# Energy conservation in partially coherent wave fields from polarization and magnetization

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We present the full expression for energy conservation for partially coherent electromagnetic fields from the randomly fluctuating and statistically stationary magnetic as well as electric sources. While the polarization gives the dominant effect in most realistic situations, magnetization could also give a non-negligible effect in various physical situations where the magnetic field is strong. The formula we derive contains terms not only from electric and magnetic sources alone but also from the interference between them. We also confirm that this conservation is valid for correlation-induced spectral changes as was proved in the previous study for the electric source alone.

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## I. INTRODUCTION

The fields generated from realistic electromagnetic sources fluctuate randomly in space and in time as do the sources. Due to these fluctuations, we need to consider the ensemble average with respect to time for any relevant physical quantities such as the Poynting vector, correlations of field vectors, and spectra of sources and fields. Also, the source correlation induces the coherence of the electromagnetic fields and may result in the change of the spectrum of the radiation on propagation.

There has been significant progress in the study of the coherence theory of electromagnetic fields with randomly fluctuating sources since the establishment of the coherence theory [1] and the theory of correlation-induced spectral changes [2,3]. For electromagnetic fields, the conservation of energy was discussed in [4–7] and the conservation of momentum has been discussed recently in [8,9].

In the conservation of electromagnetic energy, the correlation-induced spectral changes attracted the interest of physicists. The spectral changes may be different at different positions and may appear differently at each direction of observation. Several previous works demonstrated that these spectral changes do not violate the energy-conservation law under various circumstances and/or in the presence of a nonmagnetic source [4–7].

The formulation of the conservation of energy for electromagnetic fields has been focused on the case of a nonmagnetic source. In this work, we also include a magnetic source in the formulation and thus extend the previous work to more general sources. The resulting formula can be useful when the magnetization of the source is strong enough to have a non-negligible contribution to that from polarization.

The extended formalism for the conservation of energy in electromagnetic fields is discussed in detail in Sec. II. The agreement between the source spectrum and the spectrum from the radiated field in this generalized formulation of energy-conservation law is shown in Sec. III. In this research, we used the Gaussian unit system and followed the notation in [7] unless specified otherwise.

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## II. ENERGY CONSERVATION OF ELECTROMAGNETIC FIELDS FROM ELECTRIC AND MAGNETIC SOURCES

We consider electromagnetic wave fields generated by randomly fluctuating but statistically stationary sources in a finite region represented as the domain $D$. We assume that the source is quasimonochromatic. Let $\langle \mathbf{F}(\mathbf{r},\omega)\rangle$ denote the ensemble average of the Poynting vector at frequency $\omega$ and at a position $\mathbf{r}$. With the complex electric field $\mathbf{E}(\mathbf{r},\omega)$ and the complex magnetic field $\mathbf{H}(\mathbf{r},\omega)$ in the space-frequency domain, the averaged Poynting vector can be expressed as [10–12]

$$\langle \mathbf{F}(\mathbf{r},\omega)\rangle = \frac{c}{8\pi}\mathrm{Re}\langle \mathbf{E}^*(\mathbf{r},\omega)\times\mathbf{H}(\mathbf{r},\omega)\rangle. \tag{1}$$

From Maxwell's equations, we have the following relations for $\mathbf{E}$ and $\mathbf{H}$:

$$\begin{aligned}\nabla\times\mathbf{E}(\mathbf{r},\omega) &= ik[\mathbf{H}(\mathbf{r},\omega)+4\pi\mathbf{M}(\mathbf{r},\omega)],\\ \nabla\times\mathbf{H}(\mathbf{r},\omega) &= -ik[\mathbf{E}(\mathbf{r},\omega)+4\pi\mathbf{P}(\mathbf{r},\omega)].\end{aligned} \tag{2}$$

Since physical observables are real, we can drop the pure imaginary part in the divergence of the averaged Poynting vector. We then obtain the following result:

$$\begin{aligned}\nabla\cdot\langle \mathbf{F}(\mathbf{r},\omega)\rangle = -\frac{kc}{2}\mathrm{Im}[&\langle \mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{P}(\mathbf{r},\omega)\rangle\\ &+\langle \mathbf{H}^*(\mathbf{r},\omega)\cdot\mathbf{M}(\mathbf{r},\omega)\rangle].\end{aligned} \tag{3}$$

The outgoing electric and the magnetic fields in the far region can be expressed in terms of polarization and magnetization [13] as

$$\begin{aligned}\mathbf{E}(\mathbf{r},\omega) = [k^2+\nabla(\nabla\cdot)]&\int_D \mathbf{P}(\mathbf{r}',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\\ &+ik\nabla\times\int_D \mathbf{M}(\mathbf{r}',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r',\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbf{H}(\mathbf{r},\omega) = [k^2+\nabla(\nabla\cdot)]&\int_D \mathbf{M}(\mathbf{r}',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\\ &-ik\nabla\times\int_D \mathbf{P}(r',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'.\end{aligned} \tag{5}$$

 

We then substitute Eqs. (4) and (5) into Eq. (3) to obtain the expression for the divergence of the Poynting vector in terms of polarization and magnetization:

$$
\begin{aligned}
&\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle\\
&= -\frac{kc}{2}\mathrm{Im}\Bigg[\Bigg\langle k^2\int_D \mathbf{P}(\mathbf{r},\omega)\cdot\mathbf{P}^*(\mathbf{r}',\omega)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\Bigg\rangle\\
&\quad+\Bigg\langle \mathbf{P}(\mathbf{r},\omega)\cdot\nabla\int_D \mathbf{P}^*(\mathbf{r}',\omega)\cdot\nabla\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\Bigg\rangle\\
&\quad+\Bigg\langle k^2\int_D \mathbf{M}(\mathbf{r},\omega)\cdot\mathbf{M}^*(\mathbf{r}',\omega)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\Bigg\rangle\\
&\quad+\Bigg\langle \mathbf{M}(\mathbf{r},\omega)\cdot\nabla\int_D \mathbf{M}^*(\mathbf{r}',\omega)\cdot\nabla\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\Bigg\rangle\\
&\quad+\Bigg\langle ik\mathbf{P}(\mathbf{r},\omega)\cdot\int_D \mathbf{M}^*(\mathbf{r}',\omega)\times\nabla\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\Bigg\rangle\\
&\quad-\Bigg\langle ik\mathbf{M}(\mathbf{r},\omega)\cdot\int_D \mathbf{P}^*(\mathbf{r}',\omega)\times\nabla\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\Bigg\rangle\Bigg]. \qquad (6)
\end{aligned}
$$

Now, we introduce the cross-spectral density tensors $W_{ij}^{(PP)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$, $W_{ij}^{(MM)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$, $W_{ij}^{(PM)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$, and $W_{ij}^{(MP)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ from the polarization and the magnetization of the source, which we will denote as $W_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ with $X = PP$, $MM$, $PM$, or $MP$. Here, we change slightly the notation of the cross-spectral density tensors to avoid confusion of the contributions from the polarization and the magnetization and their interference. The cross-spectral density tensors are defined as

$$
\begin{aligned}
W_{ij}^{(PP)}(\mathbf{r}_1,\mathbf{r}_2,\omega) &\equiv \langle P_i^*(\mathbf{r}_1,\omega)P_j(\mathbf{r}_2,\omega)\rangle,\\
W_{ij}^{(MM)}(\mathbf{r}_1,\mathbf{r}_2,\omega) &\equiv \langle M_i^*(\mathbf{r}_1,\omega)M_j(\mathbf{r}_2,\omega)\rangle,\\
W_{ij}^{(PM)}(\mathbf{r}_1,\mathbf{r}_2,\omega) &\equiv \langle P_i^*(\mathbf{r}_1,\omega)M_j(\mathbf{r}_2,\omega)\rangle,\\
W_{ij}^{(MP)}(\mathbf{r}_1,\mathbf{r}_2,\omega) &\equiv \langle M_i^*(\mathbf{r}_1,\omega)P_j(\mathbf{r}_2,\omega)\rangle.
\end{aligned} \qquad (7)
$$

The first four terms of the right-hand side of Eq. (6) with these definitions can be simplified in exactly the same way as done in [7], and the remaining two terms can also be simplified similarly. Therefore, Eq. (6) simply becomes

$$
\begin{aligned}
\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle &= -\frac{kc}{2}\mathrm{Im}\int_D\left[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega)+W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\right]\\
&\quad\times(k^2\delta_{ij}+\partial_i\partial_j)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\\
&\quad-\frac{kc}{2}\mathrm{Im}\int_D\left[W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)-W_{ik}^{(PM)}(\mathbf{r}',\mathbf{r},\omega)\right]\\
&\quad\times ik\epsilon_{ijk}\partial_j\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \qquad (8)
\end{aligned}
$$

Here and throughout the paper, unless specified otherwise, we use the Einstein summation convention in which the repeated indices are summed over.

Applying the divergence theorem on Eq. (8) with the volume $V$, which enclosed the source domain $D$, we obtain

$$
\begin{aligned}
&\int_\Sigma\langle F(\mathbf{r},\omega)\rangle\cdot\mathbf{n}\,d\Sigma\\
&= -\frac{kc}{2}\mathrm{Im}\int_D\int_D\left[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega)+W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\right]\\
&\quad\times(k^2\delta_{ij}+\partial_i\partial_j)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r'\\
&\quad-\frac{kc}{2}\mathrm{Im}\int_D\int_D\left[W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)-W_{ik}^{(PM)}(\mathbf{r}',\mathbf{r},\omega)\right]\\
&\quad\times ik\epsilon_{ijk}\partial_j\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r', \qquad (9)
\end{aligned}
$$

where $\Sigma$ denotes the surface, which encloses the source domain $D$, and $\mathbf{n}$ denotes the normal unit vector to $\Sigma$ at $\mathbf{r}$. Note that $W_{ij}^{(X)}(\mathbf{r}',\mathbf{r},\omega)=0$ for $\mathbf{r}\notin D$.

For later use, we will rewrite Eq. (9) with the help of the following property of cross-spectral density tensors:

$$
W_{ij}^{(PM)}(\mathbf{r}',\mathbf{r},\omega) = W_{ji}^{(MP)*}(\mathbf{r},\mathbf{r}',\omega), \qquad (10)
$$

and $e^{-ik|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|$ has the mutual sign difference to the partial derivatives with respect to $\mathbf{r}$ and $\mathbf{r}'$. Therefore, Eq. (9) can now be written as

$$
\begin{aligned}
&\int_\Sigma\langle F(\mathbf{r},\omega)\rangle\cdot\mathbf{n}\,d\Sigma\\
&= \frac{k^2c}{2}\int_D\int_D\left[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega)+W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\right]\\
&\quad\times(k^2\delta_{ij}+\partial_i\partial_j)\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r'\\
&\quad-k^3c\int_D\int_D\mathrm{Im}\left[W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)\right]\\
&\quad\times\epsilon_{ijk}\partial_j\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r'. \qquad (11)
\end{aligned}
$$

This is the integral form of the energy-conservation law in Eq. (8). We can see from the equation that the total energy rate over a surface $\Sigma$ radiated from the source depends on the second-order correlation properties from the source polarization, magnetization, and the interference between polarization and magnetization, which are represented by the above cross-spectral density tensors. The cross term in Eq. (11) may bring an interesting interference effect in partially coherent wave fields. Also, we want to emphasize that this interference does not appear in the conservation of momentum as in [9].

## III. SPECTRA OF THE SOURCE AND THE RADIATED FIELD

Now, let us examine the correlation-induced spectral changes and see if the field spectrum satisfies the energy-conservation law. We consider electric and magnetic fields in the far zone from the source, at which the position is expressed as $R\mathbf{u}$, where $\mathbf{u}$ is a unit vector pointing outward.

After imposing this condition on the electric and the magnetic fields, these fields can be expressed as [7,14]

$$E_i(R\mathbf{u},\omega) \approx (2\pi)^3 k^2 \frac{e^{ikR}}{R}(\delta_{ij} - u_i u_j)\tilde{\mathbf{P}}_j(k\mathbf{u},\omega) - (2\pi)^3 k^2 \frac{e^{ikR}}{R}\epsilon_{ijk} u_j \tilde{\mathbf{M}}_k(k\mathbf{u},\omega), \quad (12)$$

$$H_i(R\mathbf{u},\omega) \approx (2\pi)^3 k^2 \frac{e^{ikR}}{R}(\delta_{ij} - u_i u_j)\tilde{\mathbf{M}}_j(k\mathbf{u},\omega) + (2\pi)^3 k^2 \frac{e^{ikR}}{R}\epsilon_{ijk} u_j \tilde{\mathbf{P}}_k(k\mathbf{u},\omega), \quad (13)$$

where $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{M}}$ are the spatial Fourier transforms of the source polarization and the source magnetization, respectively, defined as

$$\tilde{\mathbf{P}}(\mathbf{k},\omega) \equiv \frac{1}{(2\pi)^3}\int_D \mathbf{P}(\mathbf{r},\omega)e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3r, \quad (14)$$

$$\tilde{\mathbf{M}}(\mathbf{k},\omega) \equiv \frac{1}{(2\pi)^3}\int_D \mathbf{M}(\mathbf{r},\omega)e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3r. \quad (15)$$

As we defined the cross-spectral density tensor $W_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ in Eq. (7), we can similarly introduce the cross-spectral density tensors $W_{ij}^{(E)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and $W_{ij}^{(H)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ of the fields as

$$W_{ij}^{(E)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle, \quad (16)$$

$$W_{ij}^{(H)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \langle H_i^*(\mathbf{r}_1,\omega)H_j(\mathbf{r}_2,\omega)\rangle. \quad (17)$$

It is straightforward to show that the field correlations in the far zone have the following forms by substituting Eqs. (12) and (13) into Eqs. (16) and (17):

$$\begin{aligned} W_{ij}^{(E)}&(R\mathbf{u}_1,R\mathbf{u}_2,\omega) \\ &= \frac{(2\pi)^6 k^4}{R^2}\big[(\delta_{im} - u_{1i}u_{1m})(\delta_{jn} - u_{2j}u_{2n}) \\ &\quad\times \tilde{W}_{mn}^{(PP)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega) + \epsilon_{imp}\epsilon_{jnq}u_{1m}u_{2n} \\ &\quad\times \tilde{W}_{pq}^{(MM)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega) - \epsilon_{jnq}u_{2n}(\delta_{im} - u_{1i}u_{1m}) \\ &\quad\times \tilde{W}_{mq}^{(PM)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega) - \epsilon_{imp}u_{1m}(\delta_{jn} - u_{2j}u_{2n}) \\ &\quad\times \tilde{W}_{pn}^{(MP)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega)\big], \end{aligned} \quad (18)$$

$$\begin{aligned} W_{ij}^{(H)}&(R\mathbf{u}_1,R\mathbf{u}_1,\omega) \\ &= \frac{(2\pi)^6 k^4}{R^2}\big[(\delta_{im} - u_{1i}u_{1m})(\delta_{jn} - u_{2j}u_{2n}) \\ &\quad\times \tilde{W}_{mn}^{(MM)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega) + \epsilon_{imp}\epsilon_{jnq}u_{1m}u_{2n} \\ &\quad\times \tilde{W}_{pq}^{(PP)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega) + \epsilon_{jnq}u_{2n}(\delta_{im} - u_{1i}u_{1m}) \\ &\quad\times \tilde{W}_{mq}^{(MP)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega) + \epsilon_{imp}u_{1m}(\delta_{jn} - u_{2j}u_{2n}) \\ &\quad\times \tilde{W}_{pn}^{(PM)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega)\big], \end{aligned}$$

where $u_{\alpha i}$ $(i = 1,2,3)$ is the $i$th component of the unit vector $\mathbf{u}_\alpha$. The six-dimensional Fourier transforms of the cross-spectral densities of polarization, magnetization, and the interference between them of the source can be defined as

$$\tilde{W}_{ij}^{(X)}(\mathbf{k}_1,\mathbf{k}_2,\omega) \equiv \frac{1}{(2\pi)^6}\int_D\int_D W_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \times e^{-i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)}d^3r_1\, d^3r_2. \quad (19)$$

Let us denote the power spectrum of the field at the far distance $R$ from the source in the direction $\mathbf{u}$ as $S^\infty(R\mathbf{u},\omega) \equiv c\langle U^{(\infty)}(R\mathbf{u},\omega)\rangle$, where $\langle U\rangle$ is the ensemble average of the energy density. The spectrum then becomes

$$\begin{aligned} S^{(\infty)}(R\mathbf{u},\omega) &= \frac{c}{16\pi}\langle E_i^*(R\mathbf{u},\omega)E_i(R\mathbf{u},\omega)\rangle \\ &\quad+ \frac{c}{16\pi}\langle H_i^*(R\mathbf{u},\omega)H_i(R\mathbf{u},\omega)\rangle \\ &= \frac{c}{16\pi}\big[W_{ii}^{(E)}(R\mathbf{u},R\mathbf{u},\omega) + W_{ii}^{(H)}(R\mathbf{u},R\mathbf{u},\omega)\big]. \end{aligned} \quad (20)$$

By using the expressions of the fields in Eqs. (12) and (13) and the definition in Eq. (19), the spectrum in the far zone can be written as

$$\begin{aligned} S^{(\infty)}&(R\mathbf{u},\omega) \\ &= \frac{8\pi^5 k^4 c}{R^2}\{(\delta_{ij} - u_i u_j) \\ &\quad\times \big[\tilde{W}_{ij}^{(PP)}(-k\mathbf{u},k\mathbf{u},\omega) + \tilde{W}_{ij}^{(MM)}(-k\mathbf{u},k\mathbf{u},\omega)\big] \\ &\quad+ \epsilon_{ijk}u_j\big[\tilde{W}_{ki}^{(PM)}(-k\mathbf{u},k\mathbf{u},\omega) + \tilde{W}_{ik}^{(MP)}(-k\mathbf{u},k\mathbf{u},\omega)\big]\}. \end{aligned} \quad (21)$$

From the usual definition of spectrum, we define the extended spectral densities as

$$S_i^{(X)}(\mathbf{r},\omega) \equiv W_{ii}^{(X)}(\mathbf{r},\mathbf{r},\omega) \quad (i;\ \text{no summation}). \quad (22)$$

Accordingly, the spectral degree of coherence can also be defined as

$$\mu_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \frac{W_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega)}{\sqrt{S_i^{(X)}(\mathbf{r}_1,\omega)}\sqrt{S_j^{(X)}(\mathbf{r}_2,\omega)}} \quad (i,j;\ \text{no summation}), \quad (23)$$

where $0 \leqslant |\mu_{ij}^{(X)}| \leqslant 1$ and $X = PP$, $MM$, $PM$, or $MP$ as we mentioned earlier.

By considering the six-dimensional Fourier transforms in Eq. (21), the spectrum in the far zone is given by the expression

$$\begin{aligned} S^{(\infty)}(R\mathbf{u},\omega) = \frac{1}{8\pi}\frac{k^4 c}{R^2}\bigg\{&(\delta_{ij} - u_i u_j)\int_D\int_D\big[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega) \\ &+ W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\big]e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\, d^3r' \\ &+ \epsilon_{ijk}u_j\int_D\int_D\big[W_{ki}^{(PM)}(\mathbf{r}',\mathbf{r},\omega) \\ &+ W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)\big]e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\, d^3r'\bigg\}. \end{aligned} \quad (24)$$

Equation (24) can be expressed in terms of the spectral degrees of coherence $\mu$ in Eq. (23) from

$$W_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \mu_{ij}^{(X)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\sqrt{S_i^{(X)}(\mathbf{r}_1,\omega)}\sqrt{S_j^{(X)}(\mathbf{r}_2,\omega)} \quad (i,j;\ \text{no summation}). \tag{25}$$

Equations (24) and (25) show that the spectra of the fields in the far zone depend on both the source spectra and the correlations of Cartesian components of polarization, magnetization, and their interference. Thus, we once again find that the spectrum of the far field is different from the source spectrum and depends on the position of observation as was found in [7].

Since source correlations induce spectral changes, we should examine whether they also obey the energy-conservation law. The total spectrum in the far zone can be obtained by integrating $S^{(\infty)}$ over all the surface $\Sigma^{(\infty)}$. The infinitesimal surface element at a distance $R$ is $d\Sigma^{(\infty)} = R^2\, d\Omega$. After the integration of Eq. (24), we have

$$\begin{aligned}\int_{\Sigma^{(\infty)}} & S^{(\infty)}(R\mathbf{u},\omega)\, d\Sigma^{(\infty)} \\ &= \frac{1}{8\pi}k^4 c\left\{\int d\Omega(\delta_{ij} - u_i u_j)\int_D\int_D \left[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega)\right.\right. \\ &\quad \left. + W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\right]e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\, d^3r' \\ &\quad + \int d\Omega\, \epsilon_{ijk}u_j \int_D\int_D \left[W_{ki}^{(PM)}(\mathbf{r}',\mathbf{r},\omega)\right. \\ &\quad \left.\left. + W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)\right]e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\, d^3r'\right\}. \end{aligned} \tag{26}$$

Since the exponential function, differentiated by position, brings a component of $\mathbf{u}$ multiplied by $k$, then $ku_i$ and $ku_j$ can be replaced by differential operators $\partial_i$ and $\partial_j$. Therefore, this equation becomes

$$\begin{aligned}\int_{\Sigma^{(\infty)}} & S^{(\infty)}(R\mathbf{u},\omega)\, d\Sigma^{(\infty)} \\ &= \frac{1}{8\pi}k^2 c\int d\Omega \int_D\int_D \left[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega)\right. \\ &\quad \left. + W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\right](k^2\delta_{ij} + \partial_i\partial_j)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\, d^3r' \\ &\quad + \frac{i}{8\pi}k^3 c\int d\Omega \int_D\int_D \left[W_{ki}^{(PM)}(\mathbf{r}',\mathbf{r},\omega)\right. \\ &\quad \left. + W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)\right]\epsilon_{ijk}\partial_j e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\, d^3r'. \end{aligned} \tag{27}$$

We need to verify the equivalence of Eqs. (11) and (27). The obtained expression for the conservation of energy in Eq. (11) contains a sine function instead of an exponential function. We have a useful identity for that purpose, i.e.,

$$\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi}\int d\Omega\ e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{28}$$

We can reduce the interference terms in Eq. (27) into a term by using the property of spectral-density tensors in Eq. (10) and the fact that $\mathbf{r}$ and $\mathbf{r}'$ are dummy variables and, thus, we can exchange them in the integration of $W_{ik}^{(PM)}$. Note that there is a sign change when we apply the differentiations $\partial_j$ and $\partial_j'$ to the exponential function in Eq. (27). In the process of the calculation, the real and imaginary parts should be traced carefully:

$$\begin{aligned}\int_{\Sigma^{(\infty)}} & S^{(\infty)}(R\mathbf{u},\omega)\, d\Sigma^{(\infty)} \\ &= \frac{k^2 c}{2}\int_D\int_D \left[W_{ij}^{(PP)}(\mathbf{r}',\mathbf{r},\omega) + W_{ij}^{(MM)}(\mathbf{r}',\mathbf{r},\omega)\right] \\ &\quad \times (k^2\delta_{ij} + \partial_i\partial_j)\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}d^3r\, d^3r' - k^3 c \\ &\quad \times \int_D\int_D \mathrm{Im}\left[W_{ik}^{(MP)}(\mathbf{r}',\mathbf{r},\omega)\right]\epsilon_{ijk}\partial_j\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}d^3r\, d^3r'. \end{aligned} \tag{29}$$

As shown here, this result is identical to the one in Eq. (11). In other words, although source correlation induces spectral changes, the total sum of energy passing the closed surface per unit time is always conserved.

## IV. CONCLUSIONS

We extend the previous work [7] on the conservation of energy for electromagnetic fields from a randomly fluctuating, statistically static, and quasimonochromatic electric source to an electric and non-negligible magnetic source. The full expressions have been derived for the conservation of energy. The expressions from the polarization and the magnetization alone are similar to those of the previous work for the electric source. However, there is an additional term from the interference between polarization and magnetization while such interference does not appear in the case of momentum conservation [9]. As shown in the previous work, we find that the energy-conservation law for electromagnetic fields for the more generalized source is also consistent with the correlation-induced spectrum.

Magnetization is usually negligible in ordinary materials in the optically visible region. However, unusual magnetic (meta)materials have been investigated recently [15–19], although the frequencies of the electromagnetic waves could be mostly in the infrared or the far-infrared regions. For some cases in astronomy, the magnetization effect may be comparable or even dominant. Strongly magnetized neutron stars (so-called magnetars) produce a very strong magnetic field. (An observed exceptionally bright flare is thought to provide evidence of the existence of magnetars [20].) However, the magnetization and its interference effect might be hard to detect in this case. In these respects, the effect of magnetization still needs to be explored experimentally.

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