Comparing different non-Markovianity measures in a driven qubit system

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We consider two recently proposed measures of non-Markovianity applied to a particular quantum process describing the dynamics of a driven qubit in a structured reservoir. The motivation for this study is twofold: on one hand, we study the differences and analogies of the non-Markovianity measures, and on the other hand, we investigate the effect of the driving force on the dissipative dynamics of the qubit. In particular we ask if the driving force introduces new channels for energy and/or information transfer between the system and the environment or if it amplifies existing ones. We show under which conditions the presence of the driving force slows down the inevitable loss of quantum properties of the qubit.

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I. INTRODUCTION

The inevitable interaction of a quantum system with its environment leads to dissipation of energy and loss of quantum coherence [1,2]. Unfortunately these phenomena can fundamentally limit the potential of quantum technologies, whose power is based on quantum effects [3]. In suitable surroundings, however, the quantum system may temporarily regain some of the previously lost energy and/or information due to non-Markovian effects in the system dynamics. This is one of the reasons why the study of non-Markovian quantum dynamics has received an increasing amount of interest in the last several years [4]. It is surprising, then, that the definition of non-Markovianity still remains elusive and, in some sense, controversial.

Markovianity is well defined for classical stochastic processes: a stochastic process has the *Markov property* if the probability distribution for the future states of the process depends only on the present state. Loosely speaking, dependence on past states should then be a characteristic trait of non-Markovian processes. The mathematical definition of non-Markovianity does not, however, translate easily into the widely used language of density matrices and master equations in quantum physics. In the case of an open quantum system evolution described by a Lindblad master equation [5,6] one can find a stochastic description for the system dynamics, e.g., with the Monte Carlo wave function method, where future dynamics of individual quantum trajectories only depend on their present states [7,8]. In this spirit some physicists have attributed non-Markovianity to the breakdown of the semigroup property [9] or a generalization thereof, the divisibility property [10]. However, finding a stochastic description that corresponds to more general master equations is not, in general, a straightforward task [11].

With the aim of extending the concept of non-Markovianity to more general quantum dynamics, efforts have been made to clarify the very definition of non-Markovianity in the context of open quantum systems and to quantify the amount of non-Markovianity in a given quantum process. One viewpoint associates non-Markovianity to dynamical dependence on past evolution. This may mean either dependence of the evolution on all past states of the system, as in the memory kernel approach [12,13], or dependence of the asymptotic state of the system on its initial state [14]. Yet another definition equates non-Markovianity with a partial recovery of previously lost information [15–17]. It is worth stressing that the definitions of non-Markovianity are generally not the same, nor are they mutually exclusive, and that they may agree perfectly about the non-Markovian character for some models and disagree about others [18].

In this paper we consider two recently proposed measures of non-Markovianity introduced by Rivas, Huelga, and Plenio (RHP) [10] and by Breuer, Laine, and Piilo (BLP) [15] and apply them to a quantum process describing the short and intermediate time-scale dynamics of a driven qubit in a structured reservoir [19,20]. The comparison allows us, on one hand, to elucidate the differences and analogies between the non-Markovianity measures in a physically interesting model and, on the other hand, to gain new understanding of the effect of the driving force on dissipative dynamics. For the sake of concreteness in the rest of the paper we refer to the case of a laser-driven atom. All the conclusions, however, are valid for any driven two-state system. Due to the presence of the driving force, the commonly used secular approximation does not always hold, especially in the short non-Markovian time scales. The importance of the nonsecular terms has been recently demonstrated in the context of superconducting circuits in Refs. [21,22].

The article is organized as follows. In Sec. II we introduce our model, namely, the driven qubit, and the master equation describing its dynamics in a structured environment. In Secs. III and IV we introduce and study the RHP and BLP measures of non-Markovianity using this model. In Sec. V we compare the non-Markovian properties of a driven qubit with those of a nondriven qubit and finally, in Sec. VI, we summarize the results and present our conclusions.

II. THE DRIVEN QUBIT

We consider a qubit with energy separation ω_A ($\hbar = 1$), driven by a laser of frequency ω_L almost resonant with the transition frequency of the qubit: $|\Delta| = |\omega_A - \omega_L| \ll \omega_A$. The

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qubit is embedded in a zero-T environment. The strengths of the interaction between the qubit and the laser and between the qubit and the environment are quantified by the Rabi frequency Ω and the coupling constant α, respectively. We assume that $\Omega \ll \omega_A$, a condition that is typically satisfied in quantum optical situations. When α is the smallest relevant frequency, that is, the qubit couples weakly to the reservoir, the dynamics of the qubit is given by the following local-in-time completely positive master equation, derived by one of us in Ref. [19]:

$$\frac{d\rho(t)}{dt} = -i[H,\rho(t)] + \mathcal{D}_S(t)\rho(t) + \mathcal{D}_{NS}(t)\rho(t), \tag{1}$$

where the unitary evolution of the qubit is generated by the system Hamiltonian

$$H = \frac{\omega}{2}\sigma_z, \quad \omega = \sqrt{\Delta^2 + \Omega^2}, \tag{2}$$

and the dissipative terms have been divided into a secular contribution $\mathcal{D}_S$ and a nonsecular contribution $\mathcal{D}_{\mathrm{NS}}$. Introducing the shorthand notation

$$\mathcal{L}[A(t),\gamma(t)] = \gamma(t)\big(A(t)\rho(t)A^\dagger(t) - \tfrac{1}{2}[A^\dagger(t)A(t),\rho(t)]\big) \tag{3}$$

to describe a Lindblad-type decay channel with a jump operator $A(t)$ and a corresponding decay rate $\gamma(t)$, we can express the dissipative terms as

$$\begin{aligned}\mathcal{D}_S(t)\rho(t) = {} & C_+^2\mathcal{L}[\sigma_-,\gamma_+(t)] + C_-^2\mathcal{L}[\sigma_+,\gamma_-(t)] \\ & + C_0^2\mathcal{L}[\sigma_z,\gamma_0(t)],\end{aligned} \tag{4}$$

$$\begin{aligned}\mathcal{D}_{NS}(t)\rho(t) = {} & \Gamma_-(t)\{C_-C_0[\sigma_+\rho(t)\sigma_z - \sigma_z\sigma_+\rho(t)] \\ & + C_+C_-[\sigma_+\rho(t)\sigma_+ - \sigma_+\sigma_+\rho(t)]\} \\ & + \Gamma_+(t)\{C_+C_0[\sigma_-\rho(t)\sigma_z - \sigma_z\sigma_-\rho(t)] \\ & + C_+C_-[\sigma_-\rho(t)\sigma_- - \sigma_-\sigma_-\rho(t)]\} \\ & + \Gamma_0(t)\{C_-C_0[\sigma_z\rho(t)\sigma_- - \sigma_-\sigma_z\rho(t)] \\ & + C_+C_0[\sigma_z\rho(t)\sigma_+ - \sigma_+\sigma_z\rho(t)]\} + \mathrm{H.c.},\end{aligned} \tag{5}$$

where $C_\pm = (\Delta \pm \omega)/2\omega$, $C_0 = \Omega/2\omega$ and H.c. denotes Hermitian conjugation. The master equation is given in the dressed state basis $|\psi_\pm\rangle = \pm\sqrt{C_\pm}|e\rangle + \sqrt{C_\mp}|g\rangle$, where $|e\rangle$ and $|g\rangle$ are the excited and ground states of the qubit. The decay rates in $\mathcal{D}_{\mathrm{NS}}$ and $\mathcal{D}_S$ are connected via the relation

$$\Gamma_\xi(t) = \left[\frac{\gamma_\xi(t)}{2} - i\lambda_\xi(t)\right], \tag{6}$$

where $\gamma_\xi,\lambda_\xi \in \mathbf{R}$ and $\xi \in \{-,0,+\}$. For the sake of concreteness we focus on a qubit embedded in a reservoir with a Lorentzian spectral density $J(\omega) = \alpha/(2\pi)\,\lambda^2/[\lambda^2 + (\omega-\omega_0)^2]$, where ω_0 is the center frequency and λ is the width of the Lorentzian. For this case the decay rates in Eqs. (4) and (5) take the form [20]

$$\begin{aligned}\gamma_\xi(T) &= \frac{\alpha^2}{2\big(1+q_\xi^2\big)}(1 - e^{-T}\cos q_\xi T + e^{-T} q_\xi \sin q_\xi T), \\ \lambda_\xi(T) &= \frac{\alpha^2}{1+q_\xi^2}(-q_\xi + e^{-T} q_\xi \cos q_\xi T + e^{-T}\sin q_\xi T),\end{aligned} \tag{7}$$

where $T = \lambda t$ and $q_\xi = s - \xi p$, again with $\xi = \{-,0,+\}$. Different dynamical regimes are defined in terms of the parameters

$$s = \frac{\omega_0 - \omega_L}{\lambda}, \quad p = \frac{\tau_C}{\tau_S} = \frac{\omega}{\lambda}. \tag{8}$$

The parameter s quantifies the detuning between the qubit and the center frequency of the Lorentzian. The parameter p expresses the relationship between the reservoir correlation time scale $\tau_C = \lambda^{-1}$ and the typical system time scale $\tau_S = \omega^{-1}$ and it determines the border between secular and nonsecular regimes. Note that here we refine the typical textbook notion of the secular approximation, which requires that the typical time scale of the system is negligible in comparison with the relaxation time scale and results in a coarse-graining of the relaxation dynamics. Since in this work we are interested in the short time-scale dynamics we assume a stronger condition, namely, that the typical time scale of the driven qubit is much smaller than the other time scales and, in particular, the reservoir correlation time scale τ_C. We call the limit $\tau_S \ll \tau_C$ the strong secular limit.

A. Strong secular limit

When $p \gg 1$ the secular approximation holds and the nonsecular dissipation term $\mathcal{D}_{\mathrm{NS}}$ can be neglected from Eq. (1). Consequently, in the language of Markovian [7,8] and non-Markovian [23,24] quantum jumps, the master equation comprises three Lindblad-like terms describing phase flips and jumps between the eigenstates of the driven qubit, with the direction of the jump (regular or reversed) depending on the sign of the corresponding decay rate (positive or negative). We have shown in Ref. [20] that, in the secular regime, the decay rates $\gamma_\pm(t)$ always oscillate, taking temporarily negative values, whereas $\gamma_0(t)$ is positive for small values of s and has periods of negativity when $s \gtrsim 3.6$.

B. Nonsecular limit

In the opposite regime, defined by $p \ll 1$, we have to consider the full master equation including the nonsecular terms to obtain a proper description of the short time-scale dynamics. However, in this limit the decay rates take a simpler form. More precisely, $\gamma_\pm(t) \approx \gamma_0(t) \equiv \gamma(t)$ and $\lambda_\pm(t) \approx \lambda_0(t) \equiv \lambda(t)$, and again $\gamma(t)$ is positive for small values of s and takes temporarily negative values when $s \gtrsim 3.6$, while $\lambda(t) \leqslant 0$ for all times t. With this approximation the master equation can be cast into a remarkably simple form with a single Lindblad-type decay channel:

$$\frac{d\rho(t)}{dt} = -i[H + H',\rho(t)] + \mathcal{L}[A,\gamma(t)], \tag{9}$$

where the jump operator is

$$A = C_-\sigma_+ + C_+\sigma_- + C_0\sigma_z \tag{10}$$

and the additional term in the coherent evolution is

$$H' = \lambda(t)C_0(C_+ - C_-)(\sigma_- + \sigma_+). \tag{11}$$

III. SEMIGROUP AND DIVISIBILITY

Historically non-Markovianity has been closely associated with deviations from the Lindblad master equation. The completely positive trace preserving (CPTP) map Φ: $\rho(0) \mapsto \rho(t) = \Phi(t)\rho(0)$ corresponding to a Lindblad master equation always satisfies the semigroup property $\Phi(t+s) = \Phi(t)\Phi(s)$. For example Wolf, Eisert, Cubitt, and Cirac propose to call a map Markovian if it is a CPTP map satisfying the semigroup property [9]. The master equation (1) for the driven qubit in any structured reservoir is not in the Lindblad form due to the time dependency of the decay rates and therefore the dynamical map is not an element of a single-parameter semigroup and Markovian in this sense.

The measure for non-Markovianity proposed by Rivas, Huelga, and Plenio is based on a property of maps more general than the semigroup property, namely, divisibility [10]. The CPTP map $\Phi(t+\tau,0)$ is divisible if it can be written as a decomposition of two CPTP maps, $\Phi(t+\tau,t)$ and $\Phi(t,0)$:

$$\Phi(t+\tau,0) = \Phi(t+\tau,t)\Phi(t,0). \tag{12}$$

When the dynamical map is homogenous in time, $\Phi(t+\tau,t) = \Phi(\tau,0) \equiv \Phi(\tau)$, Eq. (12) reduces to the semigroup property.

RHP define a map to be Markovian exactly when it is divisible. This amounts to requiring that $\Phi(t+\tau,t)$ be completely positive. It is shown in Ref. [10] that the quantity

$$g(t) = \lim_{\epsilon\to 0} \frac{\|\Phi(t+\epsilon,t)\otimes \mathbf{I}\,|\phi\rangle\langle\phi|\| - 1}{\epsilon} \geqslant 0 \tag{13}$$

is strictly positive if and only if Φ is indivisible and hence $g(t)$ identifies non-Markovian processes, according to their definition. Here $|\phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d}|i\rangle|i\rangle$, where d is the dimension of the quantum system; in our case $|\phi\rangle = (|11\rangle + |00\rangle)/\sqrt{2}$ is the maximally entangled Bell state. The quantity $g(t)$ gives rise to a measure for non-Markovianity defined as

$$\mathcal{N}_{\mathrm{RHP}}(\Phi) = \frac{\mathcal{I}}{\mathcal{I}+1}, \quad \mathcal{I} = \int ds g(s), \tag{14}$$

with the property that $\mathcal{N}_{\mathrm{RHP}} \in [0,1]$. The lower and upper limits correspond to Markovian and maximally non-Markovian maps, respectively.

For the local-in-time master equation $d\rho(t)/dt = \mathcal{L}(t)\rho(t)$ the dynamical map, in the limit $\epsilon \to 0$, is formally given by $\Phi(t+\epsilon,t) = \exp[\mathcal{L}(t)\epsilon]$. Consequently, RHP propose that, in this case, one may check the non-Markovianity of the dynamical process by means of a simplified quantity:

$$g(t) = \lim_{\epsilon\to 0} \frac{\|\{\mathbf{I} + \epsilon[\mathcal{L}(t)\otimes \mathbf{I}]\}|\phi\rangle\langle\phi|\| - 1}{\epsilon}. \tag{15}$$

A. Secular regime

A direct calculation of Eq. (15) in the case of a driven qubit in the strong secular regime gives

$$g(t) = \frac{C_+^2 P[\gamma_+(t)] + C_-^2 P[\gamma_-(t)] + 2C_0^2 P[\gamma_0(t)]}{2}, \tag{16}$$

where we define the auxiliary function P: $P(x) = 0$ when $x \geqslant 0$ and $P(x) = -x$ when $x < 0$. In general, whenever any of the decay rates is negative, divisibility is lost. In the Lorentzian case this is, indeed, the situation in the secular regime where $\gamma_\pm(t) < 0$ always for some periods of time. Thus we prove that according to RHP the dynamics of the driven qubit in the secular regime and in the Lorentzian case is always non-Markovian.

B. Nonsecular regime

In the nonsecular regime we find

$$g(t) = \frac{(C_+^2 + C_-^2 + 2C_0)P[\gamma(t)]}{2}, \tag{17}$$

that is, the dynamics is indivisible and hence non-Markovian according to RHP whenever the decay rate $\gamma(t)$ takes temporarily negative values. Again the result is valid for any reservoir, in the weak coupling limit. Specifically, for a laser-driven qubit in a Lorentzian reservoir divisibility is broken whenever $s = (\omega_0 - \omega_L)\lambda \gtrsim 3.6$; that is, the laser and the atom are sufficiently detuned from the center of the Lorentzian, as discussed in Sec. II.

IV. INFORMATION BACKFLOW

As a second measure of non-Markovianity we consider the one proposed by Breuer, Laine, and Piilo in Ref. [15]. The BLP measure aims at identifying non-Markovian dynamics with certain dynamical physical features of the system-reservoir interaction, rather than with the mathematical properties of the dynamical map of the open system. In particular, BLP define non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system. This regain of information manifests itself as an increase in the distinguishability of pairs of evolving quantum states. Hence, the dynamical map Φ is non-Markovian according to BLP if there exists a pair of initial states, $\rho_1(0)$ and $\rho_2(0)$, such that for some time t the distinguishability of $\rho_1(t)$ and $\rho_2(t)$ increases; that is,

$$\sigma(\rho_1,\rho_2,t) = \frac{d}{dt}D[\rho_1(t),\rho_2(t)] > 0, \tag{18}$$

where $D(\rho_1,\rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\|$ is the distinguishability of ρ_1 and ρ_2, with $\|A\| = \sqrt{AA^\dagger}$ being the trace distance, and $\rho_{1/2}(t) = \Phi(t,0)\rho_{1/2}(0)$. When the distinguishability of two states increases information flows from the environment back to the system and vice versa. With this definition one can quantify the amount of non-Markovianity in the quantum process as follows

$$\mathcal{N}_{\mathrm{BLP}}(\Phi) = \max_{\rho_{1,2}(0)} \int_{\sigma>0} dt\,\sigma(\rho_1,\rho_2,t); \tag{19}$$

that is, $\mathcal{N}_{\mathrm{BLP}}$ gives the maximum amount of information that can flow back to the system for the given process.

All divisible maps continuously reduce the distinguishability of quantum states; that is, for a divisible Φ we get $\mathcal{N}_{\mathrm{BLP}} = 0$. Hence the connection between the BLP and RHP measures is the following: if a map is Markovian according to RHP, it is Markovian also according to BLT. The converse is, in general, not true.

A. Secular regime

A straightforward application of Eq. (18) to the solution of the master equation (1) in the secular regime gives

$$\sigma(t) = -\frac{e^{-2\Lambda(t)}\Lambda'(t)(\delta x^2+\delta y^2)+e^{-2\Gamma(t)}\Gamma'(t)\delta z^2}{2\sqrt{e^{-2\Lambda(t)}(\delta x^2+\delta y^2)+e^{-2\Gamma(t)}\delta z^2}}, \quad (20)$$

where $\delta x = x_1(0) - x_2(0)$, $\delta y = y_1(0) - y_2(0)$, and $\delta z = z_1(0) - z_2(0)$ are the differences in the x, y, and z components of the Bloch vector representations of $\rho_1(0)$ and $\rho_2(0)$, respectively, and

$$\Gamma(t) = \frac{1}{2}\int_0^t ds\left[C_+^2\gamma_+(s) + C_-^2\gamma_-(s) + 4C_0^2\gamma_0(s)\right],$$
$$\Lambda(t) = \int_0^t ds[C_+^2\gamma_+(s) + C_-^2\gamma_-(s)]. \quad (21)$$

The expression in Eq. (20) is valid for a generic structured reservoir and the non-Markovian properties of the driven qubit strongly depend on the choice of the environment. We now focus on the Lorentzian case. Let us choose a pair of initial states such that $\delta z = 0$. Then the sign of σ only depends on $\Lambda'(t) = C_+^2\gamma_+(t) + C_-^2\gamma_-(t)$. Recall from Sec. II that in the secular regime $\gamma_\pm(t)$ have negative periods for all possible parameter values. Then $\sigma(t) > 0$ always for some periods of time. Therefore, in the secular regime the process describing the dynamics of a driven qubit in a Lorentzian reservoir is always non-Markovian according to BLP.

It is worth stressing that although the BLP and RHP measures both agree that the dynamics of the driven qubit in a Lorentzian reservoir is non-Markovian, the result need not hold when the driven qubit is embedded in some other structured reservoir. Indeed, negativity of one of the decay rates does not guarantee that $\sigma(t) > 0$, although it immediately breaks the divisibility. It is therefore natural to ask if we can find some general parameter regions when the dynamics of the driven qubit is Markovian according to BLP, but non-Markovian according to RHP. It turns out that the question is nontrivial from many points of view. For one, showing that the reduced dynamics of an open system is non-Markovian according to BLP is simple in comparison to showing that the dynamics is Markovian, since the former case amounts to finding one pair of initial states whose distinguishability increases for some time interval, while the latter case requires that distinguishability always decreases for all possible pairs of states. Such an optimization procedure is possible in practice only numerically, making it hard to identify the parameter regimes for which the system is non-Markovian according to BLP. Note that the BLP non-Markovianity quantifier depends strongly on the spectral density of the environment via linear combinations of the three decay rates and their integrals; so identifying the parameters for which the two measures deviate requires the specification of the environmental spectral density and involves extensive numerical study with several free parameters. So far we have been unable to find instances where the two measures disagree for the model considered here.

B. Nonsecular regime

In the nonsecular regime the master equation (1) is not, in general, solvable analytically and we can evaluate the quantity σ only numerically. We can, however, consider the special case of resonance between the atom and the laser. In this case the master equation is solvable and we find

$$\sigma(t) = -\frac{\gamma(t)e^{-2\int_0^t ds\gamma(s)}[(\delta x^2-\delta z^2)+2\delta y^2]}{\sqrt{2}\sqrt{(\delta z^2+\delta x^2)+e^{-2\int_0^t ds\gamma(s)}[(\delta x^2-\delta z^2)+2\delta y^2]}}. \quad (22)$$

Thus in the resonant nonsecular case the process is non-Markovian according to BLP if and only if $\gamma(t) < 0$ for some time t. Numerical studies of the off-resonant case $\Delta \neq 0$ indicate that this result holds more generally, that is, in the nonsecular regime $\mathcal{N}_{\mathrm{BLP}}(t) = 0$ if and only if $\gamma(t) \leqslant 0$.

V. LASER-INDUCED NON-MARKOVIANITY

In this section we compare the non-Markovian character of the driven dissipative qubit and the analytically solvable model of an unperturbed dissipative qubit when both are embedded in a Lorentzian reservoir [1,25]. The master equation for the qubit in a Lorentzian reservoir, in the special case when the qubit is in resonance with the Lorentzian, $\omega_A = \omega_0$, is

$$\frac{d\rho(t)}{dt} = \gamma(t)\left\{\sigma_-\rho(t)\sigma_+ - \frac{1}{2}[\sigma_+\sigma_-,\rho(t)]\right\}, \quad (23)$$

where, in contrast to the master equation of Eq. (1), the operators are in the bare state basis $\{|e\rangle,|g\rangle\}$. The time-dependent decay rate is

$$\gamma(t) = \frac{2\alpha\lambda\sinh(dt/2)}{d\cosh(dt/2)+\lambda\sinh(dt/2)}, \quad (24)$$

where $d = \sqrt{\lambda^2 - 2\alpha\lambda}$. Applying this model to Eqs. (18) and (15) and introducing the quantity $\Gamma(t) = \int_0^t ds\gamma(s)$, we get

$$\sigma(t) = -\frac{\gamma(t)[2e^{-2\Gamma(t)}\delta z^2 + e^{-\Gamma(t)}(\delta x^2+\delta y^2)]}{4\sqrt{e^{-2\Gamma(t)}\delta z^2 + e^{-\Gamma(t)}(\delta x^2+\delta y^2)}}, \quad (25)$$

$$g(t) = \tfrac{1}{2}P[\gamma(t)]. \quad (26)$$

Equations (25) and (26) show that, in absence of the driving laser, the dynamics is non-Markovian for both the RHP and the BLP measures if and only if $\gamma(t) < 0$ for some time $t \in \mathbf{R}_+$. This happens exactly when $\lambda < 2\alpha$, that is, when the qubit couples strongly to the environment.

Consider now a situation where the qubit couples weakly to the Lorentzian reservoir and the transition frequency of the qubit is resonant with the center of the Lorentzian. The analysis above shows that, without the driving laser, the dynamics of the qubit is Markovian according to both RHP and BLP. If, instead, we drive the qubit with a laser, the results obtained in Secs. IV and III show that we can induce non-Markovianity in the system dynamics when the laser parameters are suitably chosen. More precisely, the laser should couple to the qubit in such a way that the condition $p \gg 1$ holds; that is, we are in the secular regime. The secular regime can be achieved with a large enough Rabi frequency Ω and/or when the laser—and consequently the atom—are sufficiently detuned from the center of the Lorentzian. When this happens the laser interferes with the dynamics of the dissipative qubit so much that it can temporarily reverse the flow of information from the qubit to the environment.

VI. DISCUSSION AND CONCLUSIONS

We have studied the non-Markovian character of a driven qubit in a structured reservoir in different dynamical regimes, using two different non-Markovianity measures, and we have analyzed the effect of the driving laser on the non-Markovian properties of the dissipative qubit.

In the secular regime both measures confirm that the dynamics of the driven qubit is always non-Markovian when it is embedded in a Lorentzian reservoir. However, in general, this need not be the case and deviations between the two measures could be discovered when the driven qubit is embedded in another reservoir. So far we have been unable to find a spectral density for which the two measures disagree. This might be an indication that there are no realistic physical situations where the deviation appears, even if the two mathematical definitions do not rule it out. It remains an interesting open question whether the two non-Markovianity measures always agree in the case of a driven qubit embedded in a generic structured reservoir; indeed, in general, deviations between the two measures are neither characterized nor well understood to date.

In the nonsecular regime the non-Markovianity measures agree perfectly. In this regime the appearance of non-Markovian features is strongly dependent on the way the laser is coupled to the qubit. Neither the presence of the structured reservoir nor the presence of the driving laser is sufficient to guarantee non-Markovianity in the system dynamics in the nonsecular regime. In Sec. V we investigated the origins of these non-Markovian effects by comparing driven and nondriven qubits in a Lorentzian reservoir; it was shown that non-Markovian effects are not possible without a strong driving laser and/or detuning between the driving laser and the center frequency of the Lorentzian distribution. This discovery has a clear physical interpretation; only a strong enough driving laser can induce a feedback of information from the environment into the system. In fact, non-Markovianity in the nonsecular regime occurs in the far off-resonant case. Far off-resonance dynamics is usually associated with the presence of virtual processes which may be at the origin, in this case, of the backflow of information.

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