Effective method of calculating the non-Markovianity $\mathcal{N}$ for single-channel open systems

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We propose an effective method which can simplify the optimization of the increase of the trace distance over all pairs of initial states in calculating the non-Markovianity $\mathcal{N}$ for single-channel open systems. For the amplitude-damping channel, we can unify the results of Breuer *et al.* [Phys. Rev. Lett. **103**, 210401 (2009)] in the large-detuning case and the results of Xu *et al.* [Phys. Rev. A **81**, 044105 (2010)] in the resonant case; furthermore, for the general off-resonant cases, we can obtain a very tight lower bound of $\mathcal{N}$. As another application of our method, we also discuss $\mathcal{N}$ for the non-Markovian depolarizing channel.

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I. INTRODUCTION

Inevitable interaction with the external environment may lead to the phenomenon of decoherence for open quantum systems. In general, the assumption of Markovian approximation usually was applied to dynamical evolution of system. However, recently people found that the Markovian processes without memory and non-Markovian processes with memory can lead to distinctly different effects on decoherence and disentanglement of open systems. Thus, the non-Markovian dynamics have become increasingly important and are extensive studied in both discrete-variable [1–9] and continuous-variable [10–15] systems. It has been found that the non-Markovian effect of environment can extend significantly the entanglement time of the qubits [1] and has been experimentally observed [7]. Because of the importance of the non-Markovian effect of environment, some authors [16–19] have developed some measures to detect the non-Markovianity of open systems from different points of view. Breuer *et al.* [17] proposed a computable measure $\mathcal{N}$ to detect the non-Markovianity of open systems. The idea is based on the distinguishability of quantum states, which results from information flow between the open system and its environment. Rivas *et al.* [18] also proposed a measure of non-Markovianity, which is based on the fact that negative rates are linked to whether entanglement between the system and an ancilla can increase. Lu *et al.* [19] defined a measure of non-Markovianity using quantum Fisher information flow. We know that evaluation of $\mathcal{N}$ requires optimization of the total increase of the trace distance over all pairs of initial states, which is very difficult to accomplish. In Ref. [17], Breuer *et al.* considered a two-level system interacting with a reservoir which possesses the Lorentzian spectral property, and in the large detuning case they found by numerical simulation the pair of initial states $\rho_1(0) = |e\rangle\langle e|$ and $\rho_2(0) = |g\rangle\langle g|$ which optimize the total increase of the trace distance. (It should be noted that in Ref. [17] they use $|+\rangle$ and $|-\rangle$ instead of $|e\rangle$ and $|g\rangle$ to represent the excited and ground states, respectively.) Later, for the same two initial states, Li *et al.* [20] obtained the analytical expression of the trace distance $D(\rho_1(t),\rho_2(t)) = |h(t)|^2$. Very recently, for the same model in the resonant case, Xu *et al.* [21] found two different initial states which optimize the total increase of the trace distance by using an analytical method and the corresponding trace distance $D(\rho_1(t),\rho_2(t)) = |h(t)|$. It should be noted that in Ref. [21] they use $b(t)$ instead of $h(t)$ to represent the amplitude damping of the excited state $|e\rangle$. It is intriguing that these two results are quite different; that is, in these two cases the two initial states which make the optimization are different, and the corresponding trace distance is also different. Because the optimization is very difficult to accomplish, until now we have not seen any reports about the non-Markovianity for the general off-resonant case. In Refs. [17,21], the authors separately dealt with the optimization by different methods, one numerical and the other analytical. For the same model, only for different parameter regimes, the results are quite different; therefore, a unified understanding of these results is in demand.

In this paper, we propose an effective method which can easily optimize the increase of the trace distance over all pairs of initial states in calculating the non-Markovianity $\mathcal{N}$ for single-channel open systems. For the amplitude-damping channel, we analytically derive the results of Ref. [17] in the large-detuning case and the results of Ref. [21] in the resonant case; furthermore, in the general off-resonant cases, we can obtain a very tight lower bound for $\mathcal{N}$. Thus, a unified understanding of the results of Refs. [17] and [21] is given. As another application of our method, we also discuss $\mathcal{N}$ for the non-Markovian depolarizing channel.

The paper is organized as follows. In Sec. II, we introduce our method. The non-Markovian amplitude-damping channel and the non-Markovian depolarizing channel are examined by our method in Sec. III. Finally, we give the conclusion of our results in Sec. IV.

II. METHOD OF OPTIMIZING THE TOTAL INCREASE OF TRACE DISTANCE

Recently, Breuer *et al.* [17] proposed a measure to detect the non-Markovian behavior of quantum processes in open systems based on the distinguishability of quantum states. The trace distance D describing the distinguishability between the two states is defined as [22]

$$D(\rho_1,\rho_2) = \tfrac{1}{2}\mathrm{tr}|\rho_1 - \rho_2|, \tag{1}$$

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where $|M| = \sqrt{M^\dagger M}$ and $0 \leqslant D \leqslant 1$. If $D = 0$, the two states are the same, and if $D = 1$, the two states are totally distinguishable.

Considering a quantum process $\Phi(t), \rho(t) = \Phi(t)\rho(0)$, where $\rho(0)$ and $\rho(t)$ denote the density operators at time $t = 0$ and at any time $t > 0$, respectively, then the non-Markovianity $\mathcal{N}$ is defined as

$$\mathcal{N}(\Phi) = \max_{\rho_{1,2}(0)} \int_{\sigma>0} dt\sigma(t, \rho_{1,2}(0)), \tag{2}$$

where $\sigma(t, \rho_{1,2}(0))$ is the rate of change of the trace distance defined as

$$\sigma(t, \rho_{1,2}(0)) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)). \tag{3}$$

As we known that $\sigma(t, \rho_{1,2}(0)) \leqslant 0$ corresponds to all dynamical semigroups and all time-dependent Markovian processes, a process is non-Markovian if there exists a pair of initial states and at certain time t such that $\sigma(t, \rho_{1,2}(0)) > 0$. Physically, this means that for non-Markovian dynamics the distinguishability of the pair of states increases at certain times.

In view of Eq. (3), the non-Markovianity $\mathcal{N}$ also can be written as the following form:

$$\mathcal{N}(\Phi) = \max_{\rho_{1,2}(0)} \sum_n \left[D\left(\rho_1\left(\tau_n^{\max}\right), \rho_2\left(\tau_n^{\max}\right)\right) - D\left(\rho_1\left(\tau_n^{\min}\right), \rho_2\left(\tau_n^{\min}\right)\right)\right], \tag{4}$$

where $\tau_n^{\max}$ and $\tau_n^{\min}$ correspond to the time points of the local maximum and minimum of $D(\rho_1(t), \rho_2(t))$, respectively. $\mathcal{N}(\Phi)$ can be calculated as follows: One first derives the increment of the trace distance over each time interval $[\tau_n^{\min}, \tau_n^{\max}]$ for any pairs of initial states, then sums up the total contributions of all intervals, and finally performs the maximization for all pairs of initial states.

Generally, it is very difficult to make the maximization in Eq. (4). In this paper, we want to find easy ways to maximize the increase of trace distance. Our idea is this: First, we find the two specific initial states which make the maximization of the quantity $\mathcal{N}_n(\Phi)$ at each time interval. The $\mathcal{N}_n(\Phi)$ is defined as the difference between the local maximum and local minimum of the trace distance for arbitrary time interval $[\tau_n^{\min}, \tau_n^{\max}]$. That is,

$$\mathcal{N}_n(\Phi) = \max_{\rho_{1,2}(0)} \left[D\left(\rho_1\left(\tau_n^{\max}\right), \rho_2\left(\tau_n^{\max}\right)\right) - D\left(\rho_1\left(\tau_n^{\min}\right), \rho_2\left(\tau_n^{\min}\right)\right)\right]. \tag{5}$$

Apparently, it is much easier to find the two initial states which make the maximization in Eq. (5) than to find the two initial states which make the summation in Eq. (4) maximal. Then, for a specific non-Markovian channel we try to prove that the two initial states we found can also make the summation in Eq. (4) maximal. Generally, it is not easy to prove this. If it cannot be proved, we still believe that the pair of initial states which make the increase of the trace distance in single time interval maximal will also optimize the summation in $\mathcal{N}$. Of course, this is not rigorous. If we are strict enough, at least in this case, we can find a lower bound of $\mathcal{N}(\Phi)$, and we argue that this lower bound is tight. We show in the following that this method is very effective.

III. APPLICATIONS

Based on this idea, we can calculate the non-Markovianity $\mathcal{N}$ for the non-Markovian amplitude-damping channel and the non-Markovian depolarizing channel.

A. Non-Markovian amplitude-damping channel

We consider a two-level system (qubit) interacting with a zero-temperature reservoir. The Hamiltonian of the total system under the rotating-wave approximation is given by ($\hbar = 1$)

$$\hat{H} = \omega_0 \hat{\sigma}_+ \hat{\sigma}_- + \sum_{k=1}^{N} \omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_{k=1}^{N} (g_k \hat{\sigma}_- \hat{a}_k^\dagger + g_k^* \hat{\sigma}_+ \hat{a}_k), \tag{6}$$

where $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ are the Pauli raising and lowering operators for the two-level system, respectively; ω_0 is the Bohr frequency of the two-level system; $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and creation operators for reservoir mode k; ω_k is the frequency of the mode k of the reservoir; and g_k is the coupling constant. The Hamiltonian of Eq. (6) can describe various systems. For a concrete discussion, we take a two-level atom interacting with the reservoir formed by the quantized modes of a high-$\mathcal{Q}$ cavity. The dynamics of the reduced density matrix for the two-level atom can be written as [23]

$$\rho^S(t) = \begin{pmatrix} \rho_{ee}^S(0)|h(t)|^2 & \rho_{eg}^S(0)h(t) \\ \rho_{eg}^{S*}(0)h^*(t) & 1 - \rho_{ee}^S(0)|h(t)|^2 \end{pmatrix} \tag{7}$$

in the basis $\{|e\rangle, |g\rangle\}$, where the superscript S represents the two-level atom. Corresponding $h(t)$ denotes the amplitude of the upper level $|e\rangle$ of the atom initially prepared in $|e\rangle$ and satisfies the following integrodifferential equation:

$$\frac{d}{dt} h(t) = -\int_0^t dt_1 f(t - t_1) h(t_1), \tag{8}$$

where the kernel $f(t - t_1) = \int d\omega J(\omega) \exp[i(\omega_0 - \omega)(t - t_1)]$ is related to the spectral density $J(\omega)$ of the reservoir. The model describes the damping of a two-level atom in a cavity. In this paper, we restrict ourselves to the case in which the atom-cavity system has only one excitation and suppose that $J(\omega)$ takes the Lorentzian spectral density [23] with detuning, namely

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \delta - \omega)^2 + \lambda^2}. \tag{9}$$

Here $\delta = \omega_0 - \omega_c$ is the detuning of the center frequency of the cavity ω_c and the Bohr frequency of the two-level atom ω_0, the parameter λ defines the spectral width of the coupling, which is associated with the reservoir correlation time by the relation $\tau_B = \lambda^{-1}$, and the parameter γ_0 is related to the relaxation time scale τ_R by the relation $\tau_R = \gamma_0^{-1}$. Therefore, the analytic expression of $h(t)$ can be obtained as

$$h(t) = e^{-(\lambda - i\delta)t/2} [\cosh(dt/2) + (\lambda - i\delta) \sinh(dt/2)/d] \tag{10}$$

with $d = \sqrt{(\lambda - i\delta)^2 - 2\gamma_0 \lambda}$.

Based on the Hermiticity and unit trace of a physical density matrix, any pair of initial states can be defined as [21]

$$\rho_1^S(0)=\begin{pmatrix}\alpha & \beta\\ \beta^* & 1-\alpha\end{pmatrix},\qquad \rho_2^S(0)=\begin{pmatrix}\mu & \nu\\ \nu^* & 1-\mu\end{pmatrix}, \tag{11}$$

with $|\beta|^2\leqslant\alpha(1-\alpha),|\nu|^2\leqslant\mu(1-\mu)$ corresponding to the semipositivity of a density matrix, $\beta,\nu\in\mathbb{C},0\leqslant\alpha,\mu\leqslant 1$, and $\alpha,\mu\in\mathbb{R}$. Thus, the evolution of the corresponding density matrix can be obtained:

$$\rho_1^S(t)=\begin{pmatrix}\alpha\,|h(t)|^2 & \beta h(t)\\ \beta^* h^*(t) & 1-\alpha\,|h(t)|^2\end{pmatrix},\qquad \rho_2^S(t)=\begin{pmatrix}\mu\,|h(t)|^2 & \nu h(t)\\ \nu^* h^*(t) & 1-\mu\,|h(t)|^2\end{pmatrix}. \tag{12}$$

The combination of Eqs. (1) and (12) immediately provides the expression of the trace distance at any time $t\geqslant 0$,

$$D\big(\rho_1^S(t),\rho_2^S(t)\big)=\sqrt{|h(t)|^4(\alpha-\mu)^2+|h(t)|^2|\beta-\nu|^2}, \tag{13}$$

which has been obtained in Ref. [21]. It is noted that the maximization of trace distance in the resonant case has been given in Ref. [21]; however, that method cannot be extended to the general off-detuning case because the condition $|h(\tau_n^{\min})|=0$ cannot always be guaranteed at each local minima. Using Eq. (5), we can easily achieve the maximization of $\mathcal{N}_n$ analytically, namely the optimization of the trace distance difference between the local maximum and local minimum in the time interval $[\tau_n^{\min},\tau_n^{\max}]$, by choosing two specific initial states. According to Eqs. (5) and (13), $\mathcal{N}_n$ can be written as

$$\mathcal{N}_n=\max_{\rho_{1,2}(0)}\Big[\big|h\big(\tau_n^{\max}\big)\big|\sqrt{\big|h\big(\tau_n^{\max}\big)\big|^2(\alpha-\mu)^2+|\beta-\nu|^2}-\big|h\big(\tau_n^{\min}\big)\big|\sqrt{\big|h\big(\tau_n^{\min}\big)\big|^2(\alpha-\mu)^2+|\beta-\nu|^2}\Big]. \tag{14}$$

When $t=0$, $|h(0)|=1$, and from Eq. (13), $D=\sqrt{(\alpha-\mu)^2+|\beta-\nu|^2}\leqslant 1$. Furthermore, the condition is equivalent to these parameterized conditions $(\alpha-\mu)=r\cos\theta,\beta-\nu=re^{i\phi}\sin\theta$ ($r\leqslant 1,\theta\in[0,2\pi]$ and $\phi\in[0,\pi]$). Substituting these parameterized conditions into Eq. (14) and considering that the maximization is over all pairs of initial states, we can determine that the maximization condition requires $\sqrt{(\alpha-\mu)^2+|\beta-\nu|^2}=1$ corresponding to $r=1$. Then the problem becomes the maximization of the following $\mathcal{N}_n^o(\theta)$:

$$\mathcal{N}_n^o(\theta)=\big|h\big(\tau_n^{\max}\big)\big|\sqrt{\big|h\big(\tau_n^{\max}\big)\big|^2\cos^2\theta+\sin^2\theta}-\big|h\big(\tau_n^{\min}\big)\big|\sqrt{\big|h\big(\tau_n^{\min}\big)\big|^2\cos^2\theta+\sin^2\theta}. \tag{15}$$

From the equation

$$\frac{\partial N_n^o(\theta)}{\partial\theta}=0, \tag{16}$$

we can obtain the extrema, which are $\mathcal{N}_{n1}^o=A^2-B^2$ when $\theta=0$; $\mathcal{N}_{n2}^o=A-B$ when $\theta=\pi/2$ or $\theta=3\pi/2$; and $\mathcal{N}_{n3}^o=A^2\sqrt{\frac{1-A^2}{(B^2-1)(A^2+B^2-1)}}-B^2\sqrt{\frac{1-B^2}{(A^2-1)(A^2+B^2-1)}}$ when $\theta=\arccos[\sqrt{\frac{-A^2+2A^4-A^6+B^2-2B^4+B^6}{(A^2-1)(B^2-1)(A^4-A^2+B^2-B^4)}}]$, where $A=|h(\tau_n^{\max})|,B=|h(\tau_n^{\min})|$, and $A>B$. So $\mathcal{N}_n$ can be represented as

$$\mathcal{N}_n=\max\big\{\mathcal{N}_{n1}^o,\ \mathcal{N}_{n2}^o,\ \mathcal{N}_{n3}^o\big\}. \tag{17}$$

From numerical calculation, we find that for any A and B satisfying $0<B<A<1$, $\mathcal{N}_{n3}^o$ is always less than $\mathcal{N}_{n1}^o$ and $\mathcal{N}_{n2}^o$, so $\mathcal{N}_n=\max\{\mathcal{N}_{n1}^o,\ \mathcal{N}_{n2}^o\}$. From the definitions of $\mathcal{N}_{n1}^o$ and $\mathcal{N}_{n2}^o$, we can determine that when $0\leqslant A+B<1$, $\mathcal{N}_{n1}^o<\mathcal{N}_{n2}^o$; when $A+B=1$, $\mathcal{N}_{n1}^o=\mathcal{N}_{n2}^o$; and when $A+B>1$, $\mathcal{N}_{n1}^o>\mathcal{N}_{n2}^o$.

(i) In the resonant case, that is, $\delta=0$, it is obvious that $A+B=|h(\tau_n^{\max})|+|h(\tau_n^{\min})|<1$ because $B=|h(\tau_n^{\min})|=0$ at $\tau_n^{\min}=2[n\pi-\arctan(d't/2)]/d'$ with $n=1,2,3,\ldots$, and $d'=\sqrt{|\lambda^2-2\gamma_0\lambda|}$. So, in any time interval $[\tau_n^{\min},\tau_n^{\max}]$, $\mathcal{N}_n=\max\{\mathcal{N}_{n1}^o,\ \mathcal{N}_{n2}^o\}=\mathcal{N}_{n2}^o=A-B=|h(\tau_n^{\max})|$. For $\mathcal{N}_{n2}^o,\theta=\pi/2$ or $3\pi/2$ and the two initial states correspond to $\alpha=\mu$ and $|\beta-\nu|=1$. Because in this case $\mathcal{N}_n$ reaches its maximum for any n, the same two initial states, that is, $\alpha=\mu$ and $|\beta-\nu|=1$, are also the two initial states which make the summation in Eq. (4) maximal. It is worth noting that these conditions $\alpha=\mu,|\beta-\nu|=1$ together with $|\beta|^2\leqslant\alpha(1-\alpha)$ and $|\nu|^2\leqslant\mu(1-\mu)$ are equivalent to the conditions $\alpha=\mu=1/2,|\beta|=|\nu|=1/2$, and $|\beta-\nu|=1$ obtained in Ref. [21], which can be explained as follows. Our conditions can be changed into $\alpha=\mu,|\beta-\nu|=1,(\alpha-1/2)^2+|\beta|^2\leqslant(1/2)^2$ and $(\mu-1/2)^2+|\nu|^2\leqslant(1/2)^2$. From the geometric point of view, the new conditions indicate that the two points $(\alpha,|\beta|)$ and $(\mu,|\nu|)$ are in (or on the circumference of) the same circle

$$(x-1/2)^2+|y|^2=(1/2)^2, \tag{18}$$

with $x\in\mathbb{R}$ and $y\in\mathbb{C}$. It is easy to check that the conditions $\alpha=\mu$ and $|\beta-\nu|=1$ are just $\alpha=\mu=1/2$, $|\beta|=|\nu|=1/2,|\beta-\nu|=1$. In summary, in the resonant case our results are consistent with the results of Ref. [21].

(ii) In the off-resonant case, that is, $\delta\neq 0,A+B=|h(\tau_n^{\max})|+|h(\tau_n^{\min})|$ may be less than 1, equal to 1, or more than 1 depending on the values of γ_0,λ, and δ. We have proved that depending on the value of $A+B$ there are only two pairs of initial states which maximize $\mathcal{N}_n$ for each n. Next, we give our effective and practical method to calculate $\mathcal{N}$ for any fixed parameters γ_0,λ, and δ. From Eqs. (2) and (3), we can use the two pairs of initial states to obtain $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively: $\mathcal{N}_1=\int_{\sigma_1>0}dt\sigma_1(t,\theta=0)$, $\mathcal{N}_2=\int_{\sigma_2>0}dt\sigma_2(t,\theta=\pi/2\text{ or }3\pi/2)$. Correspondingly, the expressions of $\sigma_1(t,\theta=0)$ and $\sigma_2(t,\theta=\pi/2\text{ or }3\pi/2)$ are given by

$$\sigma_1(t)=e^{-\lambda t}\{\mu[\cosh(at)-\cos(bt)]+\nu\sinh(at)-\xi\sin(bt)\}, \tag{19}$$

$$\sigma_2(t)=\frac{e^{-\frac{\lambda t}{2}}\{\mu[\cosh(at)-\cos(bt)]+\nu\sinh(at)-\xi\sin(bt)\}}{2\sqrt{\eta\sinh(at)+\chi\sin(bt)+\kappa\cosh(at)+\varsigma\cos(bt)}}, \tag{20}$$

where a and b denote the real part and imaginary part of d, respectively, $\mu=\frac{1}{2|d|^2}(\lambda a^2-\lambda b^2-\lambda\delta^2-\lambda^3-2ab\delta)$, $\nu=$

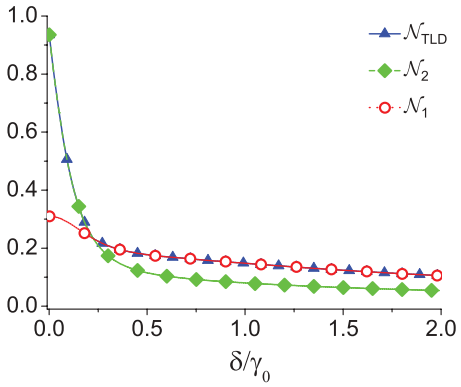

FIG. 1. (Color online) $\mathcal{N}_{\rm TLD}$, $\mathcal{N}_1$, and $\mathcal{N}_2$ as a function of $\delta,\lambda = 0.1\gamma_0$.

$\frac{1}{2|d|^2}(a^3+ab^2-\lambda^2 a+a\delta^2+2b\delta\lambda)$, $\xi=\frac{1}{2|d|^2}(b^3+ba^2+\lambda^2 b-b\delta^2+2a\delta\lambda)$, $\eta=\frac{1}{2|d|^2}(2a\lambda-2b\delta)$, $\chi=\frac{1}{2|d|^2}(2b\lambda+2a\delta)$, $\kappa=\frac{1}{2|d|^2}(\lambda^2+\delta^2+a^2+b^2)$, $\varsigma=\frac{1}{2|d|^2}(a^2+b^2-\lambda^2-\delta^2)$, and $|d|$ denotes the absolute value of d. It is worth noting that Eq. (19) has been obtained in Ref. [20]. We cannot prove but we believe that one of the two pairs of initial states we found can also make the optimization in the summation of Eq. (4); thus, $\mathcal{N}=\max\{\mathcal{N}_1,\mathcal{N}_2\}$. If we are strict enough, at least it is a very tight lower bound (TLD) for the genuine $\mathcal{N}$,

$$\mathcal{N}_{\rm TLD}=\max\{\mathcal{N}_1,\mathcal{N}_2\}. \tag{21}$$

We plot $\mathcal{N}_{\rm TLD}$, $\mathcal{N}_1$, and $\mathcal{N}_2$ as functions of detuning δ for $\lambda=0.1\gamma_0$ in Fig. 1 and $\mathcal{N}_{\rm TLD}$, $\mathcal{N}_1$, and $\mathcal{N}_2$ as functions of λ for $\delta=0.1\gamma_0$ in Fig. 2. From Fig. 1, we can see that there exists a critical point for δ at which the pair of initial states change from $\theta=\pi/2$ or $\theta=3\pi/2$ to $\theta=0$. More specifically, when $\delta<\delta_c$, $\mathcal{N}_{\rm TLD}=\mathcal{N}_2$ corresponds to the two initial states $\theta=\pi/2$ or $\theta=3\pi/2$; when $\delta=\delta_c$, $\mathcal{N}_{\rm TLD}=\mathcal{N}_1=\mathcal{N}_2$ corresponds to the two initial states $\theta=0$ and $\theta=\pi/2$ or $3\pi/2$; and when $\delta>\delta_c$, $\mathcal{N}_{\rm TLD}=\mathcal{N}_1$ corresponds to the two initial states $\theta=0$. Similarly, it can be seen from Fig. 2 that there also exists a critical point for λ at which the pair of initial states changes from $\theta=0$ to $\theta=\pi/2$ or $\theta=3\pi/2$.

It is worth noting that in the large detuning case, by numerical simulation Breuer *et al.* [17] have performed the optimization of the total increase of the trace distance. In the large detuning case from Eq. (10), we know $A+B=|h(\tau_n^{\max})|+|h(\tau_n^{\min})|>1$ for any $n=1,2,3,\ldots$. Therefore, we can obtain $\mathcal{N}_n=\max\{\mathcal{N}_{n1}^o,\mathcal{N}_{n2}^o\}=\mathcal{N}_{n1}^o=A^2-B^2=|h(\tau_n^{\max})|^2-|h(\tau_n^{\min})|^2$ for any n, and the two initial states satisfy $\alpha-\mu=1$ and $|\beta-\nu|=0$ corresponding to $\theta=0$. Because the two initial states ($\theta=0$) optimize $\mathcal{N}_n$ for any n, they also optimize the summation in Eq. (4). Similar to the resonant case, the optimization conditions including $\alpha-\mu=1$, $|\beta-\nu|=0,|\beta|^2\leqslant\alpha(1-\alpha)$, and $|\nu|^2\leqslant\mu(1-\mu)$ can be simplified into $\alpha=1,\mu=0$, and $\beta=\nu=0$. Evidently, the two initial states are $\rho_1(0)=|e\rangle\langle e|$ and $\rho_2(0)=|g\rangle\langle g|$, which has been obtained in Ref. [17]. It is noted that the condition is obtained by numerical simulation in [17], but here we obtain the same condition using an analytic method. Then we also can obtain the trace distance $D=|h(t)|^2$ for these two initial states, which is given by Ref. [20]. In summary, in the large-detuning case, our results are consistent with the results reported in Refs. [17,20].

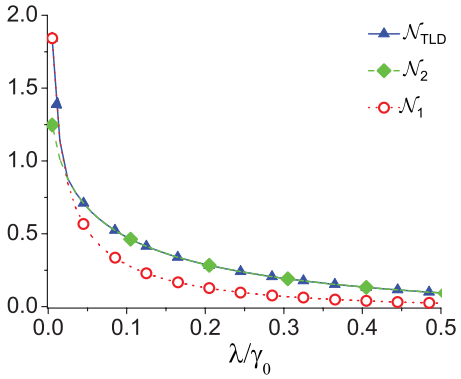

FIG. 2. (Color online) $\mathcal{N}_{\rm TLD}$, $\mathcal{N}_1$, and $\mathcal{N}_2$ as a function of $\lambda,\delta = 0.1\gamma_0$.

B. Non-Markovian depolarizing channel

As another application of our method, we consider the non-Markovianity $\mathcal{N}$ for a non-Markovian depolarizing channel. The dynamical property of this system and in particular the conditions of complete positivity of the map corresponding to a master equation have been studied in detail by Daffer *et al.* [24]. For this model, the time-dependent Hamiltonian that corresponds to a two-level system subjected to random telegraphic noise is

$$H(t)=\hbar\sum_{i=1}^{3}\Gamma_i(t)\sigma_i, \tag{22}$$

where $\Gamma_i(t)=a_i n_i(t)$ are independent random variables and σ_i are the usual Pauli operators. The term $n_i(t)$ has a Poisson distribution with a mean equal to $t/2\tau_i$, while a_i is an independent random variable taking the values $\pm a_i$. For the time-dependent Hamiltonian of Eq. (22), the corresponding equation of motion for the density operator is governed by the von Neumann equation $\dot\rho=-(i/\hbar)[H(t),\rho]=-i\,\Sigma_k\Gamma_k(t)[\sigma_k,\rho]$, which has the following formal solution:

$$\rho(t)=\rho(0)-i\int_0^t\sum_k\Gamma_k(s)[\sigma_k,\rho(s)]\,ds. \tag{23}$$

By substituting the formal solution, Eq. (23), into the von Neumann equation and performing a stochastic average, one can obtain the following memory kernel master equation:

$$\dot\rho(t)=-\int_0^t\sum_k e^{-(t-t')/\tau_k}a_k^2[\sigma_k,[\sigma_k,\rho(t')]]\,dt', \tag{24}$$

where the correlation function of the random telegraph signal $\langle\Gamma_j(t)\Gamma_k(t')\rangle=a_k^2e^{-|t-t'|/\tau_k}\delta_{jk}$ contributes to the memory kernel. It has been pointed out [24] that the system density operator with an exponential memory kernel obeys a homogeneous Volterra equation after averaging over the reservoir variables and also been proven that when two of a_i are zero

and only one direction has the noise, the map $\Phi(\rho)$ can be written in Kraus operator form [25], namely

$$\rho'(t) = \Phi_t(\rho) = \sum_{k=1}^{4} A_k^\dagger \rho A_k. \tag{25}$$

For simplicity, in this paper we consider only the case of the z direction with noise and x and y directions without noise. Therefore, $A_1 = 0, A_2 = 0, A_3 = \sqrt{[1-\Lambda(\nu)]/2}\sigma_3$, and $A_4 = \sqrt{[1+\Lambda(\nu)]/2}I$, where $\Lambda(\nu) = \exp(-\nu)[\cos(\mu\nu) + \sin(\mu\nu)/\mu]$ with $\mu = \sqrt{(4a\tau)^2 - 1}$, and $\nu = t/2\tau$ is a dimensionless time. The term a is the coupling strength of the system with the external environment while τ determines which frequencies the system prefers most. For convenience, we let $\lambda = 1/\tau$, and then $\Lambda(\nu)$ can be rewritten as

$$\Lambda(t) = \begin{cases} e^{-\lambda t/2}\left[\cosh\left(\frac{\varepsilon t}{2}\right) + \frac{\lambda}{\varepsilon}\sinh\left(\frac{\varepsilon t}{2}\right)\right] & (16a^2 < \lambda^2) \\ e^{-\lambda t/2}\left[1 + \frac{\lambda t}{2}\right] & (16a^2 = \lambda^2) \\ e^{-\lambda t/2}\left[\cos\left(\frac{\varepsilon t}{2}\right) + \frac{\lambda}{\varepsilon}\sin\left(\frac{\varepsilon t}{2}\right)\right] & (16a^2 > \lambda^2), \end{cases} \tag{26}$$

where $\varepsilon = \sqrt{|16a^2 - \lambda^2|}$.

By using the same two initial states of Eq. (11), from Eq. (25) we can obtain the evolutions of the two density matrices:

$$\rho_1^S(t) = \begin{pmatrix} \alpha & \beta\Lambda(t) \\ \beta^*\Lambda^*(t) & 1-\alpha \end{pmatrix}, \qquad \rho_2^S(t) = \begin{pmatrix} \mu & \nu\Lambda(t) \\ \nu^*\Lambda^*(t) & 1-\mu \end{pmatrix}. \tag{27}$$

Therefore, the trace distance can be obtained:

$$D\left(\rho_1^S(t), \rho_2^S(t)\right) = \sqrt{(\alpha-\mu)^2 + |\Lambda(t)|^2|\beta-\nu|^2}. \tag{28}$$

From Eqs. (5) and (28), $\mathcal{N}_n$ can be expressed as

$$\mathcal{N}_n = \max_{\rho_{1,2}(0)}\left[\sqrt{(\alpha-\mu)^2 + \left|\Lambda\left(\tau_n^{\max}\right)\right|^2|\beta-\nu|^2} - \sqrt{(\alpha-\mu)^2 + \left|\Lambda\left(\tau_n^{\min}\right)\right|^2|\beta-\nu|^2}\right]. \tag{29}$$

As in Sec. III A, after parameterizing $(\alpha-\mu)$ and $|\beta-\nu|$, the problem becomes the maximization of the following $\mathcal{N}_n^o(\theta)$:

$$\mathcal{N}_n^o(\theta) = \sqrt{\cos^2\theta + \left|\Lambda\left(\tau_n^{\max}\right)\right|^2\sin^2\theta} - \sqrt{\cos^2\theta + \left|\Lambda\left(\tau_n^{\min}\right)\right|^2\sin^2\theta}. \tag{30}$$

Then we can obtain the extrema of $\mathcal{N}_n^o$: $\mathcal{N}_{n1}^o = 0$ when $\theta = 0$ and $\mathcal{N}_{n2}^o = A - B$ when $\theta = \pi/2$ or $3\pi/2$, where $A = |\Lambda(\tau_n^{\max})|, B = |\Lambda(\tau_n^{\min})|$, and $A > B$. Obviously, $\mathcal{N}_n = \max\{\mathcal{N}_{n1}^o, \mathcal{N}_{n2}^o\} = \mathcal{N}_{n2}^o = A - B$. Because the pair of initial states corresponding to $\theta = \pi/2$ or $3\pi/2$ makes the increase of trace distance $\mathcal{N}_n^o$ maximal for any n, it also makes the summation in Eq. (4) maximal. The condition $\theta = \pi/2$ or $3\pi/2$ means $\alpha - \mu = 0$ and $|\beta - \nu| = 1$, and similar to the discussion in Sec. III A, the condition together with $|\beta|^2 \leqslant \alpha(1-\alpha)$ and $|\nu|^2 \leqslant \mu(1-\mu)$ is also equivalent to $\alpha = \mu = 1/2, |\beta| = |\nu| = 1/2, |\beta - \nu| = 1$. The trace distance after choosing the two initial states can be given as

$$D = |\Lambda(t)|. \tag{31}$$

It is noted that there is some similarity between the non-Markovian depolarizing channel and the amplitude-damping channel. In the non-Markovian depolarizing channel, the maximal trace distance is a function of the decoherence factor $|\Lambda(t)|$, while in the amplitude-damping channel it is a function of the amplitude damping factor $|h(t)|$. It is very interesting that the trace distance of the non-Markovian depolarizing channel for the specific pair of initial states which make the optimization is very similar to that of the amplitude-damping channel in the resonant case; that is, for the former, the trace distance for the specific pair of initial states is equal to the decoherence factor $|\Lambda(t)|$, while for the latter it is equal to the amplitude-damping factor $|h(t)|$. Furthermore, in both cases the two initial states which make the optimization are the same.

We can obtain the rate of the change of the trace distance using Eq. (3):

$$\sigma(t, \rho_{1,2}(0)) = \frac{\Lambda(t)\frac{d}{dt}\Lambda(t)}{|\Lambda(t)|}. \tag{32}$$

Having these preparations, we can easily calculate the non-Markovianity $\mathcal{N}$ from Eq. (2). We plot $\mathcal{N}$ as a function of a in Fig. 3. From Fig. 3, we can see that there is a threshold $a_c = \lambda/4$. When $a \leqslant a_c$, $\mathcal{N} = 0$, which means that the process is Markovian. When $a > a_c$, $\mathcal{N}$ increases with a, which means that the process is non-Markovian. This can be easily understood: because a represents the coupling strength of the system and the reservoir, it is obvious that the non-Markovianity $\mathcal{N}$ will become stronger with the increasing of a in the non-Markovian regime. It is worth noting that recently Mazzola *et al.* [26] elucidated that the memory kernel master equation does not ensure the presence of non-Markovian behavior in the evolution of dynamics. For example, in Ref. [26] they have shown that the non-Markovian behavior does not appear in the memory kernel master equations (4) and (10) of the same reference. $\xi_M(R,t)$ and $\xi_P(R,t)$ are two functions corresponding to two models, Eqs. (4) and (10) of Ref. [26], respectively, which play the central role in the dynamics of the system. It has been proved that $\xi_M(R,t)$ and $\xi_P(R,t)$ are positive, monotonically decreasing functions under the condition of positivity of the density matrix [27]. Thus, the quantum processes corresponding to Eqs. (4) and (10) of Ref. [26] are Markovian. However, the model we used can also be described by the memory kernel master equation (24), but we clearly see from Eq. (26) that the

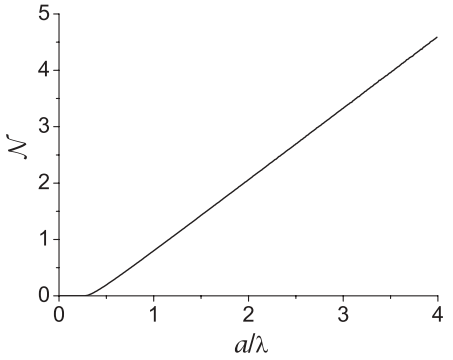

FIG. 3. Non-Markovianity $\mathcal{N}$ as a function of a.

function $\Lambda(t)$, which plays a central role in our model, is not a monotonous function with time but is a damped oscillating function in some parameter regimes. In this case, the quantum process we consider is non-Markovian.

IV. CONCLUSIONS

We know that the definition of the non-Markovianity $\mathcal{N}$ needs an optimization over all pairs of initial states, and generally it is very hard to do this. In this paper, we proposed a method which can simplify this optimization. The main idea is this: First, we find the pair of initial states which make the maximization of the difference between the local maximum and local minimum of the trace distance in arbitrary nth time interval, and then we try to prove that this pair of initial states can also optimize the summation over all pairs of initial states in calculating the non-Markovianity $\mathcal{N}$. Using this method, we have analytically obtained the pair of initial states which make the optimization in Eq. (4) and the corresponding trace distance for the amplitude-damping channel in both the resonant case and the large-detuning case, and we have unified the results of Breuer *et al.* [17] and Xu *et al.* [21]. We also have analytically obtained the pair of initial states which make the optimization in Eq. (4) and the corresponding trace distance for the non-Markovian depolarizing channel. Generally, it cannot be proved that the pair of initial states which maximize the nth difference between the local maximum and local minimum of the trace distance also optimize the summation in $\mathcal{N}$. Here, we argue that the pair of initial states which make the increase of the trace distance maximal in one time interval will also optimize the summation in $\mathcal{N}$. We cannot prove this in this paper, and this needs further investigations. If we are strict enough, at least we can obtain a very tight lower bound for $\mathcal{N}$. For example, for the amplitude-damping channel, we have found only two pairs of initial states depending on the system parameters which make the increase of the trace distance maximal in an arbitrary single time interval. Then it is easy to calculate $\mathcal{N}$ for these two pairs of initial states, and the larger one is the tight lower bound of $\mathcal{N}$. In this paper, we mainly focus on the single-channel case, and the generalization to the multichannel case may be a more challenging task. We have simplified the problem of finding a pair of initial states which optimize the summation of the increase of trace distance in many time intervals into a problem of optimizing the increase of trace distance in just one single time interval.

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