac{l\nabla_T^2}{2k},\varphi]$ and $[[\frac{l\nabla_T^2}{2k},\varphi],\varphi]$; see Eq. (9). The commutators involving higher powers of φ are shown to vanish. The exponentials of these commutators effect a modification of the intensity profile $f_i^2(\mathbf{x}_i)$ and a modification of the phase profile $e^{-i\varphi(z,\mathbf{x}_T)}$, respectively.

The main cause for the trade-off between ϕ and F is the dependence of the interaction-induced phase $\varphi(z,\mathbf{x}_T)$ on the transverse relative position $\mathbf{x}_T$. In fact, most of the behavior shown in Fig. 2 can be understood by setting the phase in the integrand of Eq. (14) as $-\frac{C}{v}\frac{1}{\mathbf{x}_T'^2}$, which is quite accurate for $l \gg \sigma$. It gives rise to transverse mode mixing in the form

$$\begin{aligned} &e^{-i\varphi(z,\mathbf{x}_T)}\psi_0(x_1)\psi_0(y_1)\psi_0(x_2)\psi_0(y_2) \\ &\quad = \sum_{m,n,l,k} C_{mnlk}\psi_m(x_1)\psi_n(y_1)\psi_l(x_2)\psi_k(y_2) \\ &\quad \neq \left\{\sum_{m,n} c_{mn}\psi_m(x_1)\psi_n(y_1)\right\}\left\{\sum_{l,k} d_{lk}\psi_l(x_2)\psi_k(y_2)\right\}, \qquad (15)\end{aligned}$$

leading to the deviation from the ideal output two-particle function $e^{i\phi}\psi_0(x_1)\psi_0(y_1)\psi_0(x_2)\psi_0(y_2)$.

The above analysis shows that the transverse mode mixing (or transverse mode entanglement) will develop even if the pulses are initially in a single transverse mode. Our analysis also clarifies that the pulse diffraction in the transverse direction has no direct impact on the performance of a photon-photon gate. It influences gate performance through its interplay with the interaction between pulses, i.e., by the third factor in Eq. (5). Compared with the effect of the transverse mode mixing shown in Eq. (15), such a diffraction-interaction interplay is insignificant in the regime considered here, where the medium length l is much smaller than the Rayleigh length r. For example, for $l/r = 0.2$ (as in our calculations), it induces corrections only at the few-percent level.

V. POTENTIAL SOLUTIONS

We have seen that the transverse multimode character of the quantum fields, which leads to a transverse relative position dependence of the interaction-induced phase shifts, has very significant consequences for the fidelity and phase achievable in photon-photon gates. We will now discuss two potential solutions for this problem. The first is applicable only to the case of long-range interactions. It consists of separating the paths of the two photons by a transverse distance D which is much greater than the transverse size σ of the pulses; i.e., the initial profiles of the pulses will be, for example, $f_1(\mathbf{x}_1) = \psi_0(x_1)\psi_0(y_1)\psi_0(z_1)$ and $f_2(\mathbf{x}_2) = \psi_0(x_2 - D)\psi_0(y_2)\psi_0(z_2 - l)$. With increasing D one will approach a situation where the transverse degrees of freedom of the photons can effectively be treated as pointlike. As a consequence, the effect studied here will diminish. This is shown in Fig. 3. We have again chosen a medium length $l = 4\pi\sigma$. Interaction-diffraction interplay effects are at or below the 10^{-3} level in this case because they depend on the gradients in the interaction-induced phase across the wave packets, which decrease with increasing transverse separation.

When adopting this solution, one has to take into account that the interaction-induced phase decreases as the transverse separation is increased. For example, achieving a conditional phase shift $\phi = \pi$ with a fidelity $F = 0.9$ requires $R = \frac{D}{\sigma} =$

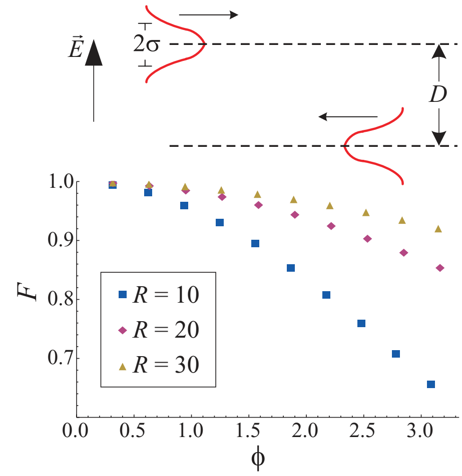

FIG. 3. (Color online) Introducing a transverse separation between the two pulses greatly relaxes the trade-off between F and ϕ, such that a phase of order π becomes compatible with high F. The dimensionless separation R is defined as $\frac{D}{\sigma}$. The interaction strength has to be increased significantly in order to compensate for the transverse separation.

26. By comparing to Ref. [16], this means that one could work with a principal number of the Rydberg state $n \simeq 75$, a transverse wave packet size $\sigma = 7\ \mu$m, and a group velocity $v = 4$ m/s. Achieving $\phi = \pi$ with a fidelity $F = 0.99$ requires $R = 79$, which is possible provided that, for example, n can be increased to 100, v reduced to 1 m/s, and σ reduced to 5 μm. These requirements are realistic with current technology; in particular Rydberg states with $n = 79$ were already used in the experiment of Ref. [26].

The second potential solution, which is applicable both to short-range and long-range interactions, consists of imposing strong transverse confinement. If the confinement energy is much greater than the interaction energy, then excitations to higher-order transverse modes are largely suppressed. All that the interaction can do in this case is multiply the lowest-order transverse mode by an almost uniform phase factor (since a nonuniform phase would imply non-negligible amplitudes in higher-order transverse modes), thus allowing high-fidelity quantum gates. Sufficiently strong confinement could be achieved for example using hollow core photonic crystal fibers [29] or optical nanofibers [30].

VI. SUMMARY

The importance of the multimode character of quantum fields for the implementation of photon-photon gates has been recognized in the past, but only for the longitudinal degree of freedom. Here we have shown that transverse degrees of freedom also play a significant role, imposing important constraints on the performance of potential quantum gates. For contact interactions the effective phase is essentially always zero, no matter how strong the interaction. For long-range interactions the situation is more favorable, but there are still significant trade-offs between the achievable phase and fidelity. We discussed two potential solutions. One is to have a significant transverse separation between the two wave packets, which is possible only for long-range interactions, and which requires an even stronger interaction. The second potential solution is to impose very strong transverse confinement, which may be possible using hollow fibers or nanofibers. In any case it will be essential for future implementations of photon-photon gates to take transverse multimode effects into account.

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