Entanglement created by spontaneously generated coherence

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We propose a scheme that is able to generate on demand a steady-state entanglement between two nondegenerate cavity modes. The scheme relies on the interaction of the cavity modes with driven two- or three-level atoms, which act as a coupler to build entanglement between the modes. We show that, in the limit of a strong driving, it is crucial for the generation of entanglement between the modes to imbalance populations of the dressed states of the driven atomic transition. In the case of a three-level *V*-type atom, we find that a stationary entanglement can be created on demand by tuning the Rabi frequency of the driving field to the difference between the atomic transition frequencies. The resulting degeneracy of the energy levels, together with the spontaneously generated coherence, generate a steady-state entanglement between the cavity modes. It is shown that the condition for the maximal entanglement coincides with the collapse of the atomic system into a pure trapping state. We also show that the creation of entanglement depends strongly on the mutual polarization of the transition atomic dipole moments.

I. INTRODUCTION

The generation of continuous-variable (CV) entangled light has attracted significant interest because of a potential application in quantum information science, specifically in quantum teleportation [1], quantum telecloning [2], and quantum dense coding [3]. Continuous variables offer the possibility to create entanglement deterministically, and different nonlinear processes have been proposed to generate CV two-mode entangled beams [4–8], including nondegenerate parametric down conversion [9,10] and nondegenerate four-wave mixing processes [11–15]. Recently, the four-wave mixing process has been proposed as a potential source of narrow-band entangled beams and as an important resource for quantum memory storage [16] and long-distance communications [17].

Of particular interest for CV entanglement are cavity quantum electrodynamic (QED) systems, where entanglement between cavity modes can be created by coupling the modes to an atomic system or nonlinear crystal located inside the cavity [18–20]. It was shown that, for the generation of entanglement between cavity modes, it is essential to create a coherence in the coupling (or entangling) system. Typical systems for entangling the modes are multilevel atoms or nonlinear crystals, where the coherence can be established initially by a preparation of the atoms in a linear superposition of their energy states or can be created dynamically by a suitable driving of the atoms through four-wave mixing [11–15] or Raman-type processes [21–24].

The coherence is subjected to dissipation as a result of the decoherence process, and over a long time it might be difficult to maintain a coherence large enough for entangling the cavity modes. The main source of decoherence is spontaneous emission resulting from the interaction of the atoms with the environment. On a microscopic scale, the spontaneous emission can be reduced or even completely eliminated, but it

could be difficult to eliminate on a macroscopic scale where one would like to create entanglement using macroscopic atomic ensembles. This raises an important question of how to eliminate the decoherence or how to maintain a large coherence in the presence of the decoherence.

In this paper, we propose a system formed by a three-level atom located inside a two-mode cavity that can generate the maximal stationary entanglement between the cavity modes in the presence of decoherence. The atom is modeled as a *V*-type system where the dipole-allowed transitions can be independent of each other or can be correlated through the spontaneously generated coherence (SGC) [25]. The atom is driven by an external laser field coupled exclusively to only one of the atomic transitions. We use the dressed-atom approach and show that the effective three-level system of dressed states comprises a suitable medium for a nonlinear coupling between the cavity modes. We work in the strong driving limit, which assumes that the Rabi frequency of the laser field is much larger than the transition damping rates and the coupling strengths of the cavity modes to the atomic transitions. This prompts us to apply the secular approximation, which ignores the coupling of the populations of the dressed states to the coherences. It is known that nonsecular terms, although small, can have a destructive effect on coherence effects [21,22] or may even have constructive effects and lead to interesting novel features [26–28]. However, we are interested in features created by the SGC rather than features created by the coherence induced by the driving field, and therefore neglect the nonsecular terms.

We consider four scenarios, wherein the cavity modes couple to the same or different atomic transitions that could be correlated or independent of each other. The first scenario represents a situation in which the atomic transitions are independent of each other, and both cavity modes couple to the same atomic transition that, in addition, is driven by a strong and in general off-resonant laser field. Physically, this system behaves as a driven two-level system, and the driving field occurs as a dressing field for the atoms. We demonstrate that the necessary and sufficient conditions for generation of the

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maximal entanglement between the modes involve creating a complete population inversion between the dressed states of the coupling atomic system. A population difference between the dressed states occurs for an off-resonant driving field. Since, for a strong driving field, there is no coherence between the dressed states, one could conclude that the entanglement occurs without coherence in this case. However, for a detuned driving field, a coherence actually occurs between the two bare states of the system. In other words, in the bare atom picture, the entanglement is created with coherence. We find that the maximal entanglement can not be created in this scenario because it is not possible to create a large population difference between the dressed states and, at the same time, maintain a strong coupling between the cavity modes mediated by the atom.

In the second scenario, we include the coupling between the atomic transitions through the SGC, a close analog on the schemes of quantum-state engineering by dissipation [29–35]. We find that, in this case, the dissipation is used to create the required coherence in the atomic system. The maximal stationary entanglement can be created on demand for the resonant driving field by tuning the Rabi frequency of the field to the difference between the atomic transition frequencies. As a result, the atomic system evolves into a pure trapping state, which is an asymmetric superposition of the degenerate energy states. The particular pure state into which the atomic system evolves depends upon the ratio of the damping rates of the atomic transitions and the detuning of the laser frequency from the atomic transition frequency. The trapping effect results in a complete population inversion between the dressed states of the system. In other words, the maximal steady-state entanglement is generated when the population of the atomic system is trapped in a pure superposition state.

In the third scenario, we assume that the cavity modes are coupled to different atomic transitions. The new feature of this scenario is that the generation of entanglement is now independent of the population of the dressed states. The necessary condition for entanglement is the creation of coherence between the atomic transitions, a coherence that can be created by the SGC.

Finally, in the fourth scenario, we consider the most general configuration in which each of the cavity modes is coupled to both atomic transitions. We show that this scenario can be treated as a combination of the second and third scenarios, and find that the generation of entanglement now depends on the mutual polarization of the atomic dipole moments. Depending on whether the transition dipole moments are parallel or antiparallel, the entanglement can be enhanced (reduced) by the constructive (destructive) interference between the atomic transition amplitudes.

This paper is organized as follows. We begin in Sec. II with a description of the proposed schemes for the generation of entanglement between two nondegenerate cavity modes and derive the master equation for the reduced density operator of the cavity modes. In Sec. III, we study the generation and enhancement of entanglement between the cavity modes for different coupling configurations of the cavity modes to the atomic transitions. We are particularly interested in the role of the mutual polarization of the atomic dipole moments and the conditions for the generation of a large stationary entanglement

between the modes. The physical origin of entanglement between the cavity modes is explained in terms of population trapping in a linear superposition of the atomic levels. Finally, we summarize our results in Sec. IV.

II. GENERAL FORMALISM

We consider a three-level atom located inside a two-mode cavity. The atom is modeled as a V-type system with a ground state $|3\rangle$ and two excited states $|1\rangle$ and $|2\rangle$ separated in frequency by $\Delta_0 = \omega_{13} - \omega_{23}$, where ω_{13} and ω_{23} are atomic transition frequencies between states $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$, respectively. We shall assume that $\omega_{13} > \omega_{23}$ so that Δ_0 is positive. This choice, of course, involves no loss of generality. The atom acts as a coupling (or entangling) medium that couples two nondegenerate cavity modes of frequencies ω_1 and ω_2 through the interaction of the modes with the atomic dipole transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$. In addition, the transition $|2\rangle \leftrightarrow |3\rangle$ is driven by a strong laser field of angular frequency ω_L and the amplitude determined by the Rabi frequency 2Ω, as illustrated in Fig. 1. The dipole moments of the two allowed atomic transitions can be orthogonal or nonorthogonal to each other. The latter case can lead to quantum interference effects induced by the SGC. The cavity modes can simultaneously couple to one of the atomic transitions or to different transitions. One can also arrange a situation in which each of the cavity modes could couple to both of the atomic transitions. In this case, the coupling and the resulting entanglement between the modes can depend on whether the transition dipole moments are parallel or antiparallel to each other.

For an open cavity in which the atom and the cavity modes are coupled to the outside vacuum modes, the dynamics of the driven atom plus the cavity modes is conveniently described by the density operator ρ, which, in a frame rotating with the laser frequency ω_L, satisfies the following master equation ($\hbar = 1$):

$$\frac{d}{dt}\rho = -i(H_c + H_a + V, \rho) + L_c\rho + L_a\rho, \qquad (1)$$

where

$$H_c = -\delta_1 a_1^\dagger a_1 + \delta_2 a_2^\dagger a_2 \qquad (2)$$

is the free Hamiltonian of the cavity modes,

$$H_a = (\Delta_L + \Delta_0) A_{11} + \Delta_L A_{22} - \Omega (A_{23} + A_{32}) \qquad (3)$$
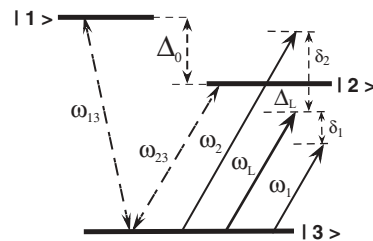

FIG. 1. Schematic diagram of the atomic levels and one of the possible coupling configurations of the laser and the cavity fields. A laser field of frequency ω_L drives the $|3\rangle \rightarrow |2\rangle$ transition with detuning Δ_L, and two nondegenerate cavity modes of frequencies ω_1 and ω_2 couple to the driven transition with detunings δ_1 and δ_2 from the laser frequency.

is the Hamiltonian of the driven atom,

$$V = (g_1 a_1 + g_2 a_2) A_{23} + (g_3 a_1 + g_4 a_2) A_{13} + \text{H.c.} \quad (4)$$

is the interaction Hamiltonian of the cavity modes with the atomic transitions, and

$$L_c \rho = \sum_{j=1}^{2} \kappa_j (2 a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j) \quad (5)$$

and

$$L_a \rho = \gamma_1 [A_{31}, \rho A_{13}] + \gamma_2 [A_{32}, \rho A_{23}]$$
$$+ \eta([A_{31}, \rho A_{23}] + [A_{32}, \rho A_{13}]) + \text{H.c.} \quad (6)$$

are operators representing the damping of the cavity-field modes by cavity decay with rates κ_1 and κ_2, and of the atomic transitions by spontaneous emission with rates γ_1 and γ_2. The parameters g_i ($i = 1,2,3,4$) are coupling strengths of the cavity modes to the atomic transitions. We assume that, in general, the modes couple with strengths g_1 and g_2 to the transition $|2\rangle \leftrightarrow |3\rangle$ and also can be simultaneously coupled to the $|1\rangle \leftrightarrow |3\rangle$ transition with strengths g_3 and g_4, respectively.

The coefficient $\eta = p\sqrt{\gamma_1 \gamma_2}$ is a measure of the amount of coherence, the so-called SGC, induced by dissipation between the $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ atomic transitions. The source of this coherence has an obvious interpretation. Namely, spontaneously emitted photons on one of the atomic transition drives the other transition. The degree of the coherence, measured by the coefficient η, depends explicitly on the mutual polarization of the transition dipole moments with $p = \cos\theta$, where θ is the angle between the two dipole moments. Thus, $p = 0$ when the transition dipole moments are orthogonal to each other, and p attains its maximal value of $p = \pm 1$ when the dipole moments are parallel or antiparallel to each other. Obviously, the SGC vanishes when $p = 0$ and attains maximal value when $p = \pm 1$.

The parameter $\Delta_L = \omega_{23} - \omega_L$ is the detuning of the laser frequency ω_L from the atomic transition frequency ω_{23}, and $\delta_1 = \omega_L - \omega_1$ and $\delta_2 = \omega_2 - \omega_L$ are detunings of the cavity modes ω_1 and ω_2 from the laser frequency, respectively; $A_{ij} = |i\rangle\langle j|$ are the atomic transition operators between energy states $|i\rangle$ and $|j\rangle$ ($i,j = 1,2,3$) of the atom.

Since the transition $|2\rangle \leftrightarrow |3\rangle$ is driven by a strong, nearly resonant laser field, it is convenient to work in the dressed-state picture [36,37]. We introduce dressed states, which are the eigenstates of the Hamiltonian (3):

$$|\tilde{1}\rangle = |1\rangle,$$
$$|\tilde{2}\rangle = \sin\phi |2\rangle - \cos\phi |3\rangle, \quad (7)$$
$$|\tilde{3}\rangle = \cos\phi |2\rangle + \sin\phi |3\rangle,$$

where

$$\cos^2\phi = \frac{1}{2} + \frac{\Delta_L}{2\Omega_0}, \quad (8)$$

and $\Omega_0 = \sqrt{\Delta_L^2 + 4\Omega^2}$ is the Rabi frequency of the detuned field. In the dressed-state basis, the operators A_{ij} are replaced by dressed-state operators $R_{ij} = |\tilde{i}\rangle\langle\tilde{j}|$, and the density operator of the system can be transformed to the dressed-atom

picture by the unitary transformation

$$\tilde{\rho} = \exp(i\tilde{H}_0 t)\rho \exp(-i\tilde{H}_0 t), \quad (9)$$

where

$$\tilde{H}_0 = (\Delta_L + \Delta_0) R_{11} + \Omega_0 R_z - \delta_1 a_1^\dagger a_1 + \delta_2 a_2^\dagger a_2, \quad (10)$$

and $R_z = (R_{22} - R_{33})/2$ is the population inversion operator between the dressed states $|\tilde{2}\rangle$ and $|\tilde{3}\rangle$.

Applying the unitary transformation (9), we find that the commutator part of the master equation for $\tilde{\rho}$ contains explicitly time-dependent terms that oscillate at frequencies δ_1 and δ_2, and the atomic dissipative part contains terms oscillating with Ω_0 and $2\Omega_0$. In the limit of large Rabi frequency $\Omega_0 \gg g_i, \gamma_i$, the oscillating terms in the dissipative part make contributions of the order γ_i/Ω_0, where $i = 1,2$. These terms can be neglected in the secular approximation. The errors of the secular approximation are of the order γ_i/Ω_0 and g_i/Ω_0. Thus, it is reasonable to neglect these terms on time scales $t \gg \gamma_i^{-1}$ when $\Omega_0 \gg g_i, \gamma_i$. This approximation permits important mathematical simplifications, and "exact" solutions for the steady-state density matrix elements may be obtained that could provide immediate insight into the physics involved in the problem.

Thus, the master equation in the dressed-atom basis and under the secular approximation simplifies to

$$\frac{d}{dt}\tilde{\rho} = -i[\tilde{V}, \tilde{\rho}] + L_d \tilde{\rho} + L_c \tilde{\rho}, \quad (11)$$

where

$$\tilde{V} = \{d_1[\sin(2\phi) R_z + \sin^2\phi R_{23} e^{i\Omega_0 t} - \cos^2\phi R_{32} e^{-i\Omega_0 t}]$$
$$+ d_2 (\sin\phi R_{13} e^{i[\Delta_0 + \frac{1}{2}(\Omega_0 + \Delta_L)]t}$$
$$- \cos\phi R_{12} e^{i[\Delta_0 - \frac{1}{2}(\Omega_0 - \Delta_L)]t})\} + \text{H.c.} \quad (12)$$

is the interaction of the dressed atom with the cavity modes, with

$$d_1 = g_1 a_1 e^{i\delta_1 t} + g_2 a_2 e^{-i\delta_2 t},$$
$$d_2 = g_3 a_1 e^{i\delta_1 t} + g_4 a_2 e^{-i\delta_2 t}, \quad (13)$$

and

$$L_d \tilde{\rho} = \gamma_1 (\sin^2\phi [R_{31}, \tilde{\rho} R_{13}] + \cos^2\phi [R_{21}, \rho R_{12}] + \text{H.c.})$$
$$+ \gamma_2 \sin^2(2\phi) ([R_z, \tilde{\rho} R_z] + \text{H.c.})$$
$$+ \gamma_2 (\sin^4\phi [R_{32}, \tilde{\rho} R_{23}] + \cos^4\phi [R_{23}, \tilde{\rho} R_{32}] + \text{H.c.})$$
$$+ \eta_0 \sin^2\phi ([R_{31}, \tilde{\rho} R_{23}] + [R_{32}, \tilde{\rho} R_{13}] + \text{H.c.})$$
$$+ \eta_0 \cos^2\phi ([R_{21}, \tilde{\rho} R_{22}] + [R_{22}, \tilde{\rho} R_{12}] + \text{H.c.}) \quad (14)$$

is an operator representing the damping of the dressed-atom system.

Obviously, the cavity damping term remains unchanged under the dressed-atom transformation, but the atomic dynamics is now determined in terms of the dressed-atom operators. In this paper, we are interested in the case of the two cavity modes being nondegenerated, i.e., $\omega_1 \neq \omega_2$, for which the time dependence of $\tilde{V}$ is quite complicated. This renders the master equation difficult to solve exactly, except in a special case of a weak coupling of the cavity modes to the atomic transitions $g_i \ll \Omega_0$. In this case, we can treat the interaction as a weak perturbation to the strong atom-laser interaction and find that,

after tracing over the atomic variables, the effective master equation for the reduced density operator of the cavity modes $\rho_c = \text{Tr}_A \tilde{\rho}$ is of the form

$$
\begin{aligned}
\frac{d}{dt}\rho_c &= i\sum_{j=1}^{2}(\delta_{12} - \bar{B}_j)[a_j^\dagger a_j, \rho_c] - i\sum_{j=1}^{2}\bar{A}_j[a_j a_j^\dagger, \rho_c] \\
&\quad + \sum_{j=1}^{2}(\tilde{B}_j + \kappa_j)(2a_j\rho_c a_j^\dagger - a_j^\dagger a_j\rho_c - \rho_c a_j^\dagger a_j) \\
&\quad + \sum_{j=1}^{2}\tilde{A}_j(2a_j^\dagger\rho_c a_j - \rho_c a_j a_j^\dagger - a_j a_j^\dagger\rho_c) \\
&\quad + \sum_{j\neq j'=1}^{2}\{C_j a_j^\dagger a_{j'}^\dagger\rho_c + D_j\rho_c a_{j'}^\dagger a_j^\dagger \\
&\quad - (C_j + D_j)a_{j'}^\dagger\rho_c a_j^\dagger + \text{H.c.}\},
\end{aligned}
\tag{15}
$$

where $\delta_{12} = (\delta_2 - \delta_1)/2$ and $\tilde{A}_j, \tilde{B}_j$ and $\bar{A}_j, \bar{B}_j$ are the real and imaginary parts of complex coefficients A_j, B_j, respectively. The coefficients $\tilde{A}_j$ and $\tilde{B}_j$ have obvious interpretation as absorption and gain rates, whereas $\bar{A}_j$ and $\bar{B}_j$ are radiative shifts of the cavity-mode frequencies. Correspondingly, the complex coefficients C_j and D_j determine terms representing desired correlations between the cavity modes. The expressions for the coefficients depend strongly on the coupling configuration of the cavity modes to the atomic transitions and also on a particular choice of other parameters. The explicit analytical forms of the coefficients for different coupling configurations of the cavity modes to the atoms will be given in Sec. III.

The master equation (15) is of a form characteristic for a system composed of two field modes coupled to a multimode squeezed vacuum [38]. For this reason, to quantify entanglement between the modes, we shall use Duan's criterion [39], which relates entanglement to squeezing between the modes. If the cavity modes were initially in a vacuum state, which is an example of a Gaussian state, the state of the modes governed by Eq. (15) will remain a two-mode Gaussian state for all times t. The quantum-statistics properties of a two-mode Gaussian state are conveniently studied in terms of quadrature operators of the two cavity modes

$$
\begin{aligned}
X_l &= \frac{1}{\sqrt{2}}(a_l^\dagger e^{i\theta_l} + a_l e^{-i\theta_l}), \\
Y_l &= \frac{i}{\sqrt{2}}(a_l^\dagger e^{i\theta_l} - a_l e^{-i\theta_l}), \quad l = 1, 2
\end{aligned}
\tag{16}
$$

where θ_l is the phase angle of the modes. If we introduce two operators

$$
u = aX_1 - \frac{1}{a}X_2, \quad v = aY_1 + \frac{1}{a}Y_2,
\tag{17}
$$

where a is a state-dependent real number, then, according to Duan's criterion, a two-mode Gaussian state is entangled if, and only if, the sum of the variances $\Sigma = \langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle$ satisfies the inequality

$$
\Sigma = 2na^2 + 2m/a^2 - 4c < a^2 + \frac{1}{a^2},
\tag{18}
$$

with $a^2 = \sqrt{(2m-1)/(2n-1)}$, $n = \langle a_1^\dagger a_1\rangle + 1/2$, $m = \langle a_2^\dagger a_2\rangle + 1/2$, and $c = |\langle a_1 a_2\rangle|$. Since the right-hand side of

Eq. (18) is a positive number, we may introduce a parameter

$$
\Upsilon = \Sigma - a^2 - \frac{1}{a^2},
\tag{19}
$$

and then the condition for entanglement between the cavity modes is that the parameter Υ must be negative.

From Eqs. (18) and (19), it is obvious that, in order to calculate the parameter Υ, it is necessary to have available the cavity-field correlation functions n, m, and c. These correlation functions are readily found using the master equation (15), from which we can derive equations of motion for the required correlation functions and find that they satisfy a set of coupled differential equations

$$
\begin{aligned}
\frac{d}{dt}\langle a_j^\dagger a_j\rangle &= -(\Gamma_j + \Gamma_j^*)\langle a_j^\dagger a_j\rangle \\
&\quad + \chi_j\langle a_1^\dagger a_2^\dagger\rangle + \chi_j^*\langle a_1 a_2\rangle + 2\tilde{A}_j, \\
\frac{d}{dt}\langle a_1 a_2\rangle &= -(\Gamma_1 + \Gamma_2)\langle a_1 a_2\rangle + \chi_2\langle a_1^\dagger a_1\rangle \\
&\quad + \chi_1\langle a_2^\dagger a_2\rangle + (C_1 + C_2),
\end{aligned}
\tag{20}
$$

where $\Gamma_j = \kappa_j + i\delta_{12} - (A_j - B_j)$ and $\chi_j = C_j - D_j$. The set of the differential equations (20) can be easily solved for arbitrary initial conditions. Since we are interested in a stationary entanglement between the cavity modes, we analyze the stability condition and find that the system is stable and reaches its steady state as $t \rightarrow \infty$ when

$$
\text{Re}[\Gamma_1 + \Gamma_2 - \sqrt{(\Gamma_1 - \Gamma_2^*)^2 + 4\chi_1\chi_2^*}] > 0.
\tag{21}
$$

The above stability condition may be simplified substantially for particular choices of the detunings and the Rabi frequency such as $\delta_1, \delta_2 \gg \gamma_i$ and $\Omega_0 \gg \gamma_i$.

III. ENTANGLEMENT BETWEEN CAVITY MODES

It is clear from Eq. (15) that the dynamics and entanglement of the cavity modes are a sensitive function of the properties of the driven atomic system. To study this dependence, we shall examine four scenarios of the coupling configuration of the cavity modes to the atomic transitions, two scenarios in which both modes couple to the same driven atomic transition and the other two in which the cavity modes are coupled to different transitions. Particular attention will be paid to the role of a specific driving of the atoms and the SGC in entangling the cavity modes.

A. The case of both modes coupled to the driven transition

In this section, we examine the entanglement properties of the cavity modes when both modes are coupled to only one of the atomic transitions, i.e., the laser-driven transition $|2\rangle \leftrightarrow |3\rangle$, as illustrated in Fig. 1. In other words, all the fields couple to only one of the atomic transitions. This is achieved by setting the coupling strengths g_3 and g_4 in the Hamiltonian (4) equal to zero. We shall be particularly interested in the generation of entanglement between the cavity modes when the coupling system is reduced to a simple two-level system and in the role of the spontaneous emission in coupling of the two-level system to the auxiliary level $|1\rangle$. Therefore, we consider separately two cases of orthogonal

($p = 0$) and nonorthogonal ($p \neq 0$) dipole moments of the atomic transitions. When the dipole moments are orthogonal to each other, $p = 0$, and then the atomic transition $|1\rangle \leftrightarrow |3\rangle$ decouples from the driven transition. In this case, the system reduces to that of a driven two-level atom. On the other hand, when the dipole moments are nonorthogonal, $p \neq 0$, and then the spontaneous emission on the $|1\rangle \leftrightarrow |3\rangle$ can influence the two-level dynamics of the driven $|2\rangle \leftrightarrow |3\rangle$ transition.

We start by introducing the explicit form of the coefficients of the master equation (15), which read

$$A_1 = g_1^2 \left[-\frac{1}{4} F_1(\delta_1) \sin 2\phi + \frac{f_1^*(-\delta_1)\rho_{33}^s}{f_{12}^*(-\delta_1) - \eta_0^2} \cos^4 \phi \right.$$
$$\left. + \frac{f_1(\delta_1)\rho_{22}^s - \eta_0 \rho_{12}^s}{f_{12}(\delta_1) - \eta_0^2} \sin^4 \phi \right],$$

$$B_1 = g_1^2 \left[-\frac{1}{4} F_2(\delta_1) \sin 2\phi + \frac{f_1(\delta_1)\rho_{33}^s}{f_{12}(\delta_1) - \eta_0^2} \sin^4 \phi \right.$$
$$\left. + \frac{f_1^*(-\delta_1)\rho_{22}^s - \eta_0 \rho_{21}^s}{f_{12}^*(-\delta_1) - \eta_0^2} \cos^4 \phi \right],$$

$$C_1 = \frac{1}{4} g_1 g_2 \sin 2\phi \left[F_2(\delta_2) + \frac{f_1(\delta_2)\rho_{33}^s}{f_{12}(\delta_2) - \eta_0^2} \right.$$
$$\left. + \frac{f_1^*(-\delta_2)\rho_{22}^s - \eta_0 \rho_{21}^s}{f_{12}^*(-\delta_2) - \eta_0^2} \right],$$

$$D_1 = \frac{1}{4} g_1 g_2 \sin 2\phi \left[F_1(\delta_2) + \frac{f_1^*(-\delta_2)\rho_{33}^s}{f_{12}^*(-\delta_2) - \eta_0^2} \right.$$
$$\left. + \frac{f_1(\delta_2)\rho_{22}^s - \eta_0 \rho_{12}^s}{f_{12}(\delta_2) - \eta_0^2} \right], \tag{22}$$

where

$$F_1(\delta_j) = [M_{32}(\delta_j) - M_{22}(\delta_j)]\rho_{22}^s - [M_{33}(\delta_j) - M_{23}(\delta_j)]$$
$$\times \rho_{33}^s + [M_{34}(\delta_j) - M_{24}(\delta_j)]\rho_{12}^s, \tag{23}$$

$$F_2(\delta_j) = [M_{32}(\delta_j) - M_{22}(\delta_j)]\rho_{22}^s - [M_{33}(\delta_j) - M_{23}(\delta_j)]$$
$$\times \rho_{33}^s + [M_{35}(\delta_j) - M_{25}(\delta_j)]\rho_{21}^s,$$

and

$$f_{12}(\pm\delta_j) = f_1(\pm\delta_j)f_2(\pm\delta_j), \quad j = 1,2 \tag{24}$$

with

$$f_1(\pm\delta_j) = \gamma_1 + \gamma_2 \cos^2 \phi + i \left(\Delta_0 + \tfrac{1}{2}(\Delta_L + \Omega_0) \pm \delta_j \right),$$
$$f_2(\pm\delta_j) = \gamma_2 \left(1 + \tfrac{1}{2} \sin^2 2\phi \right) + i(\Omega_0 \pm \delta_j). \tag{25}$$

Here, ρ_{22}^s, ρ_{33}^s, and ρ_{12}^s are the steady-state values of the atomic-density matrix elements under the condition of ignoring the effect of the weak coupling between the cavity modes and the atom, and $M_{mn}(\delta_j)$ are elements of the inverse matrix of $U(\delta_j)$:

$$U(\delta_j) = \begin{pmatrix} 2\gamma_1 + i\delta_j & 0 & 0 & \eta_0 & \eta_0 \\ -2\gamma_1 \cos^2 \phi & 2\gamma_2 \sin^4 \phi + i\delta_j & -2\gamma_2 \cos^4 \phi & -\eta_0 \cos 2\phi & -\eta_0 \cos 2\phi \\ -2\gamma_1 \sin^2 \phi & -2\gamma_2 \sin^4 \phi & 2\gamma_2 \cos^4 \phi + i\delta_j & -2\eta_0 \sin^2 \phi & -2\eta_0 \sin^2 \phi \\ \eta_0 & \eta_0 & 0 & b + i\delta_j & 0 \\ \eta_0 & \eta_0 & 0 & 0 & b^* + i\delta_j \end{pmatrix}, \tag{26}$$

where $b = \gamma_1 + \gamma_2 \sin^2 \phi + i[\Delta_0 - (\Omega_0 - \Delta_L)/2]$.

The remaining coefficients A_2, B_2, C_2, and D_2 are obtained from Eq. (22) by exchanging δ_1 with $-\delta_2$ and g_1 with g_2. We should point out here that, in the derivation of the coefficients (22), we have assumed that the states $|\tilde{1}\rangle$ and $|\tilde{3}\rangle$ are separated in energy by $\Delta_0 + (\Omega_0 + \Delta_L)/2$, while the states $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$ are separated in energy by $\Delta_0 - (\Omega_0 - \Delta_L)/2$. Thus, in general, the dressed states are nondegenerate. However, by varying the Rabi frequency Ω_0 or the splitting Δ_0, one may turn the states $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$ into degeneracy, whereas the states $|\tilde{1}\rangle$ and $|\tilde{3}\rangle$ will always remain far from resonance. This would happen when $\Delta_0 = (\Omega_0 - \Delta_L)/2$. As we shall demonstrate in this paper, the degeneracy condition is an optimal condition for entanglement between the cavity modes.

Having defined the coefficients of the master equation for the case of both cavity modes coupled to the driven atomic transition, we now turn our attention to the possibility of generating a stationary entanglement between the modes. In doing so, we shall consider separately two cases, $p = 0$ and $p \neq 0$.

1. The case of p = 0

Let us first determine how much entanglement can be generated when the atom behaves as a two-level system. The master equation (15) can be applied to this simplified case by setting $p = 0$. Figure 2 shows the entanglement measure Υ as a function of Δ_L for $\eta_0 = 0$, fixed detunings δ_1, δ_2, and the Rabi frequency Ω_0. The figure shows that, under resonant excitation, the cavity modes are separable and become entangled for an off-resonant excitation. The entanglement exhibits an interesting behavior in that it has two maxima that occur for certain nonzero values of Δ_L and then rapidly declines thereafter. A small difference $\delta_{12} = -0.61$ between the detunings δ_1 and δ_2 is introduced to cancel the effect of the Stark shifts $\bar{A}_j$ and $\bar{B}_j$. As we see from the figure, the Stark shifts have a distractive effect on entanglement.

We would like to point out that the magnitude of the entanglement is not large and that there are no parameter values at which the entanglement could reach the optimal value $\Upsilon = -1$. Moreover, the maximal entanglement occurs at large detunings $\Delta_L \approx \pm 40\gamma_1$, at which the driving field
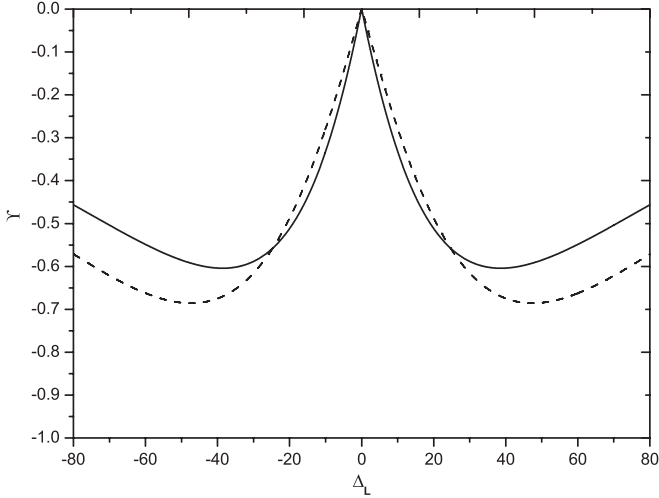
FIG. 2. The degree of entanglement Υ plotted as a function of Δ_L for the case corresponding to a two-level system $g_3 = g_4 = 0$ and $p = 0$, with $\gamma_2 = 0.02$, $\Omega = 50$, $\delta_1 \approx \delta_2 = 50$, $\kappa_1 = \kappa_2 = 0.63$, $g_1 = g_2 = 10$, and different values of δ_{12}: $\delta_{12} = 0$ (solid line) and $\delta_{12} = -0.61$ (dashed line). All parameters are normalized to γ_1.

FIG. 3. The degree of entanglement Υ plotted as a function of Δ_0 for $\Delta_L = 0$, $\gamma_2 = 0.02$, $\Omega = 50$, $\delta_1 \approx \delta_2 = 50$, $\delta_{12} = -0.61$, $\kappa_1 = \kappa_2 = 0.63$, $g_1 = g_2 = 10$, and various values of p: $p = 0.98$ (solid line), $p = 0.7$ (dashed line), $p = 0.4$ (dashed-dotted line), and $p = 0$ (dotted line). All parameters are normalized to γ_1.

is weakly coupled to the atoms. We shall demonstrate in the second scenario that the magnitude can be enhanced to its optimal value $\Upsilon = -1$ by coupling the two-level system to the third level. To summarize, we briefly discuss the parameters characterizing the system and the ranges of these parameters that are experimentally accessible. The parameters are expressed in units of the spontaneous emission rate γ. In the case of alkali atoms, γ is of the order of 10 MHz. Driving lasers used in experiments are usually tunable, providing for arbitrary detuning Δ_L, so that the range $\Delta_L \leqslant 100\gamma$ is easily accessible. The lasers are sufficiently powerful to generate Rabi frequencies up to 100γ.

2. The case of $p \neq 0$

We now turn to illustrate the role of the SGC on entanglement creation between the cavity modes. We assume that the driven transition to which the cavity modes are coupled is coupled by spontaneous emission to the auxiliary level $|1\rangle$. This coupling can occur for the case of nonorthogonal ($p \neq 0$) dipole moments of the atomic transitions, and then the spontaneous emission on the $|1\rangle \leftrightarrow |3\rangle$ can influence the two-level dynamics of the driven $|2\rangle \leftrightarrow |3\rangle$ transition.

Since the spontaneous emission on the atomic transitions occurs at different frequencies and with different rates, the created entanglement between the cavity modes may depend strongly on the splitting Δ_0. As we shall see, the crucial component for entanglement between the cavity modes is the relation between Ω_0 and Δ_0. Figure 3 illustrates the variation of Υ with gradually increasing Δ_0 for the case of resonant driving $\Delta_L = 0$. We see that the cavity modes become entangled *only* for $p \neq 0$; for a certain value of $\Delta_0 = \Omega_0/2$, the entanglement becomes optimal. In terms of the energies of the dressed states, the condition of $\Delta_0 = \Omega_0/2$ corresponds to the situation wherein the dressed state $|\tilde{1}\rangle$ becomes degenerate with the dressed state $|\tilde{2}\rangle$ [40,41]. The condition of $p \neq 0$ corresponds to the presence of direct coupling between the states $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$.
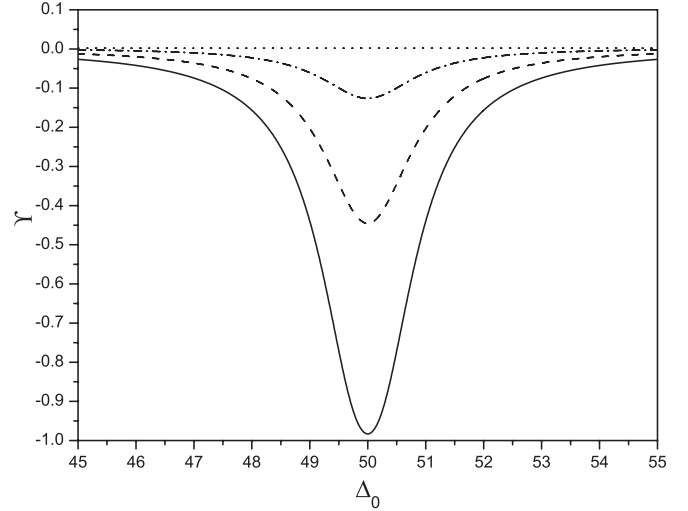
Note that this coupling is induced by the dissipative process of spontaneous emission. Since this is a resonant coupling, it creates a strong coherence between the states $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$. Under these circumstances, the modes become strongly entangled and the degree of entanglement is maximal in comparison with Fig. 2. The amount of the generated entanglement depends also on the ratio of the spontaneous emission rates γ_2/γ_1, and the maximal entanglement of $\Upsilon \approx -1$ is achieved at $\Delta_0 = \Omega_0/2$ and $p \approx 1$ for $\gamma_2 \ll \gamma_1$. In other words, a large entanglement occurs when most of the population resides in the driven transition rather than in the undriven transition. We may summarize that, by using carefully designed driving, such that $\Delta_0 = \Omega_0/2$, and carefully chosen atoms, such that $\gamma_2 \ll \gamma_1$, a large entanglement can be produced between the cavity modes via dissipation created by coherence in the atoms.

We now proceed to explain the physical origin of the process responsible for entanglement of the cavity modes predicted in the above two scenarios. As we shall see, the physics of the process can be quantitatively explained on the level of the stationary population of the atomic system. In the first instance, a simple analytical expression can be derived for the master equation as follows. When the frequency difference δ and the Rabi frequency Ω_0 are much larger than the damping rates of the atomic transitions $\delta_1 \approx \delta_2 = \delta \gg \gamma_i$ and $\Omega_0 \gg \gamma_i$, the real parts of the parameters (22) become negligible, i.e., $\tilde{A}_j = \tilde{B}_j = \tilde{C}_j = \tilde{D}_j \approx 0$, and the imaginary parts become $\bar{A}_j \approx \bar{B}_j$ and $\bar{C}_j = -\bar{D}_j$. It is then straightforward to show that the master equation (15) may be approximated by

$$\frac{d}{dt}\rho_c = -i(\delta_{12} + 2\bar{A})[a_1^\dagger a_1 + a_2^\dagger a_2, \rho_c]$$
$$- i\bar{D}[a_1^\dagger a_2^\dagger + a_1 a_2, \rho_c] + L_c\rho_c, \quad (27)$$

where

$$\bar{A} = \frac{g^2 \Omega_0 \left(1 + \cos^2 2\phi\right)}{4\left(\Omega_0^2 - \delta^2\right)} \left(\rho_{22}^s - \rho_{33}^s\right),$$

$$\bar{D} = \frac{g^2 \Omega_0 \sin^2 2\phi}{2\left(\Omega_0^2 - \delta^2\right)} \left(\rho_{22}^s - \rho_{33}^s\right),$$

(28)

and, for simplicity, we have assumed equal coupling constants $g_1 = g_2 = g$.

This shows that the atomic variables contribute to the coherent evolution of the cavity modes, and the only relaxation in the system is the damping of the cavity modes. A choice of $\delta_{12} = -2\bar{A}$ simplifies further the master equation and leaves only the parametric amplifying term in its commutator part. This term is responsible for correlations and hence for entanglement between the modes. The magnitude of entanglement attains maximal value when $\bar{D}$ maximizes. It is evident from Eq. (28) that the parameter $\bar{D}$ is different from zero only if the population is unequally distributed between the dressed states. Thus, only one factor determines the magnitude of entanglement between the cavity mode, i.e., the population must be inverted between the dressed states of the system. For the case of $p = 0$, this can be achieved if the laser frequency is detuned from the atomic transition frequency ω_{23}. It is interesting that the entanglement is created without any coherence between the dressed states. There is no coherence between the dressed states because the Rabi frequency Ω_0 is much larger than all relaxation rates, $\Omega_0 \gg \gamma_i, \kappa_i$. However, we should point out that, in the case of an off-resonant driving, there is a coherence between the bare atomic states. Thus, one can argue that the predicted entanglement actually occurs due to a nonzero coherence between the bare atomic states.

To calculate the population inversion between the dressed states, we introduce density matrix elements with respect to the three atomic dressed states in the absence of the cavity modes, denoting $\langle \tilde{1} | \tilde{\rho} | \tilde{2} \rangle$ by ρ_{12}, etc. The equations of motion are

$$\dot{\rho}_{11} = -2\gamma_1 \rho_{11} - \eta_0 \left(\rho_{12} + \rho_{21}\right),$$

$$\dot{\rho}_{22} = 2\gamma_1 \cos^2 \phi \rho_{11} + 2\gamma_2 \left(\cos^4 \phi \rho_{33} - \sin^4 \phi \rho_{22}\right)$$
$$+ \eta_0 \cos 2\phi \left(\rho_{12} + \rho_{21}\right),$$

$$\dot{\rho}_{33} = 2\gamma_1 \sin^2 \phi \rho_{11} - 2\gamma_2 \left(\cos^4 \phi \rho_{33} - \sin^4 \phi \rho_{22}\right)$$
$$+ 2\eta_0 \sin^2 \phi \left(\rho_{12} + \rho_{21}\right),$$

$$\dot{\rho}_{12} = -\left\{\gamma_1 + \gamma_2 \sin^2 \phi + i\left[\Delta_0 - \tfrac{1}{2}(\Omega_0 - \Delta_L)\right]\right\}$$
$$\times \rho_{12} - \eta_0(\rho_{11} + \rho_{22}).$$

(29)

It is evident from the above equations that the coherence ρ_{12} induced by spontaneous emission oscillates with frequency $\Delta_0 - (\Omega_0 - \Delta_L)/2$. This fact has the obvious physical meaning that the coherence attains maximal value when $\Delta_0 - (\Omega_0 - \Delta_L)/2 = 0$. For $\Delta_L = 0$, the coherence maximizes at $\Delta_0 = \Omega_0/2$ and simultaneously the factor $\sin^2 \phi$ in the coefficient $\bar{D}$ equals to 1 (consequently, the value at which the entanglement, shown in Fig. 3, attains the maximal value).

In the steady state, the dressed-state population difference can be worked out explicitly for both $p = 0$ and $p \neq 0$. For the case of $p = 0$, the steady-state population difference is given

by the expression

$$\rho_{22}^s - \rho_{33}^s = \frac{\cos^4 \phi - \sin^4 \phi}{\cos^4 \phi + \sin^4 \phi},$$

(30)

which clearly shows that the populations among the dressed states are imbalanced only for a nonzero detuning $\Delta_L \neq 0 \, (\phi \neq \pi/4)$. In this case, the parameter $\bar{D}$ responsible for the nonlinear coupling between the modes is different from zero. It is easy to check that the maximal entanglement seen in Fig. 2 is attained at the detunings corresponding to the maximal value of $\bar{D}$. Thus, we have a simple physical interpretation of the entanglement creation by a detuned laser field.

We stress that, in the case of the detuned driving ($\Delta_L \neq 0$) and in the limit $p = 0$, i.e., in the two-level situation, the population is unequally distributed between the dressed states; however, it is not possible to produce atoms in a pure dressed state in which $|\rho_{22}^s - \rho_{33}^s| = 1$ and at the same time have the coefficient $\bar{D}$ different from zero. However, for the case of three-level atoms with $p = 1$, it is possible to have $|\rho_{22}^s - \rho_{33}^s| = 1$, in which case the population is trapped in one of the dressed states. The condition of the population trapping is unique to the SGC and can be achieved even for a resonant driving $\Delta_L = 0$.

We now proceed to evaluate the population inversion when $p = 1$. A careful analysis of the steady-state solution shows that, in the case of the level crossing at $\Delta_0 = \Omega_0/2$ and in the limit $p = 1$, the population is not trapped in one of the dressed states, but rather in one of linear superpositions

$$|s\rangle = \alpha |\tilde{2}\rangle + \beta |\tilde{1}\rangle,$$
$$|a\rangle = \beta |\tilde{2}\rangle - \alpha |\tilde{1}\rangle,$$

(31)

where

$$\alpha = \left(\frac{\gamma_2 \sin^2 \phi}{\gamma_1 + \gamma_2 \sin^2 \phi}\right)^{\frac{1}{2}}, \quad \beta = \left(\frac{\gamma_1}{\gamma_1 + \gamma_2 \sin^2 \phi}\right)^{\frac{1}{2}}. \quad (32)$$

It is easy to check that, at the level crossing condition and in the limit $p = 1$, the population is trapped in the antisymmetric state $|a\rangle$, i.e., $\rho_{aa}^s = 1$ irrespective of the detuning Δ_L and the ratio between the damping rates γ_1 and γ_2. This result implies that the SGC is essential for the atomic system to be capable of achieving a pure state. In other words, the trapping effect is a direct manifestation of the presence of the SGC that can be employed to maintain the complete inversion between the dressed states, even in the case of zero detuning between the laser and the atomic transition frequencies. If we incorporate the solution $\rho_{aa}^s = 1$ into Eq. (28), we find that the resulting coefficient $\bar{D}$ takes the form

$$\bar{D} = \frac{g^2 \Omega_0}{2\left(\Omega_0^2 - \delta^2\right)} \frac{\gamma_1 \sin^2 2\phi}{\gamma_1 + \gamma_2 \sin^2 \phi},$$

(33)

from which one can easily show that the coefficient $\bar{D}$ is greatest when $\phi = \pi/4 \, (\Delta_L = 0)$ and $\gamma_2 \ll \gamma_1$. This prediction clearly explains our numerical results presented in Fig. 3.

To clarify the issue of the mechanism responsible for creation of the stationary entanglement between the cavity modes, we may refer to the equations of motion for the
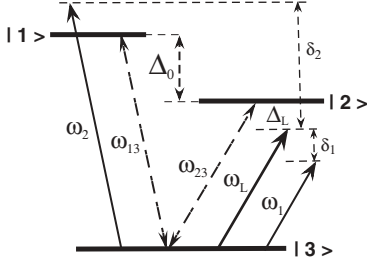
FIG. 4. Schematic diagram of the coupling configuration of the cavity modes and the driven laser field. The cavity mode of frequency ω_1 is coupled to the laser-driven transition with detuning δ_1 from the laser frequency, while the cavity mode of frequency ω_2 is coupled to the undriven transition with detuning δ_2 from the laser frequency.

correlation functions (20). It is straightforward to show that, in the limits of $\delta \gg \gamma_i$ and $\Omega_0 \gg \gamma_i$, the only damping mechanism of the correlation functions is the cavity damping. Thus, the SGC facilitates correlations between the cavity modes that then decay with the cavity damping to a stationary entangled state.

B. The case of the modes coupled to different atomic transitions

We now proceed to evaluate entanglement between the cavity modes when one of the cavity modes, a_1, is coupled to the driven $|2\rangle \leftrightarrow |3\rangle$ transition and the other mode, a_2, is coupled to the undriven transition $|1\rangle \leftrightarrow |3\rangle$, as illustrated in Fig. 4. In this case, the coupling strengths $g_2 = g_3 = 0$, then the coefficients of the master equation (15) are of the form

$$
A_1 = g_1^2 \left[-\frac{1}{4} F_1(\delta_1) \sin 2\phi + \frac{\rho_{33}^s \cos^4 \phi}{f_2^*(-\delta_1) - \eta_0^2} \right.
$$
$$
\left. + \frac{\rho_{22}^s \sin^4 \phi}{f_2^*(\delta_1) - \eta_0^2} - \frac{\eta_0 \rho_{12}^s \sin^4 \phi}{f_{12}(\delta_1) - \eta_0^2} \right],
$$

$$
B_1 = g_1^2 \left[-\frac{1}{4} F_2(\delta_1) \sin 2\phi + \frac{\rho_{33}^s \sin^4 \phi}{f_2(\delta_1) - \eta_0^2} \right.
$$
$$
\left. + \frac{f_1^*(-\delta_1)\rho_{22}^s - \eta_0 \rho_{21}^s}{f_{12}^*(-\delta_1) - \eta_0^2} \cos^4 \phi \right],
$$
(34)

$$
C_1 = g_1 g_4 \sin\phi \cos^2\phi \left[F_3(\delta_2) + \frac{f_1^*(-\delta_2)\rho_{12}^s - \eta_0 \rho_{11}^s}{f_{12}^*(-\delta_2) - \eta_0^2} \right],
$$

$$
D_1 = g_1 g_4 \sin\phi \cos^2\phi \left[F_4(\delta_2) - \frac{\eta_0 \rho_{33}^s}{f_{12}^*(-\delta_2) - \eta_0^2} \right],
$$

with $F_1(\delta_1)$ and $F_2(\delta_1)$ given in Eq. (23),

$$
F_3(\delta_2) = [M_{32}(\delta_2) - M_{22}(\delta_2)]\rho_{12}^s + [M_{35}(\delta_2) - M_{25}(\delta_2)]\rho_{11}^s,
$$
$$
F_4(\delta_2) = [M_{31}(\delta_2) - M_{21}(\delta_2)]\rho_{12}^s + [M_{35}(\delta_2) - M_{25}(\delta_2)]\rho_{22}^s,
$$
(35)

and

$$
A_2 = g_4^2 \left[h_1(\delta_2) + \frac{f_2(-\delta_2)\rho_{11}^s - \eta_0 \rho_{21}^s}{f_{12}(-\delta_2) - \eta_0^2} \sin^2\phi \right],
$$

$$
B_2 = g_4^2 \left[h_2(\delta_2) + \frac{f_2(-\delta_2)\rho_{33}^s \sin^2\phi}{f_{12}(-\delta_2) - \eta_0^2} \right],
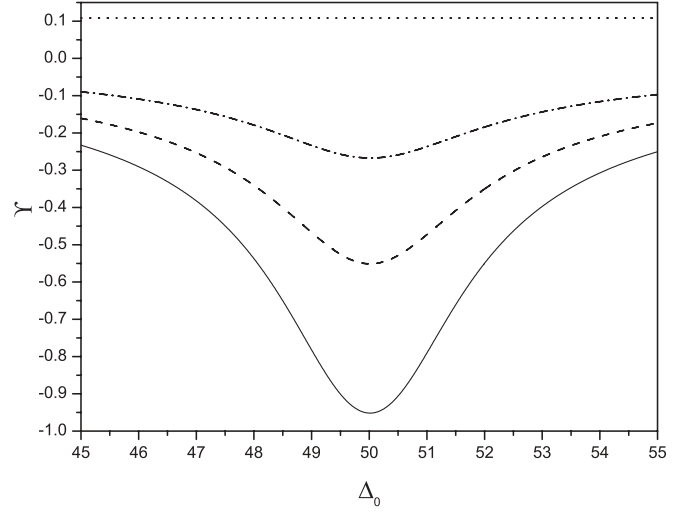$$
(36)

FIG. 5. The degree of entanglement Υ as a function of Δ_0 for the case of the cavity modes coupled to different atomic transitions $g_2 = g_3 = 0$ and $g_1 = g_4 = 10$, with $\Delta_L = 0$, $\gamma_2 = 2$, $\Omega = 50$, $\delta_1 \approx \delta_2 = 50$, $\delta_{12} = -0.38$, $\kappa_1 = \kappa_2 = 0.67$, and different values of p: $p = 0.98$ (solid line), $p = 0.7$ (dashed line), $p = 0.4$ (dashed-dotted line), and $p = 0$ (dotted line). All parameters are normalized to γ_1.

$$
C_2 = g_1 g_4 \sin\phi \cos^2\phi \left[h_3(\delta_1) - \frac{\eta_0 \rho_{33}^s}{f_{12}(-\delta_1) - \eta_0^2} \right],
$$

$$
D_2 = g_1 g_4 \sin\phi \cos^2\phi \left[h_4(\delta_1) + \frac{f_2(-\delta_1)\rho_{12}^s}{f_{12}(-\delta_1) - \eta_0^2} \right],
$$

with

$$
h_1(\delta_2) = \left[M_{42}(-\delta_2)\rho_{21}^s + M_{44}(-\delta_2)\rho_{11}^s \right]\cos^2\phi,
$$
$$
h_2(\delta_2) = \left[M_{41}(-\delta_2)\rho_{21}^s + M_{44}(-\delta_2)\rho_{22}^s \right]\cos^2\phi,
$$
$$
h_3(\delta_1) = M_{43}(-\delta_1)\rho_{33}^s - M_{42}(-\delta_1)\rho_{22}^s - M_{45}(-\delta_1)\rho_{11}^s,
$$
$$
h_4(\delta_1) = M_{43}(-\delta_1)\rho_{33}^s - M_{42}(-\delta_1)\rho_{22}^s - M_{44}(-\delta_1)\rho_{12}^s.
$$
(37)

Figure 5 shows the results for the entanglement measure Υ as a function of Δ_0 for various values of p. Since, in the case of $p = 0$, the creation of entanglement between the cavity modes was associated with a nonzero detuning $\Delta_L \neq 0$, the role of SGC is illustrated most clearly if one assumes a resonant laser field. Consequently, we choose to limit our illustration of the creation of entanglement to a situation in which $\Delta_L = 0$.

As in Sec. III A 2, the entanglement occurs for $p \neq 0$ and the optimal entanglement can be obtained at $\Delta_0 = \Omega_0/2$. However, in contrast to Sec. III A, the entanglement maximizes at $\Upsilon \approx -1$ for $\gamma_2 = 2\gamma_1$. This means that the entanglement maximizes when the transition rates of the dressed transition resonant with the undressed transition are equal.

In order to understand this behavior of entanglement, we consider the coefficients of the master equation in the limits of $\delta \gg \gamma_i$ and $\Omega_0 \gg \gamma_i$ and find that, in these limits, the master equation (15) reduces to the following form:

$$
\frac{d}{dt}\rho_c = -i(\delta_{12} + 2\bar{A})[a_1^\dagger a_1 + a_2^\dagger a_2, \rho_c]
$$
$$
+ i[\bar{D}a_1^\dagger a_2^\dagger + \bar{D}^* a_1 a_2, \rho_c] + L_c \rho_c, \qquad (38)
$$

where

$$\bar{A} = \frac{1}{4} g^2 \left[\left(\frac{\sin^4 \phi}{\Omega_0 + \delta} + \frac{\cos^4 \phi}{\Omega_0 - \delta} \right) \left(\rho_{22}^s - \rho_{33}^s \right) \right.$$
$$\left. + \frac{\sin^2 \phi}{\Omega_0 - \delta} \left(\rho_{11}^s - \rho_{33}^s \right) + \frac{\cos^2 \phi}{\delta} \left(\rho_{22}^s - \rho_{11}^s \right) \right], \quad (39)$$

$$\bar{D} = \frac{\Omega_0 g^2 \sin \phi \cos^2 \phi}{(\Omega_0 - \delta)\delta} \rho_{12}^s.$$

We may further simplify the master equation by choosing $\delta_{12} = -2\bar{A}$, which leaves only the nonlinear term in its commutator part. Note that, comparing to Sec. III A, there is a qualitative difference in the dependence of the coefficient $\bar{D}$ on the density matrix elements. The magnitude of $\bar{D}$ depends now on the coherence between the states $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$, but not on the population difference. The coherence is induced by spontaneous emission and can be different from zero only if $p \neq 0$. This means that the SGC is crucial for creation of entanglement between the cavity modes when the modes are coupled to different atomic transitions. As seen from Fig. 5, the entanglement maximizes at $\Delta_0 = \Omega_0/2$ and $p = 1$. It is easy to show from Eqs. (29) and (31) that, for $\Delta_0 = \Omega_0/2$ and $p = 1$, in the steady state the population is trapped in the antisymmetric state $|a\rangle$. Thus, similar to Sec. III A, the condition for the maximal entanglement coincides with the collapse of the atomic system into the pure trapping state. In this case, the coherence $\rho_{12}^s = -\alpha\beta$ and then the parameter $\bar{D}$ reduces to

$$\bar{D} = -\frac{\Omega_0 g^2 \sin^2 2\phi}{4(\Omega_0 - \delta)\delta} \frac{\sqrt{\gamma_1 \gamma_2}}{\gamma_1 + \gamma_2 \sin^2 \phi}. \quad (40)$$

It is easily verified that the coefficient $\bar{D}$ attains its maximal value for $\phi = \pi/4$ and $\gamma_2 = 2\gamma_1$. Thus, the simple formula in Eq. (40) predicts accurately the parameter values of the maximal entanglement in Fig. 5.

In concluding this section, we would like to point out that the qualitative features of entanglement between the cavity modes depend on whether the dipole moments of the atomic transitions are parallel ($p = 1$) or antiparallel ($p = -1$) to each other. We have already seen that, in the case of parallel dipole moments and $\Delta_0 = \Omega_0/2$, the population is trapped in the antisymmetric state irrespective of the laser detuning Δ_L and the ratio between the atomic spontaneous emission rates. However, for the antiparallel dipole moments, the situation is different. It is not difficult to show from Eqs. (29) and (31) that, for $p = -1$ and $\Delta_0 = \Omega_0/2$, the steady-state populations of the states are

$$\rho_{aa} = (\alpha^2 - \beta^2)^2, \quad \rho_{ss} = 4\alpha^2\beta^2, \quad \rho_{33} = 0, \quad (41)$$

where α and β are given in Eq. (32). It is evident that, in general, the population is redistributed between the symmetric and antisymmetric states, and only in the case of $\gamma_1 = \gamma_2 \sin^2 \phi$ is the population trapped in the symmetric superposition state. A consequence of this population redistribution is the reduction of the entanglement between the cavity modes. This is shown in Fig. 6, where we plot the entanglement measure for $p = -1$ and different ratios between the atomic spontaneous emission rates. For $\gamma_2 \neq 2\gamma_1$, the magnitude of the entanglement is reduced and attains the maximal value of $\Upsilon = -1$ for $\gamma_2 = 2\gamma_1$. This is an another demonstration

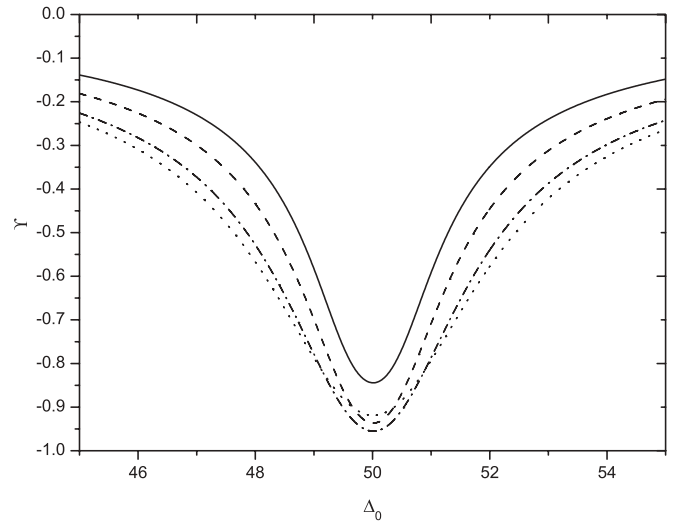FIG. 6. The degree of entanglement Υ plotted as a function of Δ_0 for the case of antiparallel transition dipole moments $p = -1$, with $\Delta_L = 0$, $\Omega = 50$, $\delta_1 \approx \delta_2 = 50$, $\kappa_1 = \kappa_2 = 0.72$, and different values of γ_2/γ_1: $\gamma_2/\gamma_1 = 0.5$ (solid line), $\gamma_2/\gamma_1 = 1.0$ (dashed line), $\gamma_2/\gamma_1 = 2.0$ (dashed-dotted line), and $\gamma_2/\gamma_1 = 3.0$ (dotted line). All parameters are normalized to γ_1.

that the maximal entanglement between the modes is achieved only when two correlated atomic transitions decay rates obey $\gamma_2 = 2\gamma_1$.

C. Other possible couplings of the modes to the atomic transitions

Finally, we briefly comment on the other possible coupling configurations of the cavity modes to the atomic transitions. The two cases discussed above predict a large entanglement at practically the same conditions, with only different conditions imposed on the damping rates of the atomic transitions. Another possible configuration is to couple the cavity mode ω_1 to the undriven transition $|1\rangle \leftrightarrow |3\rangle$ and the mode ω_2 to the driven transition $|2\rangle \leftrightarrow |3\rangle$. One can see from Fig. 4 that this configuration is obtained from Sec. III B simply by replacing δ by $-\delta$. Thus, a large entanglement could be generated in this configuration for the same condition as in Sec. III B.

The most general configuration of the coupling constants is the case corresponding to all of the cavity modes simultaneously coupled to both atomic transitions. It is easily verified that this general case can be treated as a sum of two cases of modes coupled to different atomic transitions with opposite detuning δ. By combining the two cases together, we find that the magnitude of the effective coefficient $\bar{D}$ depends strongly on the sign of the parameter p. For $p = \pm 1$, the effective coefficient $\bar{D}$ takes the following form:

$$\bar{D}_{p=\pm 1} = \frac{g^2 \Omega_0 \sin^2 2\phi}{2 \left(\Omega_0^2 - \delta^2 \right)} \frac{\sqrt{\gamma_1} \left(\sqrt{\gamma_1} \mp \sqrt{\gamma_2} \right)}{\gamma_1 + \gamma_2 \sin^2 \phi}. \quad (42)$$

We see that, depending on the sign of p, these two coupling configurations can interfere constructively or destructively, resulting in an enhanced or reduced effective magnitude of the nonlinear process. For $p = 1$, the configurations interfere destructively such that, for $\gamma_1 = \gamma_2$, the effective coefficient $\bar{D}$

vanishes. On the other hand, for $p = -1$, the configurations interfere constructively, which results in an enhanced amplitude of the nonlinear process. However, the resulting magnitude of the effective coefficient depends strongly on the ratio γ_2/γ_1 such that $\bar{D}$ is large for $\gamma_2/\gamma_1 \ll 1$; but, it becomes very small, proportional to $\sqrt{\gamma_1/\gamma_2}$ in the opposite limit of $\gamma_2/\gamma_1 \gg 1$. In other words, the three-level system can strongly entangle the cavity modes only if the spontaneous emission rate on the undriven transition is much larger than that of the driven transition.

We conclude this section with a short discussion of the possibility of creating entanglement between the cavity modes by the SGC in three-level atoms in the Lambda or cascade configurations. As we have shown, the crucial component for the maximal entanglement is to trap the population in a pure superposition state of the atoms. However, it is well known that the SGC has a constructive effect on trapping of the population in a pure state only in the V-type atoms [25]. In the Lambda or cascade-type atoms, the SGC has a destructive rather than a constructive effect on the trapping phenomenon [42,43].

The crucial components for the entanglement are three-level atoms with parallel or nearly parallel dipole moments between the two atomic transitions. It is difficult in practice to find V-type systems with parallel or antiparallel dipole moments. One of the possibilities is to use sodium dimers, which can be modeled as a five-level molecule in which transitions with parallel and antiparallel dipole moments can be selected [44,45]. An alternative solution is to engineer atomic systems with parallel dipole moments. For example, Zhou and Swain [46] showed that transitions with parallel dipole moments can be achieved in a three-level atom coupled to a cavity field with pre-selected polarization in the bad cavity limit. Agarwal [47] has demonstrated that an anisotropy in the vacuum can lead to quantum interference among the decay channels of close-lying states. Another possibility is to align the dipole moments by a slow motion of the atoms through the medium [48], or to apply a dc field to couple the upper levels of a three-level V-type atom with perpendicular dipole moments [49]. In addition, SGC arising from radiative decay of the trion into the spin states has been experimentally confirmed in charged GaAs quantum dots [50].

IV. CONCLUSIONS

We have proposed a scheme for generation on demand of a steady-state entanglement between two optical modes coupled to a V-type three-level atom. We have demonstrated that the condition for generation of the maximal entanglement between the modes is to create the complete population inversion between the dressed states of the coupling atomic system. In the case of a two-level atom composing the entangling atomic system, we have shown that a sufficient condition for entanglement between the modes is to create a population difference between dressed states of the driven atomic transition. However, we have found that the maximal entanglement can not be created in this system because it is not possible to create the complete population inversion between the dressed states and, at the same time, maintain a strong coupling between the cavity modes mediated by the atom. In the case of three-level atoms composing the entangling system, we have found that a stationary entanglement can be created on demand by tuning the Rabi frequency of the driving field to the difference between the atomic transition frequencies. The laser field mediates the spontaneously generated coherence between the atomic dipole transitions, enabling it to engineer the dissipation in such a way that the atoms evolve into a pure trapping state.

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