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Scaling laws for precision in quantum interferometry and the bifurcation landscape of the optimal state

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Phase precision in optimal two-channel quantum interferometry is studied in the limit of large photon number $N \gg 1$, for losses occurring in either one or both channels. For losses in one channel an optimal state undergoes an intriguing sequence of local bifurcations as the number of photons (or losses) increase. The optimal state has a continuous form in the Fock state basis for large N. The loss parameter limits any precision improvement over classical light to at most a constant factor independent of N. We determine a crossover value of photon number N_c beyond which supraclassical precision is progressively lost.

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It has been recognized that using quantum states of light may increase the resolution of interferometric measurements [1–3]. Particular states of N photons achieve the Heisenberg limit of phase resolution for standard error on the phase estimate $\Delta\varphi = 1/N$, an improvement over the classical (or shot-noise) limit $\Delta\varphi = 1/\sqrt{N}$ that is obtainable when N photons enter the interferometer one at a time. These bounds are derived by an application of the Cramer-Rao inequality [2] for the standard error of an unbiased estimator, $\Delta\varphi \geqslant (\nu\mathcal{F})^{-1/2}$, where $\mathcal{F}$ is the quantum Fisher information (QFI) [4] and ν is the number of repeated independent trials. Assuming any instrument is composed of three components: quantum input state, dynamics, and measurement; the functional $\mathcal{F}$ depends only on the first two — it assumes an optimal measurement choice. For pure states in a single mode $\mathcal{F}/4 = \Delta^2\hat{n} \equiv \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$ (where $\hat{n}$ is the number operator) and a familiar uncertainty relation is recovered: $\Delta n \Delta\varphi \geqslant 1/2$. Thus, for a lossless two-mode interferometer QFI and precision are greatest for the maximum variance state, or "NOON state" [5]; it saturates the Heisenberg limit. Unfortunately, it is also highly susceptible to noise, especially dissipation [6].

To mitigate this problem various two-component states were proposed [7–9], where the loss of a number of photons in the first mode does not destroy the superposition. The precision performance under dissipation of various Gaussian states, e.g., squeezed, coherent, and thermal states, has also been considered recently [10]. In all cases, the precision was found to be supraclassical for certain range of losses and N.

In the lossy case the pure input state of two oscillator modes maximizing QFI,

$$|\phi\rangle = \sum_{n=0}^{N} \phi_n |n\rangle_1 |N-n\rangle_2, \tag{1}$$

must balance supraclassical precision against robustness to photon loss. In this notation the NOON state has two nonzero components, $\phi_0 = \phi_N = 1/\sqrt{2}$. For a lossy interferometer, light propagates in each arm as a damped harmonic oscillator, with frequencies $\omega^{(1)},\omega^{(2)}$ and dissipation $\gamma^{(1)},\gamma^{(2)}$. Equivalently, losses can be introduced by beam-splitters in each mode with reflectivity $R^{(1,2)} = 1 - \exp\{-\gamma^{(1,2)}t\}$. Those lost photons siphoned out of the modes are then traced over. In the simpler case of losses in only one of the two modes, $R^{(1)} = R > 0$, $R^{(2)} = 0$, as might occur when that mode is directed through a partially transparent test sample, the state $|\phi\rangle$ decays into a mixture $\hat{\rho} = \sum_k |\psi_k\rangle\langle\psi_k|$ with

$$|\psi_k\rangle = \frac{1}{\sqrt{w_k}} \sum_n \sqrt{\Lambda_{n;k}}\, e^{in\varphi} \phi_n |n-k, N-n\rangle, \tag{2}$$

corresponding to the loss of k photons. Here w_k is the normalization factor; the phase difference is $\varphi = (\omega^{(1)} - \omega^{(2)})t$ and the loss enters via coefficients $\Lambda_{n;k} = \binom{n}{k} R^k (1-R)^{n-k}$. Fisher information of the mixed state resulting from losses is a weighted sum over pure components $\mathcal{F} = \sum_k w_k \mathcal{F}_k$, where $\mathcal{F}_k = 4\Delta^2\hat{n}_1$ for pure states $|\psi_k\rangle$ [8].

Previously, numerical optimization was used to construct optimal states for a small number of photons. Reference [8] showed sub-Heisenberg scaling of Fisher information and raised the possibility that a shot-noise (linear) scaling might set in for $N \gg 1$. This hypothesis is not easy to test by numerical effort alone: we prove the linear upper bound and observe that it is not saturated until the number of photons is quite large, e.g., $N > 10^3$ at 10% loss. Indeed, for larger N the candidate optimal states (two-component state [9,10] or spin-coherent state in high-loss regime [11]) cease to be good approximations to the true optimal state [12], the form of which we derive analytically.

Independently, shot-noise scaling was established in a recent analytical study [13] under the assumption of zero prior phase information. Our analysis has no such limitation and we prove the tightness of our bound by constructing the optimal state explicitly.

Evolution of optimal state. Here we study the analytically tractable limit $N \gg 1$ by treating $n/N \equiv x \in [0;1]$ as a continuous parameter. Examining the limits of small and large loss has revealed a scaling relationship for the optimal Fisher information

$$\mathcal{F}_{\text{opt}}(N,R) = N^2 \tilde{\mathcal{F}}\left(\frac{NR}{1-R}\right), \tag{3}$$

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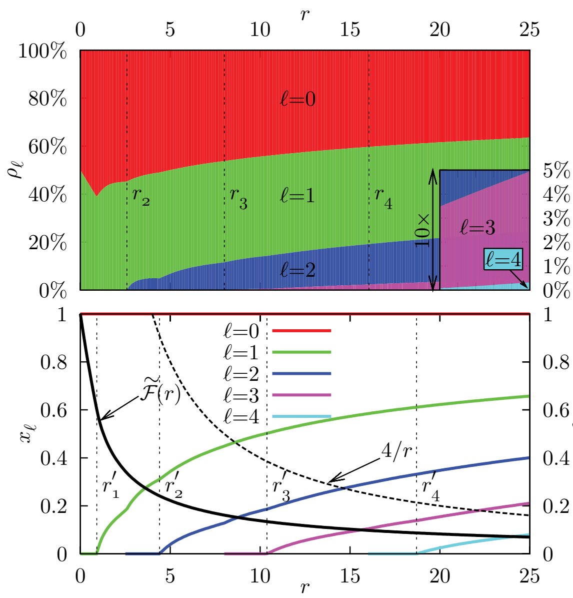

FIG. 1. (Color) Probability weights ρ_ℓ, represented as stacked histograms (top) and positions x_ℓ (bottom), as a function of r. Different components are indicated using color. Black solid line on the bottom figure is the rescaled Fisher information $\tilde{\mathcal{F}}(r)$ and the blacked dashed is its asymptote $4/r$; convergence takes place for much larger values of r. Thresholds r_2,r_3,r_4 correspond to appearances of new components at the origin. Components separate from the origin at critical values r_1',r_2',r_3',r_4'. The data for $20 \leqslant r \leqslant 25$ are magnified (top figure, lower right corner) showing components with very small weight.

that cleanly interpolates between these limits. The nontrivial dependence on N and R is captured by a single quantity: $r = NR/(1-R)$. The structure of the optimal state also depends on r alone, except for small differences due to a discrete nature of parameter $x = n/N$. We were able to demonstrate that for any finite r the optimal state can have only a finite number of components. This number increases with r as the optimal state undergoes a sequence of bifurcations: an unbalanced NOON state ceases to be optimal for $r < r_1' \approx 0.912\,957$ [14], superseded by a state $\sqrt{1-\rho_1}|N\rangle_1|0\rangle_2 + \sqrt{\rho_1}|x_1N\rangle_1|(1-x_1)N\rangle_2$. We find that for larger values of the parameter ($r > r_2$), an optimal state acquires a third component $|0\rangle_1|N\rangle_2$, which shifts away from the origin to $|x_2N\rangle_1|(1-x_2)N\rangle_2$ for $r > r_2'$ and so on. The universal set of bifurcation points $r_1' < r_2 < r_2' < r_3 < r_3' < \cdots$ as well as weights ρ_ℓ and positions x_ℓ of components in the $(m+1)$-component state $|\phi\rangle = \sum_\ell \sqrt{\rho_\ell}|x_\ell N\rangle$ are determined by solving a system of $2m-1$ or $2m$ equations. The results are shown in Fig. 1. An important caveat is that since component positions $x_\ell N$ are not integers in general, they may appear as two adjacent integer components for finite N.

The numerical results of Ref. [8] correspond to the leftmost region $r \sim 1$ of Fig. 1. In this work we are primarily interested in the regime $r \gg 1$ (not shown) as the loss parameter R, determined by the properties of the medium, is fixed while the number of photons N increases. The optimal states seemingly increase in complexity with increasing N (r) as the number of components increases. But as the density of these components increases, the optimal state $|\phi\rangle$ may be approximated by a continuous function.

Precision, quantified by $\mathcal{F}$, will always improve *at least* linearly with the photon resource N, (by sending the photons through the instrument one at a time) but the more insightful question is: How does the amount of "intrinsic" Fisher information, i.e., *per photon*, scale with N if photons are combined in some optimal quantum superposition? Examination of Eq. (3) shows that, r being proportional to N, a quadratic (Heisenberg) scaling of the Fisher information can be seen only for small r; as QFI approaches asymptote $4/r$ for larger values, it is replaced with a linear (shot-noise) scaling.

General upper bound. This general linear upper bound can be demonstrated without making an approximation of large N or $R \sim 1$. Since the variance is unaffected by a constant shift, one can rewrite the $\mathcal{F}$ as

$$4\sum_k\left[\sum_n \Lambda_{n;k}\left(n-\frac{k}{R}\right)^2|\phi_n|^2 - 4w_k\langle\psi_k|n-\frac{k}{R}|\psi_k\rangle^2\right]. \tag{4}$$

Observing that the second term is negative and performing the sum over k in the first term, we obtain the inequality

$$\mathcal{F} \leqslant 4\frac{1-R}{R}\sum_n n|\phi_n|^2 \leqslant 4\frac{1-R}{R}N = \mathcal{F}_{\text{upper}}. \tag{5}$$

This upper bound ($\mathcal{F}_{\text{upper}}$) is always valid for any $R > 0$ demonstrating that quadratic precision (at the Heisenberg limit $\mathcal{F} \propto N^2$) is only possible for $R \propto 1/N$. When R is fixed, it implies $\mathcal{F} \propto N$, scaling proportional to the shot-noise limit. This bound also appeared recently in the context of global phase estimation [13].

Limit $\mathcal{F}_{\text{upper}}$ is reachable asymptotically as can be shown by constructing a wave function that minimizes the correction $\Delta\mathcal{F} = \mathcal{F}_{\text{upper}} - \mathcal{F}$. In the limit $R \sim 1$ we approximate the true optimal state $\{\phi_n\}$ by a continuous function $\tilde{\phi}(\tilde{x})$ (with $\tilde{x} = 1 - x = 1 - n/N$), smooth on scales $\sim 1/\sqrt{N}$, and obtain, approximately,

$$\Delta\mathcal{F} \approx \frac{4N^2}{r}\int_0^\infty \left(\tilde{x}\tilde{\phi}^2(\tilde{x}) + \frac{4}{r}\tilde{\phi}'^2(\tilde{x})\right)d\tilde{x}, \tag{6}$$

where the upper limit has been set to infinity since the width of $\tilde{\phi}(\tilde{x})$ is much smaller than 1. The term proportional to $\tilde{\phi}^2(\tilde{x})$ is the first term of Eq. (4) subtracted from $\mathcal{F}_{\text{upper}}$, and the term proportional to $\tilde{\phi}'^2(\tilde{x})$ is the second term in Eq. (4) taken with the opposite sign.

Minimization of (6) subject to the boundary condition $\phi(0) = 0$ [15] and the normalization constraint produces

$$\tilde{\phi}(\tilde{x}) = \frac{(r/4)^{1/6}}{\text{Ai}'(\mu_1)}\text{Ai}\left[\left(\frac{r}{4}\right)^{1/3}\tilde{x} + \mu_1\right], \tag{7}$$

where Ai(z) is the Airy function, $\mu_1 \approx -2.338\,107\ldots$ is its first (largest) zero, and the prefactor ensures normalization. Together with the next order correction, the Fisher information of the optimal state $\mathcal{F}_{\text{opt}} = \mathcal{F}_{\text{upper}} - \Delta\mathcal{F}_{\text{min}}$ is

$$\mathcal{F}_{\text{opt}} = \frac{4N^2}{r}\left[1 - |\mu_1|\left(\frac{4}{r}\right)^{1/3} + O\left(\frac{1}{r^{1/2}}\right)\right]. \tag{8}$$

For $R \sim 1$ ($r \sim N$) the width of (7) is $O(N^{2/3})$; so is the leading correction in Eq. (8). The upper bound becomes saturated when the number of photons exceeds a value of N_c estimated by equating the principal term and the leading-order correction. This yields $N_c = r_c(1-R)/R$ with $r_c \sim 4|\mu_1|^3 \sim 50$.

Interestingly, the spin-coherent state proposed in Ref. [11] also has a continuous shape, but in the form of a Poisson distribution. Its width is comparable to that of the Airy function state for $N \sim 2.9\sqrt{\frac{R}{1-R}}$ but becomes much narrower for larger N, increasing only as a square root of mean. Unfortunately, the spin-coherent state fails to achieve supraclassical precision.

Arbitrary loss in both arms. Whenever both $R^{(1)}$ and $R^{(2)}$ are nonzero, the density matrix is a mixture of pure states $|\psi_{k_1k_2}\rangle$ resulting from the loss of k_1 and k_2 photons in modes (1) and (2), respectively. In Eq. (2) the factor $\sqrt{\Lambda_{n;k}}$ becomes $\sqrt{\Lambda^{(1)}_{n;k_1}\Lambda^{(2)}_{n;k_2}}$ and the states $|\psi_k\rangle$ become $|\psi_{k_1k_2}\rangle$.

The number of photons lost in each mode is not observed directly, although their sum $k = k_1 + k_2$ can be inferred by subtracting the detected photon number from the input N. Consequently the linear decomposition of the Fisher information serves only as an upper bound $\mathcal{F} \leqslant \sum_{k_1,k_2} w_{k_1k_2}\mathcal{F}_{k_1k_2}$ [8] and the determination of quantum Fisher information requires the diagonalization of the density matrix [4]:

$$\mathcal{F} = 4\sum_i \lambda_i\langle v_i|\hat{n}_1^2|v_i\rangle - \sum_{\substack{i,j \\ \lambda_i,\lambda_j>0}} \frac{8\lambda_i\lambda_j}{\lambda_i+\lambda_j}|\langle v_i|\hat{n}_1|v_j\rangle|^2, \tag{9}$$

where λ_i and $|v_i\rangle$ are eigenvalues and eigenvectors of the density matrix, respectively. Diagonalizations within subspaces corresponding to a fixed *total* number of lost photons $k = k_1 + k_2$ may be carried out independently. In the limit $N \gg 1$ the coefficients $\Lambda^{(1,2)}_{n;k}$ may be approximated by Gaussians so that the corresponding density matrix is also Gaussian in the continuous limit as long as $R^{(1)} \neq R^{(2)}$ and the wave function $|\phi\rangle$ is smooth on scales $\sim\sqrt{N}$. This density matrix may be expanded in terms of wave functions of harmonic oscillator with the aid of the Mehler formula [16], and the sum (9) is evaluated noting that nonzero matrix elements correspond to $j = i \pm 1$. The surprising outcome is that the exact Fisher information equals the linear upper bound ($\mathcal{F}_{\text{upper}}$ for arbitrary loss in both arms) in the asymptotic limit. This is also true in the symmetric loss case ($R^{(1)} = R^{(2)}$) as the optimal state itself turns out to be a Gaussian. This case has some import; firstly, it is relevant for balanced instruments where phases may be introduced in either arm, e.g., gyroscopes, and, secondly, the analysis has an extended applicability beyond losses in modes (1) and (2) to those occurring in any superposition of these modes. Accordingly, the discussion is applicable to losses in detection after the mode mixing.

Expressed in terms of parameters $r^{(1,2)} = NR^{(1,2)}/(1-R^{(1,2)})$, the upper bound (5) changes to $\mathcal{F} \leqslant 4N^2/(\sqrt{r^{(1)}} + \sqrt{r^{(2)}})^2 = \mathcal{F}_{\text{upper}}$.

The optimal wave function is computed by minimizing the correction to the Fisher information [$x = n/N$, $x_* = \sqrt{r^{(1)}}/(\sqrt{r^{(1)}} + \sqrt{r^{(2)}})$]:

$$\Delta\mathcal{F} \approx N^2\int_0^1\left[\frac{(x-x_*)^2}{\sqrt{r^{(1)}r^{(2)}}}\tilde{\phi}^2(x) + \frac{4\tilde{\phi}'^2(x)}{(\sqrt{r^{(1)}}+\sqrt{r^{(2)}})^4}\right]dx, \tag{10}$$

which produces a Gaussian centered at $x = x_*$ of width $\sqrt{2}(r^{(1)}r^{(2)})^{1/8}/(\sqrt{r^{(1)}} + \sqrt{r^{(2)}})$. This width scales as $N^{3/4}$ (cf. $N^{2/3}$ for single mode losses).

For moderate losses, this optimal form is attained when the number of photons is large. In the limit of large losses, this asymptotic form is reached with a small number of photons (see Fig. 2), within reach of current laboratory capabilities. The Fisher information together with the leading correction is

$$\mathcal{F}_{\text{opt}} = \frac{4N^2}{(\sqrt{r^{(1)}}+\sqrt{r^{(2)}})^2}\left[1 - \frac{2}{(r^{(1)}r^{(2)})^{1/4}} + O\left(\frac{1}{r}\right)\right]. \tag{11}$$

The correction scales as $N^{1/2}$, in contrast to the $N^{2/3}$ scaling for single-mode losses. Correspondingly, the crossover to the limiting behavior is expected for smaller N. The convergence to asymptotic precision for the case of single-mode and symmetric losses is illustrated in Fig. 3.

Summary and outlook. We have shown analytically that dissipation limits the increase in precision offered by nonclassical interferometers. This result has to be kept in mind if various proposals [17,18] for a "quantum leap" in sensitivity of gravitational wave detectors are to be implemented. Absent losses, precision can be boosted with NOON states of (theoretically) up to 10^{10} photons by estimates of Ref. [19]. Losses would imply a much lower threshold N_c; moreover the optimal state is described by a continuous wave function — a plus, as the difficulty of generating NOON states for large N stems from the need to eliminate unwanted middle components.

The existence of the upper limit leads to a new metric to describe the performance of the entire family of states: What

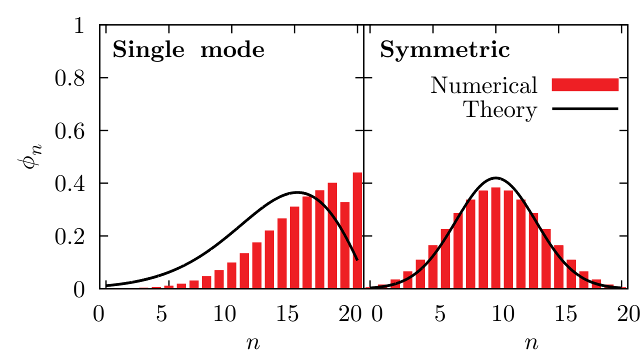

FIG. 2. (Color) Optimal 20-photon states for 95% loss ($r = 380$) in one (left) and two (right) modes. Red bars represent amplitudes ϕ_n obtained by numerical optimization. Black lines represent analytical approximation with an Airy function and a Gaussian. These optimal states offer a precision improvement (square root of Fisher information) over coherent light of just 6% (single-mode losses) and 0.4% (symmetric losses), owing to high loss amount. (For single-mode losses, using coherent light, the precision used in calculation is for the optimal reflectivity of the input beamsplitter in Mach-Zehnder interferometer.)

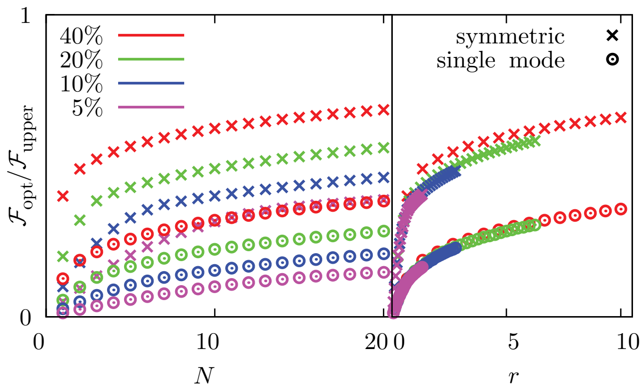

FIG. 3. (Color) Left: Fisher information for symmetric ($R^{(1)} = R^{(2)} = R$) and single-mode ($R^{(1)} = R$, $R^{(2)} = 0$) losses for optimal N-photon states as a fraction of the linear upper bound [$N(1-R)/R$ and $4N(1-R)/R$, respectively]. The curves must tend to 1 for large N, but the convergence is faster for losses in both arms. For symmetric losses we use exact Fisher information, not the approximate upper bound. Right: Collapse of data when replotted as a function of $r = NR/(1-R)$ [see Eq. (3)].

fraction of the optimum value of the Fisher information is obtained in the asymptotic limit $N \gg 1$? The two-component state attains only 47.95% of the optimum value (single-channel loss). Interestingly, the Holland-Burnett (HB) state [20] guarantees performance at 50% of the optimum in all cases, no loss, one-arm or two-arm losses, and is easier to generate.

It is an open question if the optimal quantum metrology is feasible; we give a partial answer by providing intuition necessary to construct a family of states saturating the upper bound. The optimal state has a continuous (smooth) form with a width that scales as a power of photon number N^ϵ. The smoothness property is important as we have seen; choosing $\epsilon = 2/3(3/4)$ for one- (two)-mode loss ensures faster saturation of the limit, but any value $1/2 < \epsilon < 1$ will suffice. The latter inequality is strict, e.g., symmetric losses: the HB state is suboptimal as its width $\propto N$. Similarly, the spin-coherent state (width $\propto \sqrt{N}$) fails to achieve the upper bound; in fact, it is equivalent in performance to coherent light by Caves' theorem [17]. One might envisage using an unequal number of photons, $|N-M\rangle$ and $|M\rangle$ at the input ports of an MZ interferometer: the width of the state after the first beamsplitter can be controlled by varying M between 0 (spin-coherent state) and $N/2$ (HB state). Unfortunately, the resulting state lacks the required smoothness so that at most 54.25% of the optimum can be reached with this strategy.

As a final note, we enlarge upon two proposed applications of quantum light; to free-space target acquisition and ranging, and to gravity-wave observation. For locating a target at 10 km distance, clear weather attenuation coupled with a typical 10% reflectance results in total loss of 99–99.9 %. Optimal Fisher information per *received* photon is $4/R$, thus naively one would expect a twofold improvement in phase precision $\Delta\phi$ over coherent light (having Fisher information unity per received photon) even for high losses. This comparison is for an interferometer with 50 : 50 beamsplitters; precision can be trivially increased with coherent-light inputs by optimizing the beamsplitter reflectances. Compared with this strategy, nonclassical light can improve precision by at most a factor of $(1+\sqrt{1-R})/\sqrt{R}$, i.e., by 3–10 % for losses above. This fractional advantage in the very high loss limit does not offset the high practical cost of generating those optimal states we have discovered. This result should moderate expected outcomes of such proposals.

To contrast, consider two-mode losses $R \approx 1\%$, the expected domain of advanced interferometric gravitational wave detectors with high-reflectivity mirrors and state-of-the-art photodetectors. The improvement to $\Delta\phi$ over classical light for the same $N \gg 1$ approaches a factor of $1/\sqrt{R}$. This tenfold improvement falls far short of more optimistic estimates assuming idealized conditions [18,19] but still represents a clear, nontrivial advantage for the optimal input states we have discovered. The ability to reduce intrinsic quantum noise by an order of magnitude without an associated increase in radiation pressure noise (the photon flux has not increased) is certainly of interest for gravity wave detectors.

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[15] An additional contribution from the boundary is $\propto \phi^2(0)/r^{7/6}$, which would be much larger than $\Delta\mathcal{F}$ unless $\phi(0) = 0$.
[16] $\rho(x,x') = \frac{1}{\sqrt{\pi(1-u^2)}} \exp[-\frac{1+u}{1-u}(\frac{x-x'}{2})^2 - \frac{1-u}{1+u}(\frac{x+x'}{2})^2] = \sum_{n=0}^{\infty} u^n \psi_n(x)\psi_n(x')$, where $\psi_n(x)$ is an eigenstate of the harmonic oscillator.
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