Polarization dynamics and polarization time of random three-dimensional electromagnetic fields

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We investigate the polarization dynamics of random, stationary three-dimensional (3D) electromagnetic fields. For analyzing the time evolution of the instantaneous polarization state, two intensity-normalized polarization autocorrelation functions are introduced, one based on a geometric approach with the Poincaré vectors and the other on energy considerations with the Jones vectors. Both approaches lead to the same conclusions on the rate and strength of the polarization dynamics and enable the definition of a polarization time over which the state of polarization remains essentially unchanged. For fields obeying Gaussian statistics, the two correlation functions are shown to be expressible in terms of quantities characterizing partial 3D polarization and electromagnetic coherence. The 3D degree of polarization is found to have the same meaning in the 3D polarization dynamics as the usual two-dimensional (2D) degree of polarization does with planar fields. The formalism is demonstrated with several examples, and it is expected to be useful in applications dealing with polarization fluctuations of 3D light.

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I. INTRODUCTION

The polarization state of a monochromatic beamlike electromagnetic field is specified by the orientation, shape, and helicity of the polarization ellipse, but for a random, partially polarized light the state is not invariant; instead it evolves in time [1,2]. Furthermore, for two fields with the same degree of polarization the temporal evolution of the polarization state, i.e., the polarization dynamics, can be completely different [3]. For an unpolarized field no preferred instantaneous state of polarization exists, and the state fluctuates with maximal randomness. For a partially polarized field the polarization state varies around a specific state, and the higher the degree of polarization, the smaller, on average, the range of fluctuations. For a fully polarized field the instantaneous polarization state does not change in time. Recently, the polarization dynamics of two-dimensional (2D) fields (plane waves, beams, and far fields) was analyzed in terms of the instantaneous Poincaré vectors [4] and Jones vectors [5]. Both formalisms were shown to provide the same information on the polarization dynamics, and they enabled us to define, in analogy with the coherence time and coherence length, the concepts of polarization time and polarization length for optical beams. The polarization dynamics can be used, for example, to transmit information in a fiber ring laser system [6].

In this work we extend the results of Refs. [4,5] on the polarization dynamics of 2D fields to random, statistically stationary, three-dimensional (3D) electromagnetic fields. Such fields are common in applications and in nature. For example, evanescent electromagnetic waves encountered in guided-wave optics are genuinely 3D fields. Likewise, in reflection of an unpolarized beam, the incident and reflected light generally form a 3D field. Other examples of inherently 3D fields are thermal radiation in a cavity and simply ambient light. For such fields, not only the instantaneous polarization ellipse but also the plane in which it lies can vary randomly. In order to describe the dynamics of the polarization fluctuations at a point in 3D space (consisting of any medium), we define two autocorrelation functions. The first one is obtained on the basis of geometrical arguments and relies on the instantaneous eight-component Poincaré vectors. The second approach is based on energy considerations, and it is formulated in terms of the instantaneous three-component Jones vectors. Although the two approaches are seemingly different, a connection between them is established, indicating that they are equivalent and provide the same information about the rate and strength of the polarization dynamics. Both theories can be used to define a polarization time as a time interval over which the instantaneous polarization state does not change significantly. We also show that in the case of Gaussian statistics, the polarization correlation functions can be written in terms of the 3D degree of polarization and two quantities used to characterize coherence in electromagnetic fields. Significantly, the 3D degree of polarization, as defined in Ref. [7], plays exactly the same role in the characterization of the polarization dynamics of 3D fields as the customary 2D degree of polarization does for 2D fields. We exemplify the formalism by three examples: a uniformly partially polarized and temporally Gaussian-correlated field, radiation inside a blackbody cavity, and the field at the intersection of three orthogonally propagating and orthogonally linearly polarized beams.

The paper is arranged as follows. In Sec. II we recall the tools used to describe the partial polarization and partial coherence properties of random, 3D electromagnetic fields. In Sec. III we introduce two autocorrelation functions to characterize the polarization dynamics of random 3D fields and establish a connection between the two approaches. In this section we also define a polarization time for 3D fields. The formalism is illustrated with specific examples in Sec. IV. Finally, we summarize the work in Sec. V.

II. DESCRIPTION OF ELECTROMAGNETIC COHERENCE IN 3D FIELDS

Consider a fluctuating, statistically stationary, electromagnetic field that, in general, can have three orthogonal electric vector components. The coherence properties

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of such a nonparaxial field at a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ and at time difference τ may be described using the 3×3 (electric) coherence matrix, $\boldsymbol{\mathcal{E}}(\mathbf{r}_1,\mathbf{r}_2,\tau)$, having the elements [1,8]

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \langle E_i^*(\mathbf{r}_1,t)E_j(\mathbf{r}_2,t+\tau)\rangle, \quad i,j \in (x,y,z). \tag{1}$$

In this equation $E_i(\mathbf{r},t)$, with $i \in (x,y,z)$, denotes the components of the zero-mean complex analytic signal associated with the electric field vector. Furthermore, the angle brackets and the asterisk denote averaging and complex conjugation, respectively. We assume the field to be ergodic, implying that the average can be either a time average or an ensemble average. Equation (1) indicates that the coherence matrix is Hermitian in the sense that

$$\boldsymbol{\mathcal{E}}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \boldsymbol{\mathcal{E}}^\dagger(\mathbf{r}_2,\mathbf{r}_1,-\tau), \tag{2}$$

where the dagger stands for the Hermitian adjoint.

The polarization properties of the field at a point are described by the single-point, equal-time coherence (or polarization) matrix, obtained from Eq. (1) by

$$\mathbf{J}(\mathbf{r}) = \boldsymbol{\mathcal{E}}(\mathbf{r},\mathbf{r},0). \tag{3}$$

Thus we see that the polarization matrix is purely Hermitian; i.e.,

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}^\dagger(\mathbf{r}). \tag{4}$$

Next we recall three quantities frequently employed to characterize the partial coherence and partial polarization of random 3D electromagnetic fields. First, the 3D degree of polarization $P_3(\mathbf{r})$ is defined by [2,7]

$$P_3^2(\mathbf{r}) = \frac{3}{2}\left[\frac{\mathrm{tr}\,\mathbf{J}^2(\mathbf{r})}{\mathrm{tr}^2\,\mathbf{J}(\mathbf{r})} - \frac{1}{3}\right], \tag{5}$$

where tr denotes the trace. The physical interpretation of the 3D degree of polarization is that in the reference frame in which the intensities of the electric-field components are the same, its square is equal to the average of the squared correlations between the components [7]. This feature is identical to that of the usual 2D degree of polarization [1,9] and can, in fact, be viewed as the defining property of $P_3(\mathbf{r})$. The value of $P_3(\mathbf{r})$ is invariant under unitary transformations and bounded as $0 \leqslant P_3(\mathbf{r}) \leqslant 1$, with the lower and upper limit corresponding to full unpolarization and full polarization, respectively.

Second, the electromagnetic degree of coherence, $\gamma_{\mathrm{EM}}(\mathbf{r}_1,\mathbf{r}_2,\tau)$, quantifies the spatial and temporal coherence of the field, and it has the expression [10]

$$\gamma_{\mathrm{EM}}^2(\mathbf{r}_1,\mathbf{r}_2,\tau) = \frac{\mathrm{tr}\,[\boldsymbol{\mathcal{E}}(\mathbf{r}_1,\mathbf{r}_2,\tau)\cdot\boldsymbol{\mathcal{E}}(\mathbf{r}_2,\mathbf{r}_1,-\tau)]}{\mathrm{tr}\,\mathbf{J}(\mathbf{r}_1)\,\mathrm{tr}\,\mathbf{J}(\mathbf{r}_2)}. \tag{6}$$

The degree of coherence is real-valued, invariant under unitary transformations at the two points, and bounded as $0 \leqslant \gamma_{\mathrm{EM}}(\mathbf{r}_1,\mathbf{r}_2,\tau) \leqslant 1$. The lower limit refers to complete incoherence and the upper limit to full coherence. Physically, the degree of coherence characterizes the field's ability to interfere. However, in the electromagnetic context, the interference does not show up only as intensity fringes but also as polarization modulation. The above definition is consistent with this interpretation, as it describes the modulation of all four Stokes parameters in a Young's interference arrangement [11].

Third, the intensity fringe visibility in a Young's two-pinhole experiment, $\gamma_{\mathrm{W}}(\mathbf{r}_1,\mathbf{r}_2,\tau)$, is given by [12]

$$\gamma_{\mathrm{W}}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \frac{\mathrm{tr}\,\boldsymbol{\mathcal{E}}(\mathbf{r}_1,\mathbf{r}_2,\tau)}{\sqrt{\mathrm{tr}\,\mathbf{J}(\mathbf{r}_1)\,\mathrm{tr}\,\mathbf{J}(\mathbf{r}_2)}}. \tag{7}$$

This quantity is also sometimes called the "degree of coherence" promoted by an analogous interpretation in the scalar-field context. Its magnitude is limited to the interval $0 \leqslant |\gamma_{\mathrm{W}}(\mathbf{r}_1,\mathbf{r}_2,\tau)| \leqslant 1$, but unlike Eq. (6) it is not invariant under different unitary transformations at the two points.

In this work we consider polarization and coherence properties only at a single point in space, i.e., $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, and so from now on we will not explicitly show the spatial dependence of the quantities.

III. CHARACTERIZATION OF POLARIZATION DYNAMICS IN 3D FIELDS

In this section we introduce two methods for characterizing the time evolution of the instantaneous polarization state of the field. The first, a geometrical approach, makes use of the Poincaré vectors, whereas the other one, an energy-based approach, relies on the Jones vectors. Both formalisms lead to the same conclusions on the field's polarization fluctuations.

A. Poincaré-vector formalism for 3D fields

The 3×3 polarization matrix $\mathbf{J}$ of Eq. (3) can be uniquely decomposed using the identity matrix and the eight Gell-Mann matrices [13]. The expansion coefficients are the nine (time-averaged) generalized Stokes parameters [1,7]. This result is analogous to the known expression of the 2×2 polarization matrix in terms of the unit matrix and the three Pauli spin matrices with the usual Stokes parameters of beam field as the expansion coefficients [8]. The time averages in the Stokes parameters are over a much longer interval than the time scale characterizing the polarization fluctuations, and hence they cannot, as such, provide information on the time evolution of the instantaneous polarization state of random light. For this purpose we define the instantaneous 3D Stokes parameters $\Lambda_k(t)$, $k = 0,\ldots,8$, as

$$\Lambda_0(t) = |E_x(t)|^2 + |E_y(t)|^2 + |E_z(t)|^2, \tag{8a}$$

$$\Lambda_1(t) = \tfrac{3}{2}[E_x^*(t)E_y(t) + E_y^*(t)E_x(t)], \tag{8b}$$

$$\Lambda_2(t) = \tfrac{3}{2}i[E_x^*(t)E_y(t) - E_y^*(t)E_x(t)], \tag{8c}$$

$$\Lambda_3(t) = \tfrac{3}{2}[|E_x(t)|^2 - |E_y(t)|^2], \tag{8d}$$

$$\Lambda_4(t) = \tfrac{3}{2}[E_x^*(t)E_z(t) + E_z^*(t)E_x(t)], \tag{8e}$$

$$\Lambda_5(t) = \tfrac{3}{2}i[E_x^*(t)E_z(t) - E_z^*(t)E_x(t)], \tag{8f}$$

$$\Lambda_6(t) = \tfrac{3}{2}[E_y^*(t)E_z(t) + E_z^*(t)E_y(t)], \tag{8g}$$

$$\Lambda_7(t) = \tfrac{3}{2}i[E_y^*(t)E_z(t) - E_z^*(t)E_y(t)], \tag{8h}$$

$$\Lambda_8(t) = \tfrac{\sqrt{3}}{2}[|E_x(t)|^2 + |E_y(t)|^2 - 2|E_z(t)|^2]. \tag{8i}$$

The physical interpretations of the 3D Stokes parameters are similar to those of the traditional four Stokes parameters of 2D fields [7].

In full analogy with the 2D formalism [1], we may define the instantaneous Poincaré vector as

$$\mathbf{\Lambda}(t) = \sum_{k=1}^{8} \Lambda_k(t)\hat{u}_k, \tag{9}$$

where $\hat{u}_k$, with $k = 1,\ldots,8$, are orthonormal vectors spanning an eight-dimensional real vector space. The Poincaré vector $\mathbf{\Lambda}(t)$ uniquely defines the polarization state of the field at time t. By averaging $\mathbf{\Lambda}(t)$ and making use of Eqs. (8a)–(8i), the 3D degree of polarization, Eq. (5), can be re-expressed as

$$P_3^2 = \frac{\langle \mathbf{\Lambda}(t)\rangle \cdot \langle \mathbf{\Lambda}(t)\rangle}{3\langle \Lambda_0(t)\rangle^2} = \frac{\sum_{k=1}^{8}\langle \Lambda_k(t)\rangle^2}{3\langle \Lambda_0(t)\rangle^2}. \tag{10}$$

Therefore, for partially polarized fields $\langle \mathbf{\Lambda}(t)\rangle \cdot \langle \mathbf{\Lambda}(t)\rangle \leqslant 3\langle \Lambda_0(t)\rangle^2$. Instead, for the instantaneous Poincaré vector, we find, using straightforward algebra, that

$$\mathbf{\Lambda}(t) \cdot \mathbf{\Lambda}(t) = \sum_{k=1}^{8} \Lambda_k^2(t) = 3\Lambda_0^2(t), \tag{11}$$

which may be physically interpreted to reflect the fact that at any instant of time the field has a certain polarization state. Furthermore, Eq. (11) indicates that the instantaneous Poincaré vector lies on the surface of an eight-dimensional Poincaré sphere whose radius is $\sqrt{3}\Lambda_0(t)$.

In order to separate the study of polarization fluctuations from that of intensity fluctuations, i.e., the changes in the direction of the Poincaré vector from those in its magnitude, we express the Poincaré vector defined in Eq. (9) as

$$\mathbf{\Lambda}(t) = \sqrt{3}\hat{\lambda}(t)\Lambda_0(t), \tag{12}$$

where $\hat{\lambda}(t)$ is the unit-length Poincaré vector. The vector $\hat{\lambda}(t)$ may be used to investigate the polarization fluctuations by employing the scalar product $\hat{\lambda}(t) \cdot \hat{\lambda}(t+\tau)$ as a quantitative measure of the similarity of the state of polarization at times t and $t+\tau$. If the polarization states at the two times are the same, the scalar product yields the value of 1. The minimum value for the scalar product is $-1/2$, which is encountered when the polarization states are orthogonal. This fact, which is verified later in connection with the Jones-vector formalism [Eq. (20) and Appendix], shows that not all regions on the eight-dimensional Poincaré sphere are accessible to the instantaneous Poincaré vector. For example, no polarization states exist for which the scalar product equals -1. For fluctuating light, a measure for the difference in the polarization states at the time separation τ may be established by computing the average $\langle \hat{\lambda}(t) \cdot \hat{\lambda}(t+\tau)\rangle$. In the absence of intensity fluctuations, i.e., if $\Lambda_0(t)$ is constant, this would be an entirely satisfactory quantity to characterize the time evolution of the instantaneous polarization state. However, since often the intensity fluctuates, we use the intensity-weighted average of $\hat{\lambda}(t) \cdot \hat{\lambda}(t+\tau)$ to characterize the dynamics of polarization fluctuations. This leads to a mixing of the polarization and intensity statistics, in which those instants of time with a low intensity, and hence of reduced physical importance, contribute less to the average, whereas the polarization states with high intensity give a larger contribution. Hence such a weighting accounts for the possible correlations between the intensity and the polarization state. A quantity, denoted by $\gamma_{\mathrm{P},3}(\tau)$, with the above character can be defined as

$$\gamma_{\mathrm{P},3}(\tau) = \frac{\langle \mathbf{\Lambda}(t) \cdot \mathbf{\Lambda}(t+\tau)\rangle}{3\langle \Lambda_0(t)\Lambda_0(t+\tau)\rangle}. \tag{13}$$

If the intensity and the polarization state are independent, Eq. (12) implies that $\gamma_{\mathrm{P},3}(\tau)$ reduces to the pure polarization average $\langle \hat{\lambda}(t) \cdot \hat{\lambda}(t+\tau)\rangle$. The autocorrelation function $\gamma_{\mathrm{P},3}(\tau)$ in Eq. (13) has the following properties:

$$-1/2 \leqslant \gamma_{\mathrm{P},3}(\tau) \leqslant 1, \tag{14}$$

$$\gamma_{\mathrm{P},3}(0) = 1. \tag{15}$$

The lower limit in Eq. (14) corresponds to the case that the polarization states at times separated by τ are orthogonal, whereas the upper limit indicates that the polarization states are the same. Equation (15) shows that instantaneously the field has a certain polarization state, consistent with Eq. (11).

Assuming that the field obeys Gaussian statistics, the fourth-order correlation functions in Eq. (13) can be expressed in terms of the second-order ones according to the moment theorem [8]. Doing so, the correlation function of the Poincaré vectors can be developed into the form

$$\gamma_{\mathrm{P},3}(\tau) = \frac{P_3^2 - \frac{1}{2}\gamma_{\mathrm{EM}}^2(\tau) + \frac{3}{2}\,|\gamma_{\mathrm{W}}(\tau)|^2}{1 + \gamma_{\mathrm{EM}}^2(\tau)}, \tag{16}$$

where P_3, $\gamma_{\mathrm{W}}(\tau)$, and $\gamma_{\mathrm{EM}}(\tau)$ are given in Eqs. (5), (6), and (7), respectively. Hence we have expressed $\gamma_{\mathrm{P},3}(\tau)$ using quantities that are measurable by (second-order) interferometric techniques.

Since the field is ergodic, the correlations die out for sufficiently large time separation τ. Thus, in the limit of large time difference we find from Eq. (16) that for fields of Gaussian statistics

$$\lim_{\tau\to\infty} \gamma_{\mathrm{P},3}(\tau) = P_3^2. \tag{17}$$

We want to draw attention to this result: the 3D degree of polarization plays exactly the same role in the characterization of 3D polarization dynamics as the traditional 2D degree of polarization (P_2) does for beam fields. For 2D fields the polarization correlation function defined in terms of the instantaneous three-component Poincaré vector approaches P_2^2 for large time differences [4]. This result has an intuitive explanation related to the fact that the 2×2 polarization matrix can be uniquely decomposed into two parts, one corresponding to a fully polarized and the other to a fully unpolarized field. The correlations associated with the polarized part are present at any time interval, and therefore, in the limit of infinitely large τ, the 2D polarization correlation function naturally depends on the 2D degree of polarization. It is known that the 3×3 polarization matrix cannot be written as a sum of matrices corresponding to a completely polarized and completely unpolarized field [1]. However, the value $\lim_{\tau\to\infty}\gamma_{\mathrm{P},3}(\tau)$ equals the 3D degree of polarization as defined in Ref. [4].

To have a measure to characterize the rate of polarization dynamics of a field, we introduce the polarization time as a time interval over which the polarization state remains essentially unchanged. More explicitly, we define the polarization time as the smallest (positive) solution τ_p to the equation

$$\gamma_{\mathrm{P},3}(\tau_p) = \tfrac{1}{2}. \quad (18)$$

Thus τ_p is the time during which $\gamma_{\mathrm{P},3}(\tau)$ drops from $\gamma_{\mathrm{P},3}(0) = 1$ to $1/2$. The choice of the value $1/2$ is, in principle, arbitrary, and another suitable value depending on the situation could be used as well.

B. Connection between the Poincaré and Jones vectors in 3D fields

A physically more appealing interpretation of the polarization correlation function $\gamma_{\mathrm{P},3}(\tau)$ introduced in Eq. (13) is obtained by expressing the instantaneous Poincaré vectors in terms of the Jones vectors, given at time t by $\mathbf{E}(t) = [E_x(t),\ E_y(t),\ E_z(t)]^T$, where T denotes the transpose. The time-varying polarization state is specified by a normalized Jones vector

$$\hat{e}(t) = \frac{\mathbf{E}(t)}{\sqrt{I(t)}}, \quad (19)$$

where $I(t) = \mathbf{E}^*(t)\cdot\mathbf{E}(t)$ is the intensity of the field. Recalling Eq. (8a) we note that $I(t) = \Lambda_0(t)$. Let then $\hat{\lambda}_1(t)$ and $\hat{\lambda}_2(t)$ be two arbitrary normalized Poincaré vectors and $\hat{e}_1(t)$ and $\hat{e}_2(t)$ be the corresponding normalized Jones vectors. For such vector pairs we find that (the proof is outlined in the Appendix)

$$\hat{\lambda}_1(t)\cdot\hat{\lambda}_2(t) = \tfrac{3}{2}|\hat{e}_1^*(t)\cdot\hat{e}_2(t)|^2 - \tfrac{1}{2}, \quad (20)$$

which shows that for orthogonal polarization states $\hat{\lambda}(t)\cdot\hat{\lambda}(t+\tau) = -1/2$, as remarked in connection with the Poincaré-vector analysis. By choosing $\hat{\lambda}_1(t) = \hat{\lambda}(t)$ and $\hat{\lambda}_2(t) = \hat{\lambda}(t+\tau)$ and correspondingly $\hat{e}_1(t) = \hat{e}(t)$ and $\hat{e}_2(t) = \hat{e}(t+\tau)$, multiplying both sides of Eq. (20) by $3\Lambda_0(t)\Lambda_0(t+\tau)$, taking the average, and dividing by $3\langle\Lambda_0(t)\Lambda_0(t+\tau)\rangle$, we obtain

$$\gamma_{\mathrm{P},3}(\tau) = \frac{3}{2}\frac{\langle|\mathbf{E}^*(t)\cdot\mathbf{E}(t+\tau)|^2\rangle}{\langle I(t)I(t+\tau)\rangle} - \frac{1}{2}. \quad (21)$$

Next we discuss the first term on the right-hand side of this equation.

C. Jones-vector formalism for 3D fields

Using the Jones vectors we can express, for a single realization, the fraction of intensity that at time $t+\tau$ is in the polarization state $\hat{e}(t)$ as follows [5]:

$$\gamma_{\mathrm{e}}(t,t+\tau) = \frac{|\hat{e}^*(t)\cdot\mathbf{E}(t+\tau)|^2}{I(t+\tau)} = |\hat{e}^*(t)\cdot\hat{e}(t+\tau)|^2. \quad (22)$$

It is obvious that $1-\gamma_{\mathrm{e}}(t,t+\tau)$ is the fraction of the field intensity that at time $t+\tau$ belongs to the polarization states orthogonal to $\hat{e}(t)$. When the polarization states are the same at the two times then $\gamma_{\mathrm{e}}(t,t+\tau) = 1$, whereas for orthogonal states $\gamma_{\mathrm{e}}(t,t+\tau) = 0$. If the intensity does not fluctuate, the above quantity when time-averaged would provide a physically meaningful measure for the characterization of time evolution of the instantaneous polarization state of a random 3D field. However, we are not allowed to do this simplifying assumption. Following the same logic as in defining $\gamma_{\mathrm{P},3}(\tau)$ of Eq. (13) above, we time-average the intensity-weighted version of $\gamma_{\mathrm{e}}(t,t+\tau)$ (see also Ref. [5]). This leads to the quantity

$$\begin{aligned}\gamma_{\mathrm{E}}(t,t+\tau) &= \langle I(t)I(t+\tau)|\hat{e}^*(t)\cdot\hat{e}(t+\tau)|^2\rangle\\ &= \langle|\mathbf{E}^*(t)\cdot\mathbf{E}(t+\tau)|^2\rangle,\end{aligned} \quad (23)$$

whose maximum value, $\langle I(t)I(t+\tau)\rangle$, is obtained if the polarization states at t and $t+\tau$ are the same. We use this value to normalize Eq. (23), which results in

$$\gamma_{\mathrm{J},3}(\tau) = \frac{\langle|\mathbf{E}^*(t)\cdot\mathbf{E}(t+\tau)|^2\rangle}{\langle I(t)I(t+\tau)\rangle}. \quad (24)$$

This is exactly the term on the right-hand side of Eq. (21), and physically it shows how much energy at time $t+\tau$ is, on average, in the polarization state in which the field was at time t. Therefore, the function $\gamma_{\mathrm{P},3}(\tau)$ in Eq. (21), defined originally in terms of the Poincaré vectors, is a normalized function describing the mean rate of energy exchange between orthogonal polarization states of a fluctuating 3D field.

The polarization correlation function based on the Jones vectors, Eq. (24), has the following properties:

$$0 \leqslant \gamma_{\mathrm{J},3}(\tau) \leqslant 1, \quad (25)$$

$$\gamma_{\mathrm{J},3}(0) = 1, \quad (26)$$

with the same interpretations as Eqs. (14) and (15) have. The lower and upper limits in Eq. (25) correspond to the states that are orthogonal and the same, respectively, and Eq. (26) reflects the fact that the field has a certain polarization state at any instant of time.

Assuming again Gaussian statistics, Eq. (24) can be re-expressed as

$$\gamma_{\mathrm{J},3}(\tau) = \frac{\frac{1}{3} + \frac{2}{3}P_3^2 + |\gamma_{\mathrm{W}}(\tau)|^2}{1+\gamma_{\mathrm{EM}}^2(\tau)}, \quad (27)$$

where P_3, $\gamma_{\mathrm{W}}(\tau)$, and $\gamma_{\mathrm{EM}}(\tau)$ are found from Eqs. (5), (6), and (7), respectively. We note that in the large time-interval limit

$$\lim_{\tau\to\infty}\gamma_{\mathrm{J},3}(\tau) = \left(2P_3^2+1\right)/3. \quad (28)$$

When the field is fully polarized, $P_3 = 1$, the polarization state does not evolve, and $\lim_{\tau\to\infty}\gamma_{\mathrm{J},3}(\tau) = 1$, as expected. Instead, for an unpolarized field, $P_3 = 0$, we find that $\lim_{\tau\to\infty}\gamma_{\mathrm{J},3}(\tau) = 1/3$, indicating that regardless of the initial state of polarization one third of the intensity remains, on average, in that state, and two thirds are transferred to the polarization states orthogonal to it. This result is consistent with the fact that for unpolarized 3D fields the average energy is equally distributed between any three orthogonal polarization states and the energy exchange between the states is isotropic.

Finally, we point out that the Poincaré vector approach, Eq. (13), and the Jones vector formalism, Eq. (24), are, according to Eq. (21), connected as

$$\gamma_{\mathrm{P},3}(\tau) = \tfrac{3}{2}\gamma_{\mathrm{J},3}(\tau) - \tfrac{1}{2}. \quad (29)$$

Therefore, both $\gamma_{\mathrm{P},3}(\tau)$ and $\gamma_{\mathrm{J},3}(\tau)$ may equally well be used in assessing the polarization dynamics of three-dimensional fields. Defining the polarization time τ_p in the Jones vector formalism, one can use, for example, the criterion $\gamma_{\mathrm{J},3}(\tau_p) = 2/3$, corresponding to the value 1/2 employed in connection with the Poincaré vectors [Eq. (18)].

IV. EXAMPLES

We apply the polarization-dynamics formalism presented above to three specific 3D fields: a uniformly partially polarized and temporally Gaussian-correlated field, radiation inside a blackbody cavity, and the field at the intersection of three orthogonally propagating and orthogonally linearly polarized beams. The examples are investigated using the Poincaré-vector approach presented in Sec. III A, but the analyses could be carried out equally well with the Jones-vector formalism of Sec. III C. For the polarization time we use the criterion given in Eq. (18).

A. Uniformly partially polarized, temporally Gaussian-correlated 3D field

The first example we consider is a uniformly partially polarized field endowed with Gaussian temporal correlations. The field is characterized by a 3×3 polarization matrix $\mathbf{J}$ (with the corresponding degree of polarization P_3) and a coherence time σ. The coherence matrix of such a field can be expressed as

$$\mathcal{E}(\tau) = \mathbf{J} \exp(-\tau^2/2\sigma^2). \tag{30}$$

Using Eqs. (6) and (7) we find for this field that

$$\gamma_{\mathrm{EM}}^2(\tau) = \tfrac{2}{3}\left(P_3^2 + \tfrac{1}{2}\right)\exp(-\tau^2/\sigma^2), \tag{31}$$

$$\gamma_{\mathrm{W}}(\tau) = \exp(-\tau^2/2\sigma^2), \tag{32}$$

and hence

$$\gamma_{\mathrm{P},3}(\tau) = \frac{3P_3^2 - \left(P_3^2 - 4\right)\exp(-\tau^2/\sigma^2)}{3 + \left(2P_3^2 + 1\right)\exp(-\tau^2/\sigma^2)}. \tag{33}$$

The behavior of $\gamma_{\mathrm{P},3}(\tau)$ as a function of τ/σ is illustrated in Fig. 1. The dotted, dash-dotted, dashed, and solid curves correspond to the cases of $P_3 = 0$, $P_3 = 0.5$, $P_3 = 0.95$, and $P_3 = 1$, respectively. We see that $\gamma_{\mathrm{P},3}(0) = 1$ for any P_3, as it should according to Eq. (15). Furthermore, the solid line implies that for a fully polarized field, $P_3 = 1$, the polarization state does not fluctuate, and hence $\gamma_{\mathrm{P},3}(\tau) = 1$ for all τ. The lower P_3 is, the more steeply $\gamma_{\mathrm{P},3}(\tau)$ decreases, approaching the value P_3^2 for large τ. The dotted curve indicates that for an unpolarized field, $P_3 = 0$, the state of polarization changes significantly during the coherence time σ. Therefore, for a relatively unpolarized field, the coherence time can be considered to be an estimate for the polarization time as well. The horizontal solid line in Fig. 1 identifies the criterion $\gamma_{\mathrm{P},3}(\tau_p) = 1/2$ for the polarization time τ_p. The intersections of this line and the curves for $P_3 = 0$ and $P_3 = 0.5$ are emphasized with the dashed vertical lines, corresponding to polarization times $\tau_p = 0.9\sigma$ and $\tau_p = 1.2\sigma$, respectively. In the cases of $P_3 = 0.95$ and $P_3 = 1$ the polarization time can be considered infinitely long, as the polarization state evolves only slightly or not at all as a function of τ. Similar conclusions on the polarization time were drawn in Ref. [4] for a two-dimensional, uniformly polarized, and Gaussian-correlated field using the 2D degree of polarization and the 2D Poincaré vector formalism.

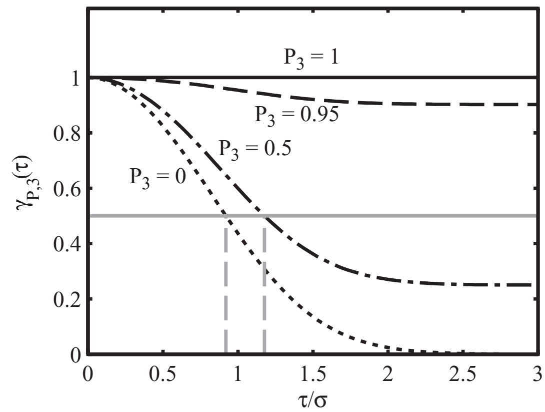

FIG. 1. Behavior of the polarization correlation function $\gamma_{\mathrm{P},3}(\tau)$ of a uniformly partially polarized and temporally Gaussian-correlated 3D field as a function of τ/σ, σ being the coherence time. The curves are for different values of the 3D degree of polarization: $P_3 = 1$ (solid), $P_3 = 0.95$ (dashed), $P_3 = 0.5$ (dash-dotted), and $P_3 = 0$ (dotted). The horizontal gray line emphasizes the criterion $\gamma_{\mathrm{P},3}(\tau_p) = 1/2$ for the polarization time τ_p, and the dashed vertical lines indicate the polarization times for $P_3 = 0$ and $P_3 = 0.5$, which are $\tau_p = 0.9\sigma$ and $\tau_p = 1.2\sigma$, respectively.

B. Blackbody field in a cavity

The polarization dynamics of blackbody radiation emanating from a small opening in a closed cavity was recently investigated, and the polarization times for various cavity temperatures were assessed [4,5]. For such a far-field analysis the 2D Poincaré and Jones vector formalisms are adequate. Here we investigate the time evolution of the polarization state of the blackbody field *inside* the cavity, which necessitates a three-dimensional treatment introduced in Sec. III.

The electric mutual coherence tensor for blackbody field inside a cavity is [14]

$$\mathcal{E}_{ij}(\tau) = \frac{16}{\pi}\frac{(k_B T)^4}{(\hbar c)^3}\zeta(4, 1 + ik_B T\tau/\hbar)\delta_{ij}, \quad i,j \in (x,y,z), \tag{34}$$

where k_B is the Boltzmann constant, T is the absolute temperature, $\hbar$ is the reduced Planck constant, c is the speed of light in vacuum, $\zeta(s,a)$ is the generalized Riemann zeta function $\zeta(s,a) = \sum_{n=0}^{\infty}(a+n)^{-s}$ as given in Ref. [14], and δ_{ij} is the Kronecker delta symbol. Use of Eq. (34) in Eqs. (5), (6), and (7) implies, respectively, that

$$P_3 = 0, \tag{35}$$

$$\gamma_{\mathrm{EM}}(\tau) = |90\zeta(4, 1 + ik_B T\tau/\hbar)|/(\sqrt{3}\pi^4), \tag{36}$$

$$\gamma_{\mathrm{W}}(\tau) = 90\zeta(4, 1 + ik_B T\tau/\hbar)/\pi^4, \tag{37}$$

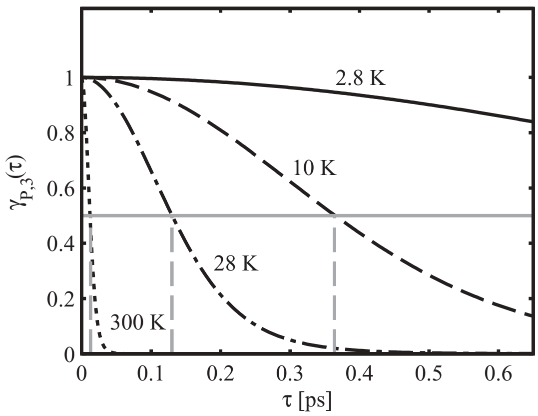

FIG. 2. Behavior of the polarization correlation function $\gamma_{\mathrm{P},3}(\tau)$ as a function of τ for a blackbody field in a cavity at several temperatures: 2.8 K (solid line), 10 K (dashed line), 28 K (dash-dotted line), and 300 K (dotted line). The horizontal solid line shows the condition $\gamma_{\mathrm{P},3}(\tau_p) = 1/2$ for the polarization time τ_p, and the vertical dashed lines indicate its values.

and therefore

$$\gamma_{\mathrm{P},3}(\tau) = \frac{4|90\zeta(4,1+ik_BT\tau/\hbar)|^2}{3\pi^8 + |90\zeta(4,1+ik_BT\tau/\hbar)|^2}, \tag{38}$$

which depends only on the temperature, as expected for a blackbody field.

The behavior of $\gamma_{\mathrm{P},3}(\tau)$ for the blackbody field inside a cavity is shown in Fig. 2. The solid, dashed, dash-dotted, and dotted lines correspond to temperatures 2.8 K, 10 K, 28 K, and 300 K, respectively. The temperature of 2.8 K is chosen as it corresponds to the cosmic microwave background radiation. In the limit $\tau \to \infty$ each curve tends to 0 since the blackbody field is unpolarized. The effect of temperature is clearly seen: the higher the temperature, the faster the polarization state evolves in time and the shorter the polarization time. Again the horizontal line indicates the criterion $\gamma_{\mathrm{P},3}(\tau_p) = 1/2$ for the polarization time τ_p, and the vertical dashed lines mark its value for various temperatures. The polarization times at temperatures 10 K, 28 K, and 300 K are, respectively, found to be 365 fs, 130 fs, and 12.2 fs. Although not shown in the figure, the polarization time for 2.8 K is 1.30 ps. The results indicate that inside a thermal cavity the polarization time of a fluctuating 3D electromagnetic field is essentially the same as that of a two-dimensional far field emitted through a small opening in the cavity (cf. Refs. [4,5]).

C. Intersection of three orthogonally propagating beams

The last example we consider is an arrangement of three orthogonally intersecting, linearly and orthogonally polarized optical beams resulting in a 3D electromagnetic field with a tunable degree of polarization. Similar setups are used in laser cooling of atoms in optical molasses.

The field at the intersection of the beams is illustrated in Fig. 3. The beam propagating in the $+x$ direction, indicated by wave vector $\mathbf{k}_x$, is y polarized, having an electric vector $\mathbf{E}^{(x)}(t) = E(t)\hat{u}_y$. The second beam is delayed by a time τ_y with respect to the first one, polarized in the z direction, and directed to travel along the $+y$ axis with wave vector $\mathbf{k}_y$; hence its electric field is $\mathbf{E}^{(y)}(t) = E(t-\tau_y)\hat{u}_z$. The third beam is delayed by τ_z with respect to the first one, x polarized, and propagates along the $+z$ axis with wave vector $\mathbf{k}_z$. Thus its electric vector is $\mathbf{E}^{(z)}(t) = E(t-\tau_z)\hat{u}_x$.

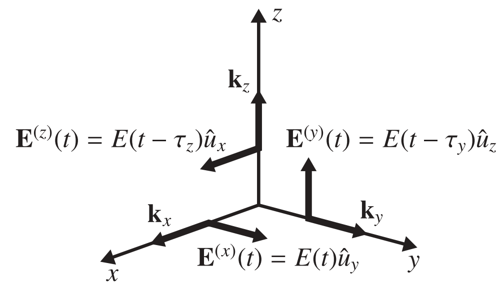

FIG. 3. The intersection of three orthogonally propagating and orthogonally polarized beams. The electric fields of the beams propagating in the x, y, and z directions, specified by wave vectors $\mathbf{k}_i$, with $i \in (x,y,z)$, are $\mathbf{E}^{(x)}(t) = E(t)\hat{u}_y$, $\mathbf{E}^{(y)}(t) = E(t-\tau_y)\hat{u}_z$, and $\mathbf{E}^{(z)}(t) = E(t-\tau_z)\hat{u}_x$, respectively. The parameters τ_y and τ_z are time delays with respect to the beam traveling along the x axis, and $\hat{u}_i$, with $i \in (x,y,z)$, are unit vectors along the Cartesian coordinate axes.

The total electric field at the intersection of the three beams is

$$\mathbf{E}(t) = [E(t-\tau_z),\ E(t),\ E(t-\tau_y)]^T, \tag{39}$$

for which the electric mutual coherence matrix is of the form

$$\begin{aligned}&\boldsymbol{\mathcal{E}}(\tau;\tau_y,\tau_z)\\ &= I\begin{bmatrix}\gamma(\tau) & \gamma(\tau+\tau_z) & \gamma(\tau-\tau_y+\tau_z)\\ \gamma(\tau-\tau_z) & \gamma(\tau) & \gamma(\tau-\tau_y)\\ \gamma(\tau+\tau_y-\tau_z) & \gamma(\tau+\tau_y) & \gamma(\tau)\end{bmatrix}.\end{aligned} \tag{40}$$

In this equation $I = \langle E^*(t)E(t)\rangle$ is the intensity of the beams, and we have introduced the normalized correlation function

$$\gamma(\tau') = \frac{\langle E^*(t)E(t+\tau')\rangle}{\langle E^*(t)E(t)\rangle}, \tag{41}$$

which obeys the relations $\gamma(0) = 1$ and $\gamma(\tau') = \gamma^*(-\tau')$. For the coherence matrix in Eq. (40) we find the parameters of Eqs. (5), (6), and (7) to be

$$P_3^2 = \tfrac{1}{3}[|\gamma(\tau_z)|^2 + |\gamma(\tau_y)|^2 + |\gamma(\tau_y-\tau_z)|^2], \tag{42}$$

$$\gamma_{\mathrm{EM}}^2(\tau) = \tfrac{1}{3}|\gamma(\tau)|^2 + \tfrac{1}{9}\xi(\tau;\tau_y,\tau_z), \tag{43}$$

$$\gamma_{\mathrm{W}}(\tau) = \gamma(\tau), \tag{44}$$

where we used the notation

$$\begin{aligned}\xi(\tau;\tau_y,\tau_z) = {} & |\gamma(\tau+\tau_z)|^2 + |\gamma(\tau-\tau_y+\tau_z)|^2 + |\gamma(\tau-\tau_y)|^2\\ & + |\gamma(\tau-\tau_z)|^2 + |\gamma(\tau+\tau_y-\tau_z)|^2\\ & + |\gamma(\tau+\tau_y)|^2\end{aligned} \tag{45}$$

for brevity.

Inserting Eqs. (42)–(44) into Eq. (16), the polarization correlation function takes on the form

$$\gamma_{P,3}(\tau) = \frac{6[|\gamma(\tau_z)|^2 + |\gamma(\tau_y)|^2 + |\gamma(\tau_y - \tau_z)|^2] - \xi(\tau;\tau_y,\tau_z) + 24|\gamma(\tau)|^2}{18 + 6|\gamma(\tau)|^2 + 2\xi(\tau;\tau_y,\tau_z)}. \quad (46)$$

The behavior of $\gamma_{P,3}(\tau)$ is illustrated in Fig. 4 in the case of Gaussian correlation function

$$\gamma(\tau) = \exp(-\tau^2/2\sigma^2), \quad (47)$$

with σ representing the coherence time. The horizontal solid line again indicates the condition $\gamma_{P,3}(\tau_p) = 1/2$ for the polarization time τ_p, and the vertical dashed lines mark its values. For all values of time delays τ_y and τ_z the polarization correlation function fulfills $\gamma_{P,3}(0) = 1$, as mandated by the definition, and for large τ it approaches P_3^2, which can be verified using Eqs. (42) and (46) and the fact that $\lim_{\tau\to\infty}\gamma(\tau) = 0$. We see from the figure that for $\tau_y = 10\sigma$ and $\tau_z = 2\sigma$ (solid curve), the field is close to a fully unpolarized 3D electromagnetic field, and the polarization time is $\tau_p = 0.89\sigma$. In fact, while at any instant of time the field has some polarization state, if $\tau_y,\tau_z,|\tau_z - \tau_y| \gg \sigma$, the orthogonal electric field components are uncorrelated, and, on average, the field is 3D unpolarized. For $\tau_y = \tau_z = 10\sigma$ (dotted curve) and $\tau_y = \tau_z = \sigma$ (dash-dotted curve) the field is 3D partially polarized, and the related polarization times are $\tau_p = 1.3\sigma$ and $\tau_p = 1.5\sigma$, respectively. In the latter case, the large-interval limit of $\gamma_{P,3}(\tau) = P_3^2$ is larger than 1/2, but the polarization time could still be defined. For relatively uncorrelated 3D fields the curves show dips at $\tau = \tau_y$, $\tau = \tau_z$, and $\tau = |\tau_y - \tau_z|$, corresponding to the time intervals at which two of the orthogonally polarized beams fully correlate. In particular, the solid curve takes on values less than zero at the dips, indicating that, when averaged over time, at these separations more than two-thirds of the energy has shifted to polarization states orthogonal to the initial state. A similar phenomenon was observed in Ref. [5] for a fluctuating 2D field. For the parameters $\tau_y = 0.5\sigma$ and $\tau_z = 0.25\sigma$ (dashed curve) the field is highly polarized, and the polarization time can be taken to be infinitely long.

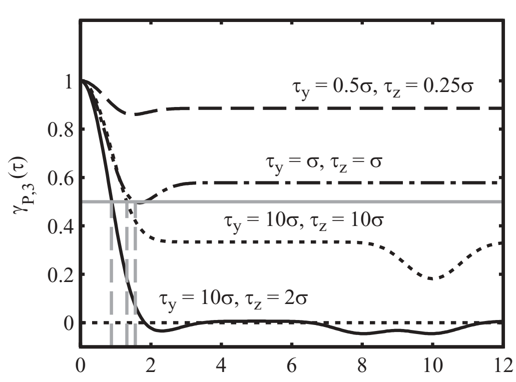

FIG. 4. Polarization correlation function $\gamma_{P,3}(\tau)$ of three intersecting orthogonally linearly polarized, orthogonally propagating, and temporally Gaussian-correlated beams. The curves are for different values of time delays τ_y and τ_z given next to the graphs. The horizontal solid line emphasizes the condition $\gamma_{P,3}(\tau_p) = 1/2$ for the polarization time τ_p, and the vertical dashed lines indicate its values.

V. SUMMARY AND CONCLUSIONS

We investigated the polarization dynamics of random, three-dimensional electromagnetic fields and introduced two intensity-normalized polarization autocorrelation functions for this purpose, one based on a geometric approach with the Poincaré vectors and the other one on energy considerations in terms of the Jones vectors. We also showed that the two approaches yield physically equivalent results on the rate and extent of the time evolution of the instantaneous polarization state, enabling us to define polarization time as a time interval over which the polarization state of the field remains essentially unchanged. In the case of Gaussian statistics, the polarization correlation functions were shown to be expressible in terms of functions characterizing partial polarization and second-order coherence of random 3D fields. We also found that the 3D degree of polarization introduced in Ref. [7] has the same meaning in the 3D polarization dynamics as the traditional 2D degree of polarization does with beam fields. The formalism was illustrated with several examples: a uniformly partially polarized and temporally Gaussian-correlated field, the field inside a blackbody cavity, and the field at the intersection of three orthogonally propagating and orthogonally polarized beams. We expect the results to be useful in applications involving polarization fluctuations in genuinely 3D fields, for instance in light-matter interaction in electromagnetic evanescent fields, intersecting beams in particle trapping and confinement, and optical systems with high numerical apertures.

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APPENDIX: DERIVATION OF EQ. (20)

The connection given in Eq. (20) between the intensity-normalized Poincaré vectors, $\hat{\lambda}(t)$, and the intensity-normalized Jones vectors, $\hat{e}(t)$, is derived in this Appendix. The scalar product of two normalized Poincaré vectors is

$$\hat{\lambda}(t_1)\cdot\hat{\lambda}(t_2) = \frac{\boldsymbol{\Lambda}(t_1)\cdot\boldsymbol{\Lambda}(t_2)}{3\Lambda_0(t_1)\Lambda_0(t_2)} = \frac{\sum_{k=1}^{8}\Lambda_k(t_1)\Lambda_k(t_2)}{3\Lambda_0(t_1)\Lambda_0(t_2)}, \quad (A1)$$

where we used Eqs. (9) and (12) and the fact that $\hat{\lambda}(t)$ is a real vector. Using the definitions of the 3D Stokes parameters given in Eqs. (8), we obtain

$$\sum_{k=1}^{8} \Lambda_k(t_1)\Lambda_k(t_2) = \frac{9}{2}\sum_{i,j} E_i^*(t_1)E_i(t_2)E_j(t_1)E_j^*(t_2) - \frac{3}{2}\sum_i |E_i(t_1)|^2 \sum_j |E_j(t_2)|^2, \quad \text{(A2)}$$

where $i,j \in (x,y,z)$. We identify the sums in the last term as the intensities $\Lambda_0(t_1)$ and $\Lambda_0(t_2)$ and thus find that

$$\hat{\lambda}(t_1)\cdot\hat{\lambda}(t_2) = \frac{3}{2}\frac{\sum_{i,j} E_i^*(t_1)E_i(t_2)E_j(t_1)E_j^*(t_2)}{\Lambda_0(t_1)\Lambda_0(t_2)} - \frac{1}{2}. \quad \text{(A3)}$$

Consider next the scalar product of normalized Jones vectors, whose squared magnitude can be expressed as

$$|\hat{e}^*(t_1)\cdot\hat{e}(t_2)|^2 = \frac{|\mathbf{E}^*(t_1)\cdot\mathbf{E}(t_2)|^2}{I(t_1)I(t_2)}, \quad \text{(A4)}$$

where we used Eq. (19). Since

$$|\mathbf{E}^*(t_1)\cdot\mathbf{E}(t_2)|^2 = \sum_{i,j} E_i^*(t_1)E_i(t_2)E_j(t_1)E_j^*(t_2), \quad \text{(A5)}$$

we observe that

$$|\hat{e}^*(t_1)\cdot\hat{e}(t_2)|^2 = \frac{\sum_{i,j} E_i^*(t_1)E_i(t_2)E_j(t_1)E_j^*(t_2)}{I(t_1)I(t_2)}. \quad \text{(A6)}$$

Since $\Lambda_0(t) = I(t)$, we see that Eqs. (A3) and (A5) imply that

$$\hat{\lambda}(t_1)\cdot\hat{\lambda}(t_2) = \tfrac{3}{2}|\hat{e}^*(t_1)\cdot\hat{e}(t_2)|^2 - \tfrac{1}{2}, \quad \text{(A7)}$$

which is Eq. (20).

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