Population kinetics in fluctuating plasmas

F. Catoire,* H. Capes, Y. Marandet, A. Mekkaoui, J. Rosato, M. Koubiti, and R. Stamm
Laboratoire de Physique des Interactions Ioniques et Moléculaires, UMR 6633 CNRS-Université de Provence, Centre de Saint Jérôme, Marseille F-13397, France
(Received 8 September 2010; published 31 January 2011)

We address a model for atomic population kinetics in fluctuating plasmas. An analytical expression for the ensemble-average populations is obtained in terms of two statistical functions that can be retrieved experimentally, namely the probability density function and the autocorrelation of the plasma fluid fields. This expression, allowing for very fast calculations, is of great interest to thoroughly analyze the radiative properties of fluctuating plasmas. Two limits, where the fluctuations are either faster or slower than the atomic relaxation time scales, are discussed in detail. Finally, an application to atomic hydrogen is presented.

DOI: 10.1103/PhysRevA.83.012518 PACS number(s): 32.70.Fw, 33.70.Fd, 52.20.Hv, 47.70.Nd

I. INTRODUCTION

Atomic population kinetics models (PKMs) are extensively used in plasma physics for interpreting experimental spectra or modeling, e.g., the ionization balance [1]. If the atomic relaxation time scales are short compared to the typical time over which the plasma parameters (density, temperature) evolve, the atomic populations follow the fluctuations of the latter adiabatically. In this case, only the stationary solution of the balance equations is of interest [1–3]. However, in fluctuating plasmas this condition is not necessarily satisfied, so that the time dependence of the populations should, in principle, be retained. The effect of plasma fluctuations on PKMs has been addressed by several groups. A common approach consists of adding a transport term to the balance equations. This approach is justified for low amplitude fluctuations, e.g., in the core of magnetically confined plasmas [4]. This transport term, resulting from a time average of the balance equations, can be strong enough to affect the averaged populations [5].

Another approach to account for fluctuations consists of modeling them by superimposing a number of harmonic functions. In [6], a numerical integration of the balance equations has shown that the time-averaged populations can be significantly modified by temperature fluctuations.

The approach we develop here consists of treating the fluctuating field as a stochastic process, characterized by the probability density function (PDF) and the autocorrelation function of the fluctuating field. Turbulence frequencies are supposed to be lower than the electron-ion collision frequency, so that the plasma is locally Maxwellian. This approach draws its inspiration from the so-called model microfield method developed for spectral line broadening in plasmas [7]. Our method leads to an analytical formula for the average populations, expressed as a function of the PDF and the autocorrelation function of the fluctuating fluid fields and is valid for any amplitude of the fluctuations. Recently [8], we have presented a preliminary application of this model to the study of radiative losses of lithium, in the context of magnetic fusion plasmas. In that work, an exponential correlation function was used. Here, we extend our approach to a broader class of correlation functions and investigate the effect of the plasma fluctuations on average atomic populations more comprehensively.

This paper is organized as follows: First we describe the general approach leading to an analytical expression of the average populations when time-dependent stochastic plasma fluid fields are taken into account. Next we present and discuss the results for two regimes corresponding to fluctuation time scales either larger or shorter than the characteristic atomic time scales. Finally, the model is illustrated by an application to atomic hydrogen and is compared to numerical methods.

II. GENERAL DESCRIPTION OF POPULATION KINETICS MODEL (PKM)

In PKM, the vector containing the level populations is noted $X(t) = (x_1(t), \ldots x_N(t))$, where $x_j(t)$ is the population of jth levels. The evolution of the vector $X(t)$ is governed by an equation of the form [1]

$$\frac{dX(t)}{dt} = M(Y(t))X(t). \tag{1}$$

In this equation the matrix M contains all the rates involving the level populations x_j, i.e., all the relevant processes leading to transitions between them. The letter Y refers to fluid fields such as the electron (ion) density, and/or electron (ion) temperature, which will be taken as time-dependent quantities in the following. The solution of Eq. (1) is given by $X(t) = G(t,0)X_0$ where $G(t,t')$ is the Green function defined by

$$\partial_t G(t,t') = M(t)\,G(t,t') \text{ with } G(t,t) = I \text{ and } G(t,t') = G(t-t'). \tag{2}$$

$X_0 = X(t=0)$ is the initial value of the population vector X and I the identity matrix. Usually, the fluid fields (Y) are assumed to evolve slowly compared to the atomic relaxation time scales, and only the stationary solution (sometime referred as quasi-steady state (QSS) for which $\frac{dX}{dt} = 0$) is considered [1]. The latter approach is valid provided that the relaxation time scales of the atomic system is short compared to the time scales over which Y evolves. If such a steady state does exist, the solution of Eq. (1), noted X_∞, is simply given

*catoire@celia.u-bordeaux1.fr; Present address: CELIA, UMR 5107 du CNRS-CEA-Université Bordeaux I, Université de Bordeaux I, 351 Cours de la Libération, F-33405 Talence, France.

by $M(Y)X_\infty = 0$. The existence of this stationary solution is conditioned by the following properties of the matrix M:

(i) $\det(M) = 0$ in order to have $X_\infty \neq 0$.

(ii) The eigenvalues (except 0) of M have to be negative in order to ensure the convergence at large times.

Since M is not invertible, the population vector X can be normalized by the total population $N_0 = \sum_i x_i$ which is a constant in this model. As a consequence, $\sum_i |x_i(t)| = 1\ \forall t$. The steady state X_∞ is given by the eigenvector of eigenvalue 0 of the matrix M [9].

III. DESCRIPTION OF MODEL

Processes considered in this work are ergodic so that the ensemble-average value is equal to the time-average one. In order to calculate the ensemble average $\langle G(t,0)\rangle$, we consider a stepwise constant process, for which the stochastic fluid field (Y) is assumed to be constant between two jumps with a value sampled according to a given probability density function, noted $p(Y)$. An example of such a process is illustrated in Fig. 1, where Y represents the electron temperature.

These jumps are assumed to be instantaneous, and the waiting time distribution (WTD) between two jumps is noted $\varphi(t)$. In the Appendix, we show that the correlation function $C(t)$ is related to the WTD by the relation

$$C(t) = \int_t^\infty dt\ \varphi(t). \tag{3}$$

Since we consider general autocorrelation functions, the WTD is not necessarily an exponential function, thus this model can describe non-Markovian processes [10,11].

The stochastic Green function is the solution of Eq. (2). Using the semigroup property of the Green function we have

$$G(t,0) = G(t,t_{N-1})\cdots G(t_2,t_1)G(t_1,0) = \prod_{i=0}^{N-1} G(t_{i+1},t_i). \tag{4}$$

The interval $[0,t]$ is split in N intervals as follows: $0 = t_0 \leqslant t_1 \leqslant t_2 \cdots \leqslant t_{N-1} \leqslant t_N = t$. For each time interval $[t_i,t_{i+1}]$, the stochastic term Y is constant and equals Y_i. All Y_i are chosen *independently* at each step. At the ith step, the Green function is written

$$G(t_{i+1},t_i) = \exp[(t_{i+1}-t_i)M(Y_i)] = G(t_{i+1}-t_i), \tag{5}$$

since the stochastic quantity, hence the matrix M, is constant during the interval $[t_i,t_{i+1}]$. Therefore for a given configuration of jumping time, the average of the Green function is given by

$$\begin{aligned}\langle G\rangle(t) &= \langle G(t,t_{N-1})\cdots\ G(t_2,t_1)G(t_1,0)\rangle \\ &= \prod_{i=0}^{N-1}\langle G(t_{i+1},t_i)\rangle = \prod_{i=0}^{N-1} G_{ST}(t_{i+1},t_i).\end{aligned} \tag{6}$$

In this expression G_{ST} represents the average of the static Green function and takes the form

$$G_{ST}(t_{i+1},t_i) = \langle G(t_{i+1},t_i)\rangle = \int dY p(Y) G_{[Y]}(t_{i+1}-t_i). \tag{7}$$

The factorization of the averages in Eq. (6) is allowed because the values of Y, before and after a jump, are *independent*. The average of the Green function is the sum of the contributions of the probabilities of having no jump, one jump, and so on. By noting $\Phi(t) = 1 - \int_0^t \varphi(t')dt'$ the probability of having no jump in the interval of time $[0,t]$, $\langle G\rangle(t)$ satisfies the following equation:

$$\begin{aligned}\langle G\rangle(t) = \Phi(t)G_{ST}(t,0) &\quad \text{No jump} \\ &+ \int_0^t \Phi(t-t_1)G_{ST}(t-t_1)\,\varphi(t_1)dt_1 G_{ST}(t_1) \quad \text{1 jump} \\ &+ \int_0^t\int_0^{t_1} \Phi(t-t_2)G_{ST}(t-t_2)\,\varphi(t_2)dt_2 G_{ST}(t_2-t_1)\,\varphi(t_1)dt_1 G_{ST}(t_1) \quad + \cdots \quad \text{2 jumps}\end{aligned} \tag{8}$$

This expression has to be understood as follows: the first term assumes that no jump occurs during the time interval $[0,t]$. The second term assumes that only one jump occurs during the same time interval. If a jump takes place at a time t_1 $(t_1 < t)$, the probability of having a jump at that time is $\varphi(t_1)dt_1$ and the average Green function, from the time 0 to t_1, is expressed by $G_{ST}(t_1)$. During the remaining time interval $[t_1,t]$, the average static Green function is given by $G_{ST}(t-t_1)$. The probability that no more jump occurs during the time interval $[t_1,t]$ is $\Phi(t-t_1)$. The third term corresponds to the case where two jumps take place, and so forth. It can be easily verified that Eq. (8) is equivalent to the following integral equation:

$$\langle G\rangle(t) = \Phi(t)G_{ST}(t) + \int_0^t dt_1 \langle G\rangle(t-t_1)\varphi(t_1)G_{ST}(t_1). \tag{9}$$

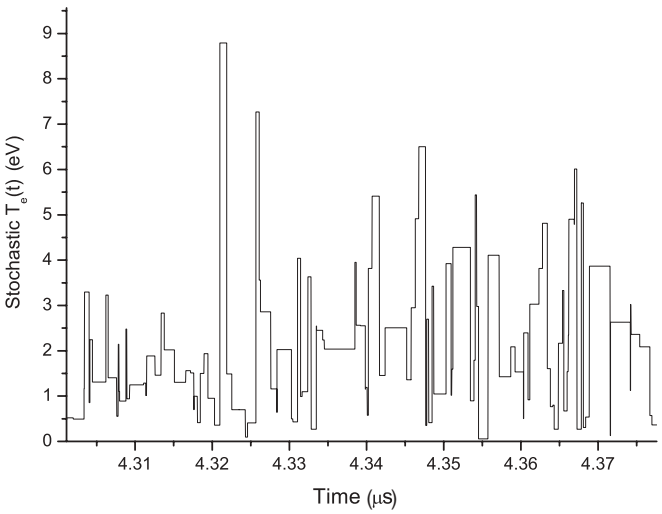

FIG. 1. Example of the stochastic evolution of a plasma fluid field as a function of time. For this illustration we consider the electron temperature as being the fluctuating parameter ($Y = T_e$). Here a Gamma distribution is considered, with $\langle T_e \rangle = 2$ eV and a fluctuation rate $r = \Delta T_e / \langle T_e \rangle = 0.9$. The duration of each step is sampled according to an exponential distribution of parameter $\nu = 1500\ \mu s^{-1}$.

This integral equation can be solved by using Laplace transform techniques. We introduce the following notations:

$$\widetilde{\langle G \rangle}(s) = \int_0^\infty dt \exp(-st) \langle G \rangle(t), \tag{10a}$$

$$\widetilde{\Phi G_{ST}}(s) = \int_0^\infty dt \exp(-st) \Phi(t) G_{ST}(t), \tag{10b}$$

$$\widetilde{\varphi G_{ST}}(s) = \int_0^\infty dt \exp(-st) \varphi(t) G_{ST}(t), \tag{10c}$$

with

$$G_{ST}(t) = \int_0^\infty dY p(Y) \exp\left(M(Y)t\right). \tag{10d}$$

The Laplace transform of Eq. (9) leads to the following solution:

$$\widetilde{\langle G \rangle}(s) = \widetilde{\Phi G_{ST}}(s) [I - \widetilde{\varphi G_{ST}}(s)]^{-1}, \tag{11}$$

where I is the identity matrix. The main advantage of this result is to express the average Green function by an analytical formula which only relies on two probability distributions, namely the PDF and the WTD, to describe the statistics of the process under consideration. The PDF and the WTD are functions which can be determined from statistical measurements of turbulence. As a reminder, the WTD is related to the autocorrelation function of the time-dependent plasma fluid field. In the case of an exponential correlation function, $\varphi(t) = \nu \exp(-\nu t)$, where ν is the inverse of the turbulence correlation time, Eq. (11) simplifies to

$$12 \widetilde{\langle G \rangle}(s) = [I - \nu \widetilde{G_{ST}}(s)]^{-1} \widetilde{G_{ST}}(s) \tag{12a}$$

with

$$\widetilde{G_{ST}}(s) = \left\langle \frac{1}{s + \nu - M} \right\rangle. \tag{12b}$$

This expression is similar to that of the Kubo-Anderson model that was obtained in the context of Stark broadening [7].

The average of the Green function given by Eqs. (11) and (12) is expressed in terms of G_{ST} given by Eq. (10d), which remains a simple function of the static value of the matrix M, even for nonexponential correlation functions. The only restriction is that the WTD is independent of the stochastic fluid field. In Sec. IV, we use the solution of Eq. (11) to extract and discuss the limits where the turbulence frequency is either larger or smaller than the atomic time scales for the nonexponential correlation function.

IV. APPLICATION TO POPULATION KINETICS

The average of the population vector $\langle X(t) \rangle$, after a long time compared to both the correlation time and the atomic relaxation time, reaches a limit $\langle X \rangle$. The purpose of this section is to make the link between the latter vector and the averaged Green function as defined by Eq. (11).

A. Calculation of the ensemble-average population

In order to calculate $\langle X \rangle$, we make use of Eq. (11) and the relation

$$\lim_{t \to \infty} \langle X(t) \rangle = \langle X \rangle = \lim_{s \to 0^+} s \widetilde{\langle X \rangle}(s) = \lim_{s \to 0^+} s \widetilde{\langle G \rangle}(s) X_0, \tag{13}$$

which links the solution in the time domain to that of Eq. (11) in the Laplace domain. Two limits can be investigated analytically, for any integrable correlation functions (i.e., having finite moments), corresponding to short and long turbulence time scales compared to characteristic atomic time scales. Let us consider the first finite moment:

$$\nu^{-1} = \int_0^\infty dt\ t \varphi(t), \tag{14}$$

where ν is the jumping frequency associated to the fluctuating fluid field Y. ν is actually the inverse of the turbulence correlation time $\langle t \rangle$.

The limit $\nu \ll \nu_{at}$ corresponds to the case where plasma parameters may be considered as constant during the atomic lifetime, characterized by the frequency ν_{at}. Inversely for $\nu \gg \nu_{at}$, a case we call the diabatic limit, many plasma field fluctuations occur during the atomic level lifetime. In the two next sections, we infer the atomic population vector corresponding to these two regimes.

B. Static limit

In order to investigate the static limit, $\nu \ll \nu_{at}$ or in practice $\nu \to 0$, let us start from Eq. (10c) which can be reformulated as

$$\widetilde{\varphi G_{ST}}(s) = \left\langle \int_0^\infty dt \exp[-(s - M)t] \varphi(t) \right\rangle = \langle \widetilde{\varphi}(s - M) \rangle, \tag{15}$$

where $\widetilde{\varphi}(s - M)$ is the Laplace transform of $\varphi(t)$ evaluated for the matrix $s - M$. Equation (10b) can be integrated by parts such that

$$\widetilde{\Phi G_{ST}}(s) = \langle (s - M)^{-1} \rangle - \langle (s - M)^{-1} \widetilde{\varphi}(s - M) \rangle. \tag{16}$$

Now we focus on the calculation of Eq. (15) in the case where $\nu \to 0$. The WTD is such that $\int_0^\infty \varphi(t) dt = 1$, meaning

that $\lim_{t\to\infty}\varphi(t)=0$. From Eq. (14), $\int_0^\infty dt(1-t\nu)\varphi(t)=0$ for all ν, where the function φ depends on ν. We split the latter integral into two terms, one integrated from 0 to $1/\nu$ and one from $1/\nu$ to ∞. A change of variable leads to $\int_0^{1/\nu} dt(1-t\nu)\varphi(t)=\int_0^\infty du\ \nu u\varphi(u+\frac{1}{\nu})$. One can show [9] that the right-hand side of the equality goes to 0 when $\nu\to 0$, so that $\lim_{\nu\to 0}\int_0^{1/\nu} dt(1-t\nu)\varphi(t)=0$. Since the integrand of the latter integral is positive for all ν and t, one gets $\lim_{\nu\to 0}\varphi(t)=0$. Translated into the laplace domain, the latter relation implies

$$\lim_{\nu\to 0}\widetilde{\varphi}(s-M)=0. \tag{17}$$

From Eqs. (11) and (16), and using Eq. (17), the following relation is obtained:

$$s\widetilde{\langle G\rangle}(s)X_0 \underset{\nu\to 0}{=} \left\langle \frac{s}{s-M}X_0\right\rangle, \tag{18}$$

since X_0 does not depend on the fluctuating quantity Y. The solution we are seeking is the limit when $s\to 0$ of Eq. (18). It can be shown (cf. [9]) that the solution of this equation is given by the average value of the eigenvector of M corresponding to the eigenvalue 0, i.e., $X_{ST}=\int dY p(Y)X_\infty[Y]$ where X_∞ is such that $MX_\infty=0$. This solution corresponds to the average of the stationary solution over the fluctuation of Y as expected intuitively and is called the static limit.

C. Diabatic limit

The diabatic limit corresponds to the case where $\nu\gg\nu_{at}$ or in practice $\nu\to\infty$. A Taylor expansion [1] of $\widetilde{\varphi}(s-M)$ for small values of s (and a fixed large value of ν) gives

$$\widetilde{\varphi}(s-M)\approx 1-\frac{(s-M)}{\nu}. \tag{19}$$

Using Eqs. (19) and (16) we get

$$\widetilde{\Phi G_{ST}}(s)\underset{\nu\to\infty}{\sim}\frac{1}{\nu} \tag{20}$$

and

$$(s-\langle M\rangle)\widetilde{\langle G\rangle}(s)\underset{\nu\to\infty}{\sim} I. \tag{21}$$

It is interesting to notice here that the expression of $\langle\widetilde{G}(s)\rangle$ in Eq. (21) is similar to that of Eq. (18) where $\langle 1/(s-M)\rangle$ is replaced by $1/(s-\langle M\rangle)$. Finally, introducing Eqs. (19) and (21) into Eq. (11), the solution in the diabatic regime is such that

$$\langle M\rangle\lim_{s\to 0}s\widetilde{\langle G\rangle}(s)X_0=\langle M\rangle X_{DB}=0,$$
$$\text{where}\quad X_{DB}=\lim_{s\to 0}s\widetilde{\langle G\rangle}(s)X_0. \tag{22}$$

As a consequence, the solution in the diabatic limit is the unique element of $\ker(\langle M\rangle)$. Unlike in the previous case, here the average over the quantity Y is performed on M. It can be shown that the two solutions corresponding to the diabatic and static limits do not depend on the initial condition X_0 [9].

In the static limit, the population reaches its steady state before a jump of Y. Since the steady state does not depend on X_0, consequently the average in the static regime does not depend on the initial conditions. In the diabatic limit, this property is more difficult to figure out. The independence of the diabatic limit on X_0 is linked to the ergodic theorem. The correlation time can be seen as the time from which a *de-correlation* with the initial conditions occurs. This is the reason why the average population does not depend on the initial conditions, even in the diabatic regime.

It must be underlined that the static and diabatic asymptotic behaviors do not depend on the WTD, as long as the WTD has finite moments. Therefore the WTD does not affect the asymptotic trends of the population kinetics but only influences the intermediate region between the static and diabatic limits.

The static and diabatic limits have been obtained for WTD having finite moments, i.e., for integrable correlation functions. The average population given by Eq. (11) has been obtained for any correlation functions and can be applied to correlation functions that are not integrable like the one corresponding to Lévy WTD. But in this case, the two limits previously analyzed need to be redefined.

V. FULL NUMERICAL INTEGRATION

In order to calculate the ensemble-average value $\langle X(t)\rangle$, a straightforward procedure consists of solving Eq. (1) numerically for a given turbulent field configuration. For comparisons with the model detailed in the next section, we have performed a numerical integration and made use of the Crank-Nicolson algorithm [12]. The turbulent field configuration was calculated by generating randomly a set of independent values of the fluid field (Y), as shown in Fig. 1.

For a time long enough (longer than both the relaxation time and the correlation time to ensure ergodicity, cf. [9]), the ensemble-average value $\langle X(t)\rangle$ becomes time-independent and the value reached corresponds to the ensemble-average population vector. Typically a few hundred samples are needed to ensure the convergence of the calculation. This is the reason why this procedure may be time consuming and the model proposed here turns out to be useful.

VI. RESULTS OF THE STOCHASTIC MODEL AND DISCUSSIONS

In this section we illustrate the stochastic model using a simple system formed by the $1s$, $2s$, and $2p$ levels of atomic hydrogen. Figure 2 represents the different processes included in our model:

(i) the spontaneous emission from $2p$ to $1s$,

(ii) electron excitation and de-excitation from $1s$ to $2s$ and $1s$ to $2p$,

(iii) ion excitation and de-excitation from $2s$ to $2p$.

Here we assume that electrons and ions have equal densities and temperatures.

[1] Here we assume that φ satisfies the Carleman criterion. If it is not the case, like for the log-normal distribution, φ cannot be expanded as a Taylor series but can still be expanded around $s=0$. Since Eq. (21) does not depend on the first moment, the result of Eq. (22) is valid as long as φ has finite moments.

The matrix $M(t)$ of Eq. (1) of the reduced system is given by

$$M(t) = \begin{pmatrix} -\Gamma_{1s\to 2s} - \Gamma_{1s\to 2p} & \Gamma_{2s\to 1s} & A_{2p\to 1s} + \Gamma_{2p\to 1s} \\ \Gamma_{1s\to 2s} & -\Gamma_{2s\to 2p} - \Gamma_{2s\to 1s} & \Gamma_{2p\to 2s} \\ \Gamma_{1s\to 2p} & \Gamma_{2s\to 2p} & -\Gamma_{2p\to 2s} - A_{2p\to 1s} - \Gamma_{2p\to 1s} \end{pmatrix}, \tag{23}$$

where $\Gamma_{i\to j} = N_e k_{i\to j}(T_e)$ and N_e, T_e are, respectively, the time-dependent electron density and temperature. The rate coefficients ($k_{i\to j}$) have been calculated from the cross section presented in [13] assuming a Maxwellian velocity distribution function. This simple model still reproduces quite well the physics of the population dynamics and the usual static regime is recovered [1]. Depending on the electron density (other parameters being fixed), three regimes can be distinguished [1,9]:

(i) High densities: the local thermodynamical equilibrium is reached and all the populations follow the Boltzmann law.

(ii) Low densities: the collisional de-excitation rate coefficients can be neglected ($k_{2p\to 1s}$ and $k_{2s\to 1s}$ equals 0) and the population density steady-state vector is given by

$$X_\infty = \begin{pmatrix} 1 \\ \dfrac{k_{2p\to 2s}N_e}{A_{2p\to 1s}k_{2s\to 2p}}(k_{1s\to 2p} + k_{1s\to 2s}) + \dfrac{k_{1s\to 2s}}{k_{2s\to 2p}} \\ \dfrac{(k_{1s\to 2s} + k_{1s\to 2p})N_e}{A_{2p\to 1s}} \end{pmatrix} x_{1s}^\infty. \tag{24}$$

x_{1s}^∞, corresponding to the value of the population density steady state of the $1s$ level, is set such that $\sum_i x_i = 1$. This regime is called the coronal regime.

(iii) Intermediate densities: for which a collisional radiative model is needed to describe the population kinetics.

A. Description of the statistics used in this work

In this section we introduce the plasma fluctuations and apply the analytical result given by Eqs. (13) and (11) to our system. As stated in Sec. II, the goal of this work is to extract the main features of the average populations when fluctuating plasma fluid fields are taken into account. In fact, for low N_e (such that $x_{1s}^\infty \approx 1$), the populations are linear in electron density, so that no effects from a fluctuating electron density are expected. Thus the electron temperature is considered as the only fluctuating quantity in this work, while the electron density is assumed to be constant. The PDF of the electron temperature is assumed to be a Gamma distribution, which allows full analytical calculations in the diabatic and static limit. For the sake of simplicity and since this is a first application study, we consider an exponential correlation function, i.e., an exponential WTD. We have shown in Sec. IV that the diabatic and static limits are independent of the WTD. Thus a nonexponential WTD would only affect our results in the intermediate regime. Other WTDs, relaxing the Markov approximation, will be investigated in a future work. The WTD $\varphi(t)$ and the PDF $p(T)$ describing the statistical properties of the electron temperature are, respectively, given by

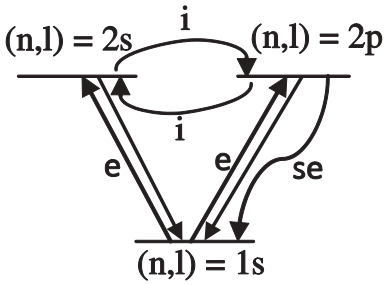

FIG. 2. Schematic of the system; e stands for electrons, i for ions, and se for spontaneous emission.

$$\varphi(t) = \nu \exp(-\nu t), \tag{25a}$$

$$p(T) = \frac{\alpha^\beta}{\Gamma(\beta)} T^{\beta-1} \exp(-\alpha T). \tag{25b}$$

The average value of the time intervals $\langle t\rangle$ on which T_e is constant is $\langle t\rangle = \int_0^\infty \tau\varphi(\tau)d\tau = 1/\nu$. The average value of the electron temperature is given by $\langle T\rangle = \int_0^\infty Tp(T)dT = \beta/\alpha$ and the temperature dispersion is given by $\Delta T = \sqrt{\langle T^2\rangle - \langle T\rangle^2} = \sqrt{\beta}/\alpha$. We introduce the fluctuation rate $r = \Delta T/\langle T\rangle = 1/\sqrt{\beta}$. The system is then exhaustively described by N_e, $\langle T_e\rangle$ for the static description of the plasma, and ν, r for the fluctuating part.

B. Average populations as a function of ν

In this paragraph, results for the average populations x_{2s} and x_{2p}, respectively, for the $2s$ and $2p$ levels of the reduced atomic hydrogen as a function of the turbulence frequency ν are presented. We set the electron density to 10^{18} m^{-3} and the average electron temperature to 2 eV (coronal regime). The fluctuation rate of the electron temperature is set to 90% ($r = 0.9$). In Fig. 3, the results obtained from the analytical Eq. (12) (black solid line) are compared to the time-dependent calculation solving Eq. (1) numerically (red dots), as explained in Sec. V.

An excellent agreement between the two methods is found. Moreover, our method is 20 times faster than the numerical integration, and is even faster for small values of ν. The gain in calculation time would have been even more significant if the turbulent fields had been calculated from a turbulence code. This highlights the interest of our approach.

The first noticeable result is the increase of populations when turbulence is introduced. This result can be explained by the spread of the PDF toward high temperatures. Since the higher the temperature the higher the population, the averaging provides an increase of population compared to the turbulence-free case.

The population of the $2s$ level differs by a factor of 2 going from the static to the diabatic limits requiring a change of two

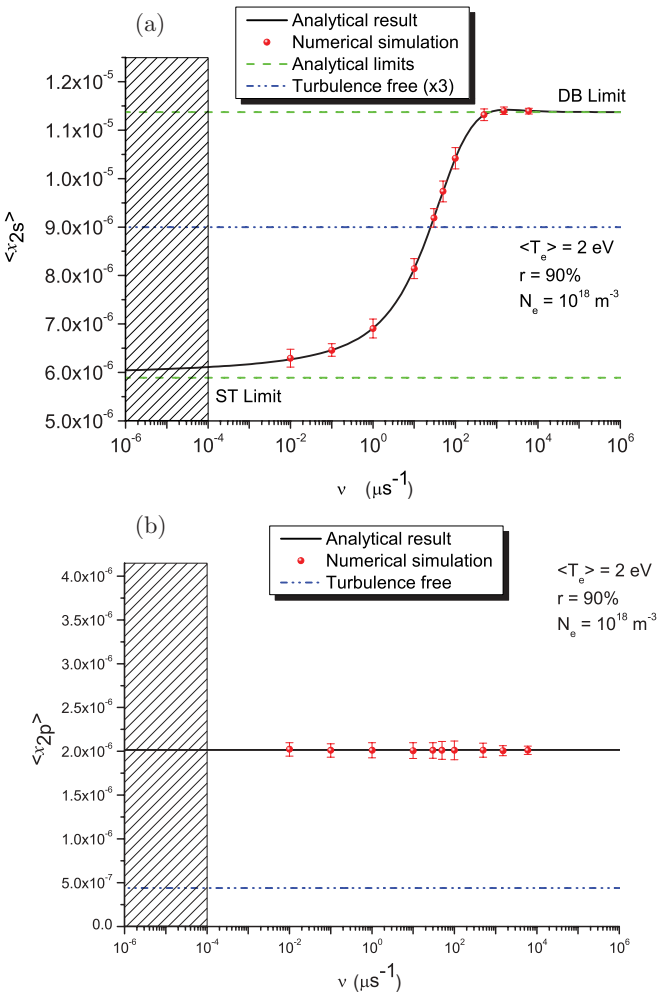

FIG. 3. (Color online) Plot of the average populations as a function of the turbulence frequency (ν). The result for the $2s$ level is given in (a) and that of the $2p$ level in (b). The dispersion around the average value is indicated by error bars which represents the deviation to ergodicity. The dashed region indicates the range of turbulence frequency for which the ionizing regime approximation is not valid. In the same way, the turbulence frequency has to be smaller than the frequency of electron-ion, electron-electron, and ion-ion collisions, in order to use the Maxwellian velocity distribution function. For this example, the lowest of those frequencies is the ion-electron collision that equals $10^{10}\ \mu s^{-1}$. The plasma frequency is equal to $3 \times 10^4\ \mu s^{-1}$. The dot-dashed curve is the turbulence-free result.

orders of magnitude of the turbulence frequency (from 1 to 100 μs^{-1}). This shows the strong effect on the population kinetics of the plasma fluctuations.

This range of turbulence frequencies may be observed in turbulent plasmas. Indeed, the plasma frequency is equal to $3 \times 10^4 \mu s^{-1}$, meaning that the highest achievable turbulence frequency (of the order of a tenth of the plasma frequency) is in the diabatic regime.

The population of the $2s$ level in the diabatic regime is higher than the one in the static regime since $x_{2s}(T_e)$ is convex (cf. Jensen inequality [14]). The transition between the two regimes occurs, as expected, at a frequency ν close to those of the average rates of the atomic system defined by $N_e \int dT_e p(T_e) k(T_e)$, where $k(T_e)$ is the rate coefficient ($N_e \langle k_{1s\to 2s}\rangle = 4.5 \times 10^{-4}\ \mu s^{-1}$, $N_e \langle k_{1s\to 2p}\rangle = 8 \times 10^{-4}\ \mu s^{-1}$, $A_{2p\to 1s} = 626\ \mu s^{-1}$, while $N_e \langle k_{2s\to 2p}\rangle = 42.7\ \mu s^{-1}$).

The values of the population in the diabatic and static limits depend on the average value of the temperature and density. For large values of the density, the diabatic and the static merge since the thermodynamical equilibrium is reached.

From Fig. 3 we show that the average population of the $2p$ level does not depend on the value of the turbulence frequency, a peculiar result that is related to the limited number of levels considered in this model. It can be checked in Eq. (24) that the diabatic and static limits are indeed identical for this particular level. Moreover, from Eq. (24) it can be seen that the $2s$ level behaves differently from the $2p$ level due to the finite value of the transition rate from $2s$ to $2p$ by ion impact.

C. Average populations as a function of $\langle T_e \rangle$ and the fluctuation rate r

In the previous paragraph, we have shown that the average populations are increased compared to the turbulence-free case. The calculation has been performed for a low average temperature (2 eV). We now focus on the effect of the average temperature and the fluctuation rate r in order to investigate whether the previous behavior remains the same. In the following, the electron density is set to $10^{19}\ m^{-3}$ and the turbulence frequency to $1000\ \mu s^{-1}$ (i.e., close to the diabatic limit). The average populations $\langle x_{2s}\rangle$ and $\langle x_{2p}\rangle$ are plotted in Fig. 4, as a function of $\langle T_e \rangle$ for different values of r.

The first feature to note is that the effect of fluctuations are significant only for quite large fluctuation rates (several tens of percents). Next the global trend of the populations remains the same, i.e., an increase of the populations when $\langle T_e \rangle$ increases, and three regimes can be globally distinguished:

(i) $\langle T_e \rangle < T_c$ for which the average populations decrease when r decreases,

(ii) $\langle T_e \rangle > T_c$ having the opposite behavior, namely the average populations decrease when r increases,

(iii) $\langle T_e \rangle \approx T_c$ for which the average populations do not depend on r.

We call T_c the "characteristic" temperature that is equal to 5.7 eV for the $2s$ level and 5.4 eV for the $2p$ level. We observed that T_c depends on N_e and ν for the $2s$ level, while both parameters do not affect the value of T_c for the $2p$ level.

T_c can be estimated by expanding Eq. (22), in the diabatic regime, i.e., using an expansion of the rate coefficients. T_c is then defined such that the average population does not depend on r up to the second order. T_c, in the diabatic regime, is found to be equal to 5.2 eV for the $2p$ level and 5.0 eV for the $2s$ level, in good agreement compared to the values found in Fig. 4 (10% discrepancy for the $2s$ level and 4% for the $2p$ level).

Actually, the same behavior survives in the static regime and is easier to interpret in this regime. Indeed, at large r and $\langle T_e \rangle$, the PDF becomes dominant at a low T_e. Since the population at low T_e is small, the averaging at large r and $\langle T_e \rangle$ reduces the value of the population. On the other hand, at low average temperature, when r increases, the contribution of high temperature becomes dominant. Since the higher the

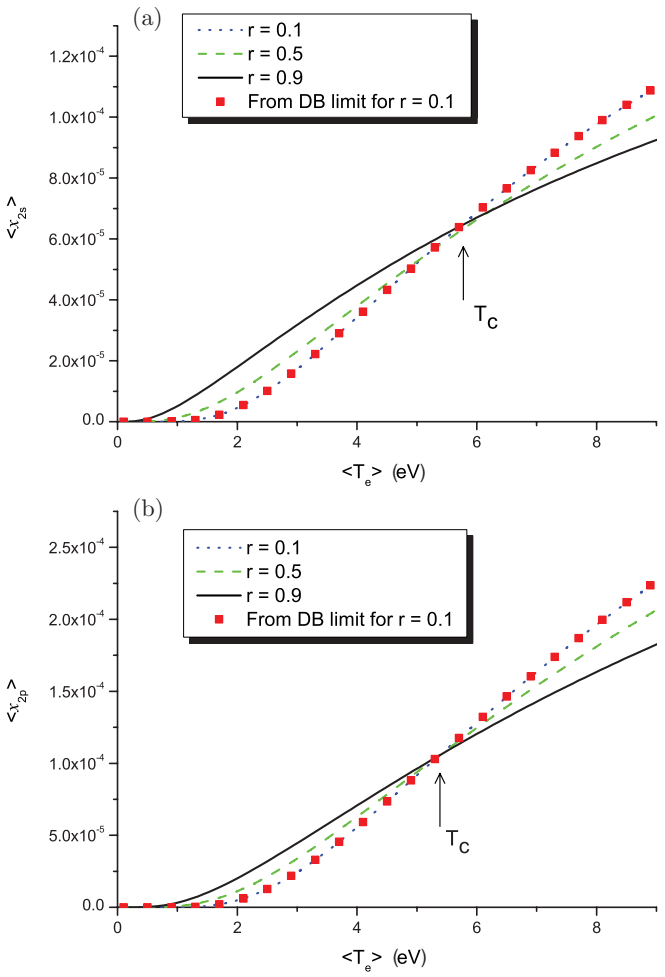

FIG. 4. (Color online) Plot of the average populations as a function of the average electron temperature. (a) is for the $2s$ level, and (b) is for the $2p$ level. The electron density and the turbulence frequency have been set to $N_e = 10^{19}$ m^{-3} and $\nu = 1000$ μs^{-1}. The dotted curve is for $r = 0.1$, the dashed curve is for $r = 0.5$, while the full curve is for $r = 0.9$. The squared curve is the solution of the diabatic limit for $r = 0.1$, which corresponds to the turbulence-free case. T_c is the characteristic temperature as defined in the text.

temperature the higher the population, the effect of the average is to increase the value of the population.

We now give a physical interpretation of this behavior by using our model in the static limit. The average population of the ith level is then given by $\langle x_i\rangle(\langle T\rangle) = \int p(T_e)\, x_i(T_e) dT_e$, where $p(T_e)$ is the temperature PDF which depends on r and $\langle T_e\rangle$. Expanding x_i around the average temperature up to the second order leads to

$$\langle x_i\rangle(\langle T_e\rangle) \approx x_i(\langle T_e\rangle) + x_i''(\langle T_e\rangle) r^2 \langle T_e\rangle^2/2. \qquad (26)$$

From Eq. (26), $\langle x_i\rangle$ does not depend on r at T_c up to the second order if

$$x_i''(T_c) = 0, \qquad (27)$$

which does correspond to the inflection point of $x(T_e)$ given by Eq. (24). In the case of the $2p$ level, the inflection point is $T_c = 5.17$ eV in very good agreement with the value found in Fig. 4 (discrepancy of 4%).

For the $2s$ level, the inflection point is found to be at 4.2 eV, corresponding to a discrepancy of 26% compared with the value issued from Fig. 4. This discrepancy is not surprising since the inflection point has been obtained in the static regime while the data of Fig. 4 have been calculated in a regime close to the diabatic limit.

As pointed out previously, T_c does not depend on N_e for the $2p$ level. Since the population in the $2p$ level is proportional to N_e, Eq. (27) defines a T_c which also does not depend on N_e.

A last point that can be well explained by Eq. (26) is the change of behavior of the average population as a function of r, whether $\langle T_e\rangle$ is larger or smaller than T_c. If $\langle T_e\rangle < T_c$, then $x_i''(\langle T_e\rangle) > 0$ so when r increases, $\langle x_i\rangle(\langle T_e\rangle)$ increases at a given value of $\langle T_e\rangle$. Conversely, if $\langle T_e\rangle > T_c$, then $x_i''(\langle T_e\rangle) < 0$; as a consequence when r increases, $\langle x_i\rangle(\langle T_e\rangle)$ decreases for a given value of $\langle T_e\rangle$, exactly as observed in Fig. 4.

VII. CONCLUSION AND FUTURE WORK

A population kinetics model accounting for plasma fluctuations has been developed. By using a stochastic description of the plasma fields, we have obtained an analytical formula for the ensemble-average populations. This stochastic description is shown to only depend on two functions characterizing the plasma fields fluctuations, namely their PDF and autocorrelation function, which can be matched to experimental data.

Two limits, corresponding to characteristic turbulent time scales either larger or smaller than the atomic time scales, have been shown to be independent of the autocorrelation function. The latter only affects the intermediate turbulence frequencies range.

As an illustration of the effect of the fluctuating plasma on the population kinetics, we have investigated a system formed of the ($1s$, $2s$, $2p$) states of atomic hydrogen. The electron temperature is assumed to be the stochastic field, described by a Gamma PDF and an exponential correlation function. We have observed a strong effect of the fluctuating temperature on the averaged atomic populations of the upper levels (up to a factor of 4). From this study we can conclude that the average populations increase or decrease, respectively, for high and low average temperatures. Consequently there is a characteristic temperature for which the average populations are not affected by turbulence. This temperature depends both on the plasma conditions and the atomic system. For this particular reduced atomic system, we have shown that the average population of the $2s$ level increases with the turbulence frequency, while that of $2p$ level remains constant, the latter behavior being the consequence of a reduced system. Our model can be used for more complex systems; this is ongoing work.

ACKNOWLEDGMENTS

This work was supported by the Agence Nationale de la Recherche (Grant No. ANR-07-BLAN-0187-01, project PHOTONITER), and is part of a collaboration (LRC DSM99-14) between the Laboratoire de Physique des Interactions Ioniques et Moléculaires (PIIM) (UMR 6633) and the Institut de Recherche sur la Fusion Magnétique, CEA Cadarache, within the framework of the Fédération de Recherche sur la Fusion par Confinement Magnétique (FRFCM).

APPENDIX: LINK BETWEEN THE AUTOCORRELATION FUNCTION AND THE WTD

In this appendix we establish the link between the autocorrelation function of the fluctuating parameter Y and the WTD, noted $\varphi(t)$. Using the same approach as in Sec. III, it can be shown that the Laplace transform of the conditional probability, $P_t(Y|Y')$ for Y at the time t given Y' at the time $t = 0$, is given by

$$\widetilde{P}_s(Y|Y') = \widetilde{\Phi}(s)\,\delta(Y - Y') + \frac{p(Y)\widetilde{\varphi}(s)}{1 - \widetilde{\varphi}(s)}\,. \tag{A1}$$

$\widetilde{\varphi}(s)$ is the Laplace transform of $\varphi(t)$ and $\widetilde{\Phi}(s)$ is the Laplace transform of $\Phi(t)$, which is the probability of having no jumps in the interval of time $[0,t]$. The covariance function of Y is

$$\langle \delta Y \delta Y' \rangle = \mathrm{Var}_Y(t) = \int\int dY dY' (Y - \langle Y \rangle) \times (Y' - \langle Y \rangle) P_t(Y|Y') p(Y') \tag{A2a}$$

with

$$\delta Y = (Y - \langle Y \rangle) \text{ and } \delta Y' = (Y' - \langle Y \rangle). \tag{A2b}$$

The Laplace transform of Eq. (A2a) gives

$$\widetilde{\mathrm{Var}_Y}(s) = \int\int dY dY' (Y - \langle Y \rangle) \times (Y' - \langle Y \rangle) \widetilde{P}_s(Y|Y') p(Y'), \tag{A3}$$

so that, inserting Eq. (A1) in Eq. (A3), we get $\widetilde{\mathrm{Var}_Y}(s) = \sigma^2 \widetilde{\Phi}(s)$. Here σ^2 is the variance of Y. The inverse Laplace transform of the latter expression gives

$$\mathrm{Var}_Y(t) = \sigma^2 \Phi(t). \tag{A4}$$

One can notice that the covariance function is only due to the first term of Eq. (A1) corresponding to the contribution of processes for which there are no jumps. The autocorrelation of Y, which is usually a quantity that can be determined experimentally, is simply proportional to $\Phi(t)$. The autocorrelation function is given by $C_Y(t) = \mathrm{Var}_Y(t)/\sigma^2$ and is simply equal to $\Phi(t)$. Since the WTD is the opposite of the derivative of $\Phi(t)$, the generalization of this model allows us to use autocorrelation functions that are not necessarily exponential and relax the Markov approximation. The link between the autocorrelation function and the WTD allows an unambiguous determination of the WTD from the autocorrelation. Moreover, it is clear from Eq. (A4) that

$$\int_0^\infty C_Y(t) = \int_0^\infty t\varphi(t) = \langle t \rangle = 1/\nu. \tag{A5}$$

This relation is valid for autocorrelation functions that are decreasing faster than $1/t$ at large values of t, meaning that the autocorrelation function is integrable. Here it must be noted that the derivative of the autocorrelation function is a PDF. As such, this model cannot describe all classes of autocorrelation fucntions. We have extended our analysis to the case where the WTD depends on the fluctuating quantity Y, i.e., $\varphi(t)$ becomes $\varphi_Y(t)$. In that case, following the same procedure as the one detailed in the text, the average Green function in the Laplace domain is then

$$\langle \tilde{G} \rangle(s) = \int dY p(Y) \tilde{K}_Y(s) + \int dY p(Y) \tilde{J}_Y(s) \times \left(1 - \int dY q(Y) \tilde{J}_Y(s)\right)^{-1} \int dY q(Y) \tilde{K}_Y(s), \tag{A6}$$

with

$$\tilde{K}_Y(s) = \int dt \exp(-st) G_{ST}(t) \Phi_Y(t) \text{ and } \tilde{J}_Y(s) = \int dt \exp(-st) G_{ST}(t) \varphi_Y(t). \tag{A7}$$

$p(Y)$ and $q(y)$ are related by

$$q(Y) = p(Y) \frac{\langle t \rangle_q}{\langle t \rangle_Y}, \tag{A8}$$

where

$$\langle t \rangle_Y = \int dt\, t \varphi_Y(t) \text{ and } \langle t \rangle_q = \int dY q(Y) \langle t \rangle_Y = \left(\int dY \frac{p(Y)}{\langle t \rangle_Y} \right)^{-1}. \tag{A9}$$

The correlation function is then

$$C(t) = \int dY p(Y) \Phi_Y(t) (Y - \langle Y \rangle)^2 / \sigma^2. \tag{A10}$$

The previous study, for which ν does not depend on Y, needs to be extended. This is the purpose of a coming work.

[1] T. Fujimoto, *Plasma Spectroscopy* (Oxford University Press, New York, 2004).

[2] K. Sawada and T. Fujimoto, Phys. Rev. E **49**, 5565 (1994).

[3] H. Summers *et al.*, Plasma Phys. Controlled Fusion **48**, 263 (2006).

[4] F. Rosmej *et al.*, Contrib. Plasma Phys. **48**, 243 (2008).

[5] P. Stangeby, *The Plasma Boundary of Magnetic Fusion Devices* (Taylor and Francis, London, 2000).

[6] V. Gervids and D. Morozov, JETP Lett. **67**, 324 (1998).

[7] A. Brissaud and U. Frisch, J. Quant. Spectrosc. Radiat. Transfer **11**, 1767 (1971).

[8] J. Rosato *et al.*, J. Nucl. Mater. (in press).

[9] F. Catoire *et al.*, Eur. Phys. J. D (to be published).

[10] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1970).

[11] P. Résibois and M. D. Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).

[12] J. Crank and P. Nicolson, Proc. Cambridge Philos. Soc. **43**, 50 (1947).

[13] R. Janev, W. Langer, K. Evans, Jr., and D. E. Post Jr., *Elementary Processes in Hydrogen-Helium Plasmas: Cross Sections and Reaction Rate Coefficients,* Springer Series on Atoms and Plasmas Vol. 4 (Springer-Verlag, Heidelberg, 1987).

[14] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products*, 7th edition (Academic, New York, 2007).