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Low-temperature thermodynamics of the unitary Fermi gas: Superfluid fraction, first sound, and second sound

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We investigate the low-temperature thermodynamics of the unitary Fermi gas by introducing a model based on the zero-temperature spectra of both bosonic collective modes and fermonic single-particle excitations. We calculate the Helmholtz free energy and from it we obtain the entropy, the internal energy, and the chemical potential as a function of the temperature. By using these quantities and the Landau's expression for the superfluid density we determine analytically the superfluid fraction, the critical temperature, the first sound velocity, and the second sound velocity. We compare our analytical results with other theoretical predictions and experimental data of ultracold atoms and dilute neutron matter.

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I. INTRODUCTION

In a system of fermions the unitary regime is the situation in which $r_e \ll n^{-1/3} \ll |a|$, where n is total number density, r_e is the effective radius of the interaction potential, and a is the s-wave scattering length [1,2]. Thus the system is dilute but the s-wave scattering length a greatly exceeds the average interparticle separation $n^{-1/3}$. It was shown experimentally with dilute and ultracold atomic vapors that such systems exist and are (meta)stable [3]. It has been suggested that also the dilute neutron matter, which is predicted to fill the crust of neutron stars [4], is close to the unitary Fermi gas at a certain density range [5]. At low temperature, the thermodynamic properties of the superfluid unitary Fermi gas can be obtained from the spectrum of elementary excitations, as done many years ago by Landau with the superfluid ^{4}He [6–9]. This approach has been adopted by Bulgac, Drut, and Magierski [10] and also by Nishida [11] to calculate the internal energy and the entropy of the unitary Fermi gas. It has been also suggested by Haussmann, Punk, and Zwerger [12], who proposed a way to calculate the lifetime of fermionic excitations at zero temperature.

In this paper we adopt the Landau approach [6–9] by introducing a thermodynamical model which uses the collective bosonic excitations of the generalized hydrodynamics [13] and the spectrum of fermionic single-particle excitations [14,15]. We calculate the Helmholtz free energy of the two-component balanced unitary Fermi gas and from it we determine the entropy, the internal energy, and the chemical potential. In addition, we use the Laudau's criterion to derive the superfluid fraction and estimate the critical temperature of the system. Finally, by using the obtained superfluid fraction and equations of state we calculate the first sound and the second sound of the unitary gas as a function of the temperature. Our results are compared with previous theoretical predictions [10,16–21] and experimental data [22–26].

II. COLLECTIVE AND SINGLE-PARTICLE EXCITATIONS

For any many-body system the weakly excited states, the so-called elementary excitations, can be treated as a noninteracting gas of excitations [7,9]. In general, these elementary excitations are the result of collective interactions of the particles of the system, and therefore pertain to the system as whole and not to its separate particles [7,9]. For the unitary Fermi gas the mean-field extended BCS theory predicts the existence of fermionic single-particle elementary excitations characterized by an energy gap Δ [1,2]. The inclusion of beyond-mean-field effects, namely quantum fluctuations of the order parameter, gives rise to bosonic collective excitations [1,2], which are density waves reducing to the Bogoliubov-Goldstone-Anderson mode in the limit of small momenta [13].

The detailed properties of these elementary excitations strongly depend on the approximations involved in the theoretical approach [1,2]. As previously stressed, in this paper we extract the details of the zero-temperature elementary excitations from a density functional approach based on fixed-node diffusion Monte Carlo calculation [13] and from recent path integral Monte Carlo simulations [14,15].

It is now well established [1,2,11] that the ground-state energy E_0 of the uniform unitary Fermi gas made of N atoms in a volume V is given by

$$E_0 = \tfrac{3}{5}\xi N\epsilon_F, \tag{1}$$

where E_0 is the ground-state internal energy, $\xi \simeq 0.4$ is a universal parameter [27], and $\epsilon_F = \hbar^2(3\pi^2 n)^{2/3}/(2m)$ is the Fermi energy with $n = N/V$ the number density and N the number of atoms of the uniform system in a volume V.

The exact dispersion relation of elementary (collective and single-particle) excitations is not fully known [1,2]. In Ref. [13] we have found the dispersion relation of collective elementary excitations as

$$\epsilon_{\mathrm{col}}(q) = \sqrt{c_1^2 q^2 + \frac{\lambda}{4m^2}q^4}, \tag{2}$$

where

$$c_1 = \sqrt{\frac{\xi}{3}}\, v_F, \tag{3}$$

is the zero-temperature first sound velocity, with $v_F = (\hbar/m)(3\pi^2 n)^{1/3}$ the Fermi velocity of a noninteracting Fermi gas (see Fig. 1). Notice that the term with λ takes into account the increase of kinetic energy due the spatial variation of the

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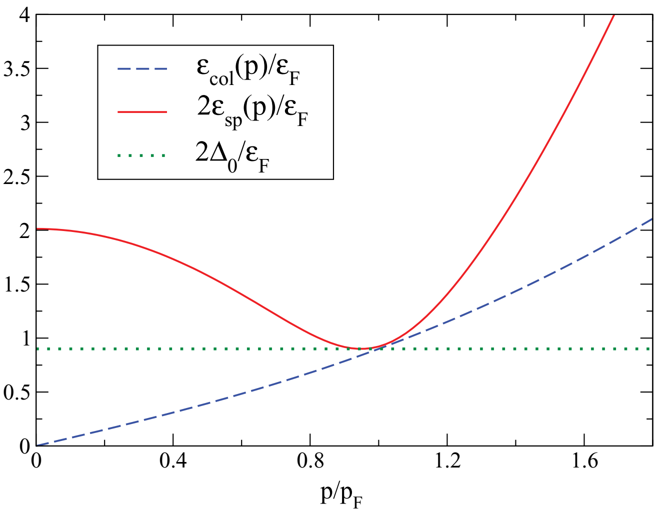

FIG. 1. (Color online) Elementary excitations of the unitary Fermi gas: bosonic collective excitations $\epsilon_{\rm col}(p)$ (dashed line) and fermionic single-particle excitations $2\epsilon_{\rm sp}(p)$ (solid line). The collective mode $\epsilon_{\rm col}(p)$ decays in the single-particle continuum when there is the breaking of Cooper pairs, namely above $\epsilon_{\rm th} = 2\Delta_0$ (dotted line). Zero-temperature parameters of elementary excitations: $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

density [13,28–32]. Expanding the dispersion relation (3) for low momenta we get

$$\epsilon_{\rm col}(q) = c_1 q + \frac{\lambda}{8m^2 c_1} q^3, \tag{4}$$

where the linear term is the familiar phonon dispersion relation (the so-called Bogoliubov-Goldstone-Anderson mode [1,2]) while the cubic correction depends on both the sound velocity c_1 and the gradient parameter λ. Recently Escobedo, Mannarelli, and Manuel [33] have included an additional dispersive term in the phonon spectrum on the basis of the ϵ expansion of the effective field theory [34]. Here the dispersive term depends on λ. In general, a gradient term with λ is essential to describe accurately the zero-temperature surface effects of a trapped system, in particular, with a small number of atoms, where the Thomas-Fermi (local density, i.e., $\lambda = 0$) approximation fails [13]. For the purposes of the present paper fixing $\xi = 0.42$, that is, the Monte Carlo prediction for a uniform gas of Astrakharchik *et al.* [35], we find that the best agreement with Monte Carlo data is obtained with $\lambda = 0.25$.

The collective modes of Eq. (4) are useful to describe correctly only the low-energy density oscillations of the system. At higher energies one expects the emergence of fermionic single-particle excitations starting from the threshold above which there is the breaking of Cooper pairs [1,2,10,14]. At zero temperature these single-particle elementary excitations can be written as

$$\epsilon_{\rm sp}(p) = \sqrt{\left(\frac{p^2}{2m} - \zeta\epsilon_F\right)^2 + \Delta_0^2}, \tag{5}$$

where ζ is a parameter which takes into account the interaction between fermions ($\zeta \simeq 0.9$ according to recent Monte Carlo results [14]) with ϵ_F the Fermi energy of the ideal Fermi gas. Δ_0 is the zero-temperature gap parameter with $2\Delta_0$ the minimal energy to break a Cooper pair [1,2]. The behavior of $\epsilon_{\rm sp}(p)$ is shown in Fig. 1, where we plot also (dotted line) the energy threshold $\epsilon_{\rm th} = 2\Delta_0$ above which there is pair breaking and the continuum of single-particle excitations [36]. Expanding $\epsilon_{\rm sp}(p)$ around the minimum momentum $p_0 = \sqrt{2m\mu} = \zeta^{1/2} p_F$, with $p_F = \sqrt{2m\epsilon_F}$ the Fermi momentum of the ideal Fermi gas, we find

$$\epsilon_{\rm sp}(p) = \Delta_0 + \frac{1}{2m_0}(p - p_0)^2, \tag{6}$$

where the effective mass m_0 is given by

$$m_0 = \frac{m\Delta_0}{2\zeta\epsilon_F}. \tag{7}$$

Notice that the gap energy Δ_0 of the unitary Fermi gas at zero temperature has been calculated with Monte Carlo simulations [14,38] and reads $\gamma = \Delta_0/\epsilon_F \simeq 0.45$.

III. ELEMENTARY EXCITATIONS AND THERMODYNAMICS

As stressed in Sec. I and in the previous section, at very low temperature the thermodynamic properties of the superfluid unitary Fermi gas can be obtained from the collective spectrum given by Eq. (2) and considering an ideal Bose gas of elementary excitations [6–8]. As T increases also the fermionic single-particle excitations, given by Eq. (5) become important. Thus there is also the effect of an ideal Fermi gas of single-particle excitations.

The Helmholtz free energy F_0 of the uniform ground state coincides with the zero-temperature internal energy E_0 and is given by

$$F_0 = E_0 = \tfrac{3}{5}\xi N\epsilon_F. \tag{8}$$

The free energy $F_{\rm col}$ of the collective excitations is instead given by (see also [6–8])

$$F_{\rm col} = \frac{1}{\beta}\sum_{\mathbf{q}} \ln\left[1 - e^{-\beta\epsilon_{\rm col}(q)}\right], \tag{9}$$

while the free energy $F_{\rm sp}$ due to the single-particle excitations is

$$F_{\rm sp} = -\frac{2}{\beta}\sum_{\mathbf{p}} \ln\left[1 + e^{-\beta\epsilon_{\rm sp}(p)}\right]. \tag{10}$$

Here $\beta = 1/(k_B T)$ with T the absolute temperature, and k_B is the Boltzmann constant. The total free energy $F = F_0 + F_{\rm col} + F_{\rm sp}$ reads

$$F = N\epsilon_F \Phi\left(\frac{T}{T_F}\right), \tag{11}$$

where $\Phi(x)$ is a function of the scaled temperature $x = T/T_F$, with $T_F = \epsilon_F/k_B$, given by

$$\Phi(x) = \frac{3}{5}\xi + \frac{3}{2}x\int_0^{+\infty} \ln\left[1 - e^{-\tilde{\epsilon}_{\rm col}(\eta)/x}\right]\eta^2 d\eta \\ - 3x\int_0^{+\infty} \ln\left[1 + e^{-\tilde{\epsilon}_{\rm sp}(\eta)/x}\right]\eta^2 d\eta. \tag{12}$$

Notice that the discrete summations have been replaced by integrals, $\tilde{\epsilon}_{\mathrm{col}}(\eta) = \sqrt{\eta^2(\lambda\eta^2 + 4\xi/3)}$, and $\tilde{\epsilon}_{\mathrm{sp}}(\eta) = \sqrt{(\eta^2-\zeta)^2+\gamma^2}$. We observe that, by using the expansions (4) and (6) for the elementary excitations, adopting the Maxwell-Boltzmann distribution for fermionic single particles instead of the Fermi-Dirac one, and under the further assumption that $\lambda = 0$, this formula becomes exactly the simple model,

$$\Phi(x) \simeq \frac{3}{5}\xi - \frac{\pi^4\sqrt{3}}{80\,\xi^{3/2}}x^4 - \frac{3\sqrt{2\pi}}{2}\zeta^{1/2}\gamma^{1/2}x^{3/2}e^{-\gamma/x}, \tag{13}$$

proposed by Bulgac, Drut, and Magierski [10]. We call this equation the BDM model.

From the Helmholtz free energy F we can immediately obtain the chemical potential μ, that is, defined as

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}. \tag{14}$$

The chemical potential reads

$$\mu = \epsilon_F\left[\frac{5}{3}\Phi\left(\frac{T}{T_F}\right) - \frac{2}{3}\frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)\right], \tag{15}$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$ and one recovers $\mu_0 = \xi\epsilon_F$ in the limit of zero temperature.

The entropy S is related to the free energy F by the formula,

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V}, \tag{16}$$

from which we get

$$S = -Nk_B\Phi'\left(\frac{T}{T_F}\right). \tag{17}$$

In addition, the internal energy E, given by

$$E = F + TS, \tag{18}$$

can be written explicitly as

$$E = N\epsilon_F\left[\Phi\left(\frac{T}{T_F}\right) - \frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)\right]. \tag{19}$$

To conclude this section we observe that the pressure P of the unitary Fermi gas is related to the free energy F by the simple expression,

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T}. \tag{20}$$

We can then write the pressure as

$$P = \frac{2}{3}n\epsilon_F\left[\Phi\left(\frac{T}{T_F}\right) - \frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)\right]. \tag{21}$$

In Fig. 2 we plot various thermodynamical quantities obtained with our model, Eq. (12), as a function of the scaled temperature T/T_F: the scaled free energy $F/(N\epsilon_F)$, the scaled entropy $S/(Nk_B)$, the scaled chemical potential μ/ϵ_F,and the scaled internal energy $E/(N\epsilon_F)$.

A. Gas of dilute and ultracold atoms

It is interesting to compare our model, given by Eqs. (11) and (12), with other theoretical approaches and also with the available experimental data.

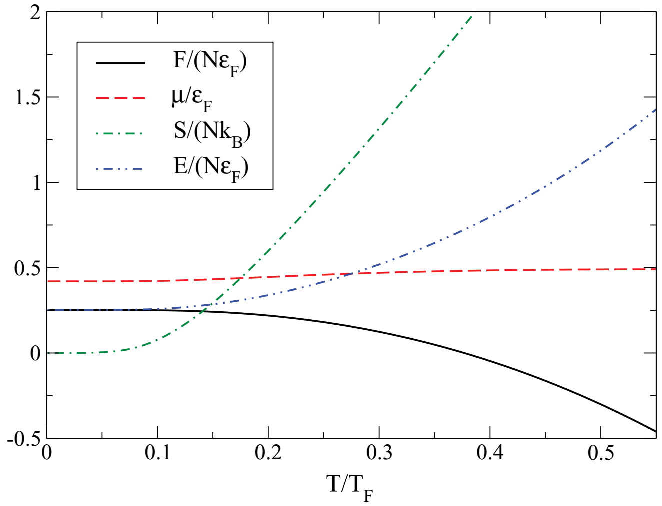

FIG. 2. (Color online) Thermodynamical quantities of the unitary Fermi gas deduced from our model. Zero-temperature parameters of elementary excitations: $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

In Fig. 3 we report the data of internal energy E (top panel) and chemical potential μ (bottom panel) obtained by Bulgac, Drut, and Magierski [10] with their Monte Carlo simulations (solid circles) of the atomic unitary gas. We insert also the very recent experimental data of Horikoshi *et al.* [25] for the unitary Fermi gas of ^{6}Li atoms but extracted from the gas under harmonic confinement (open squares with error bars). In the figure we include the results of two models: our model (solid line), that is, given by Eqs. (19) and (12); the BDM model (dashed line), that is, given by Eqs. (19) and (13).

The critical temperature T_c of the superfluid-normal phase transition has been theoretically estimated to be around $0.2T_F$. In particular, the theoretical estimations are as follows: 0.23 [10], 0.225 [16], 0.152 [17], 0.15 [14], 0.245 [20], and 0.248 [21]. Notice that these values are all much smaller than the prediction of the mean-field extended BCS theory which is $T_c/T_F = 0.50$ [1,2,16]. Recent experiments with ^{40}K [24] and ^{6}Li [25] atoms have measured the condensate fraction of the unitary Fermi gas and both suggest $T_c/T_F = 0.17$. Another very recent experiment [26] has deduced $T_c/T_F = 0.157$ from the behavior of the thermodynamic functions.

Our model is based on zero-temperature elementary excitations and its thermodynamical quantities do not show a phase transition. Nevertheless, the results shown in Fig. 3 strongly suggest that our model works quite well in the superfluid regime, but also slightly above the critical temperature ($T_c \simeq 0.15$) suggested by two theoretical groups [14,17]. We have also verified that the term with λ in Eq. (12) plays a marginal role. The main difference between our model and the BDM model is instead due to the low-momentum expansions of the elementary excitations and to the use of the Maxwell-Boltzmann distribution instead of the Fermi-Dirac one.

B. Dilute neutron matter

Quantum Monte Carlo data of the dilute neutron matter close to the unitarity limit have been produced at finite temperature by Wlazlowski and Magierski [19]. The data have been obtained for the uniform neutron matter at the density

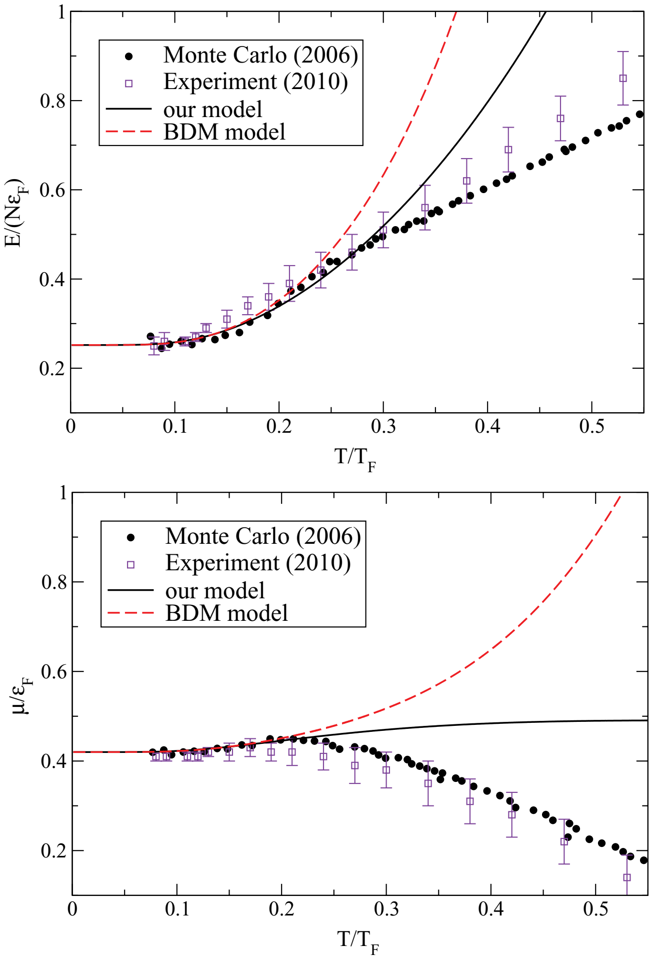

FIG. 3. (Color online) Atomic unitary Fermi gas. (Top panel) Scaled internal energy $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. (Bottom panel) Scaled chemical potential $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Solid circles, Monte Carlo simulations [10]; open squares with error bars, experimental data of Horikoshi *et al.* [25]; solid line, our model [i.e., Eq. (19) with Eq. (12)]; dashed line, BDM model [10] [i.e., Eq. (19) with Eq. (13)]. Zero-temperature parameters of elementary excitations are as follows: $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

$n = 0.003$ fm^{-3}, where $T_F \simeq 5 \times 10^{10}$ Kelvin (for comparison, $T_F \simeq 10^{-7}$ Kelvin in ultracold atomic vapors). In the neutron matter, the effective radius of the neutron-neutron interaction potential is $r_e \simeq 2.8$ fm and the neutron-neutron scattering length is $a \simeq -18.5$ fm. This means that in the calculations of Wlazlowski and Magierski [19] $r_e < n^{1/3} = d = 6.93$ fm $< |a|$. Thus this dilute neutron matter is close but not equal [5] to the unitarity Fermi gas ($r_e \ll d \ll |a|$) [5]. Consequently, the zero-temperature parameters of the elementary excitations, extracted from the spectral weight function [15], are slightly different from those of the unitary Fermi gas with a negligible effective range: $\xi \simeq 0.46$, $\zeta \simeq 0.82$, and $\gamma \simeq 0.29$ [19].

In Fig. 4 we plot the scaled internal energy $E/(N\epsilon_F)$ versus the scaled temperature T/T_F of the nuclear matter obtained by Wlazlowski and Magierski [19] with their Monte Carlo simulations (solid circles with error bars). On the basis of the known zero-temperature parameters of the elementary excitations we can compare their finite-temperature results with our model (solid line) and the BDM model (dashed line). This value is smaller than that of the atomic unitary Fermi gas because the scaled energy gap $\gamma = \Delta/\epsilon_F$ of the neutron matter at $n = 0.003$ fm^{-3} is smaller than the scaled energy gap of the (atomic) unitary Fermi gas. Moreover, the estimated critical temperature for this dilute neutron matter is $T_c/T_F \simeq 0.09$ [19].

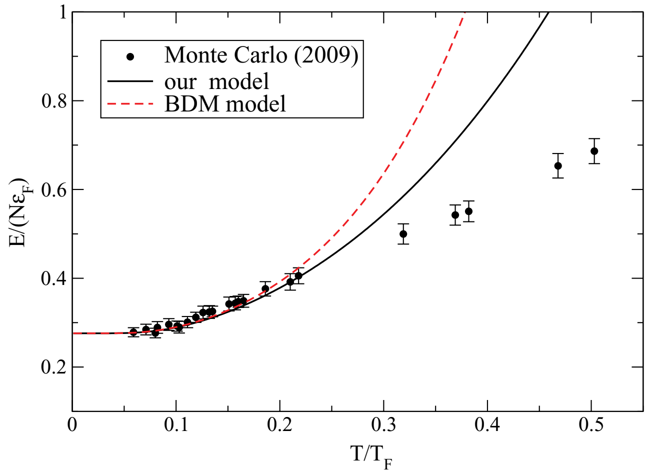

FIG. 4. (Color online) Dilute neutron matter at the density $n = 0.003$ fm^{-3}. Scaled internal energy $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Solid circles, Monte Carlo simulations [19]; solid line, our model [i.e., Eqs. (15) and (19) with Eq. (12)]; dashed line, BDM model [10] [i.e., Eqs. (15) and (19) with Eq. (13)]. Zero-temperature parameters of elementary excitations are as follows: $\xi = 0.46$, $\lambda = 0.25$, $\zeta = 0.82$, and $\gamma = 0.29$.

In agreement with the findings of Fig. 3, also the results of Fig. 4 show that our model (solid line) works quite well in the entire superfluid regime, but also above T_c.

IV. SUPERFLUID FRACTION

The total number density n of the unitary Fermi gas can be written as

$$n = n_s + n_n, \tag{22}$$

where n_s is the superfluid density and n_n is the normal density [1]. At zero temperature $n_n = 0$ and $n = n_s$, while at finite temperature the normal density n_n is finite and increases by increasing the temperature. Correspondingly, the superfluid density n_s decreases and becomes equal to zero at a critical temperature T_c. The normal density is given by

$$n_n = n_{n,\mathrm{col}} + n_{n,\mathrm{sp}}, \tag{23}$$

that is, the sum of the normal density $n_{n,\mathrm{col}}$ due to collective excitations and the normal density $n_{n,\mathrm{sp}}$ due to the single-particle excitations. According to the Landau's approach [6], the gas of collective excitations $\epsilon_{\mathrm{col}}(p)$ which move with drift velocity $\mathbf{v}$ has a distribution $f_B(\epsilon_{\mathrm{col}}(p) - \mathbf{p}\cdot\mathbf{v})$, with

$$f_B(\epsilon_{\mathrm{col}}(p)) = \frac{1}{e^{\beta\epsilon_{\mathrm{col}}(p)} - 1} \tag{24}$$

the Bose-Einstein distribution of collective excitations, and total linear momentum,

$$\mathbf{P} = m \, n_{n,\mathrm{col}} \, \mathbf{v}, \tag{25}$$

where the normal density $n_{n,\mathrm{col}}$ is given by [6–8]

$$n_{n,\mathrm{col}} = -\frac{1}{3} \int \frac{p^2}{m} \frac{df_B(\epsilon_{\mathrm{col}}(p))}{d\epsilon_{\mathrm{col}}} \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}. \tag{26}$$

Similar results hold for the normal density $n_{n,\mathrm{sp}}$ due to single-particle fermionic excitations. It is then easy to derive the superfluid fraction,

$$\frac{n_s}{n} = 1 - \Xi\left(\frac{T}{T_F}\right), \tag{27}$$

where the universal function $\Xi(x)$ of the scaled temperature $x = T/T_F$ is given by

$$\begin{aligned}\Xi(x) = {} & \frac{1}{x}\int_0^{+\infty} \frac{e^{\tilde{\epsilon}_{\mathrm{col}}(\eta)/x}\,\eta^4}{(e^{\tilde{\epsilon}_{\mathrm{col}}(\eta)/x} - 1)^2}\, d\eta \\ & + \frac{2}{x}\int_0^{+\infty} \frac{e^{\tilde{\epsilon}_{\mathrm{sp}}(\eta)/x}\,\eta^4}{(e^{\tilde{\epsilon}_{\mathrm{sp}}(\eta)/x} + 1)^2}\, d\eta,\end{aligned} \tag{28}$$

a where $\tilde{\epsilon}_{\mathrm{col}}(\eta) = \sqrt{\eta^2(\lambda\eta^2 + 4\xi/3)}$, and $\tilde{\epsilon}_{\mathrm{sp}}(\eta) = \sqrt{(\eta^2 - \zeta)^2 + \gamma^2}$. The function $\Xi(x)$ can be approximated as

$$\Xi(x) \simeq \frac{3\sqrt{3}\pi^4}{40\xi^{5/2}} x^4 + \sqrt{\frac{2\pi\gamma}{x}}\, \zeta^{3/2} e^{-\gamma/x}, \tag{29}$$

by using the expansions (4) and (6) for the elementary excitations, adopting the Maxwell-Boltzmann distribution for fermionic single particles instead of the Fermi-Dirac one, and assuming $\lambda = 0$.

In Fig. 5 we plot the superfluid fraction n_s/n of the unitary Fermi gas as a function of the scaled temperature T/T_F, obtained by using Eq. (27) with Eq. (28) (solid line) and Eq. (29) (dashed line). The figure shows that the superfluid fraction becomes zero at $T_c/T_F = 0.34$. This value clearly overestimates the critical temperature with respect to all other beyond-mean-field determinations [10,14,16,17,20,21]. Remarkably, the approximate formula, Eq. (29), is very close to the full one, Eq. (27), up to $T/T_F \simeq 0.15$.

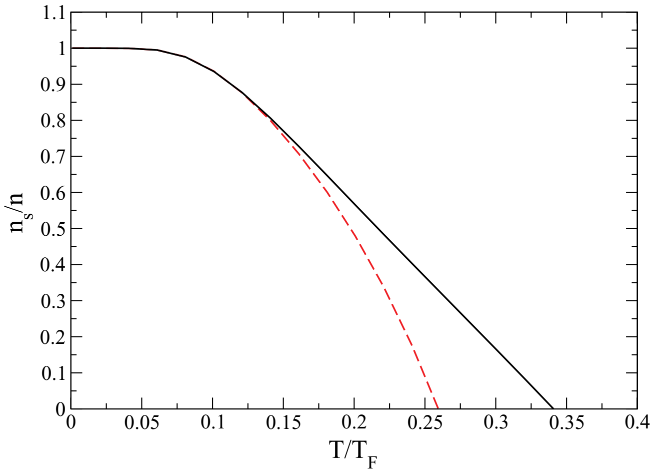

FIG. 5. (Color online) Superfluid fraction n_s/n of the unitary Fermi gas as a function of the scaled temperature T/T_F, obtained by using Eq. (27) with Eq. (28) (solid line) and Eq. (29) (dashed line). Parameters of the elementary excitations are as follows: $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

V. SOUND PROPAGATION AT FINITE TEMPERATURE

The analysis of the sound propagation in the superfluid unitary Fermi gas at finite temperature can be done on the basis of the equations of superfluid hydrodynamics [6,8], where superfluid and normal densities and velocities depend on space and time. In our problem the constitutive equations to be inserted in the equations of superfluid hydrodynamics are Eq. (11) of the entropy S and Eq. (21) of the pressure P.

According to Landau [6,8] any superfluid system admits a density wave, the first sound, where the velocities of superfluid and normal components are in-phase, and the first sound velocity is given by

$$u_1 = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{\bar{S},V}}, \tag{30}$$

where $\bar{S} = S/N$ is the entropy per particle. In addition, the superfluid system supports a temperature wave, called second sound [6,8], where the velocities of superfluid and normal components are out-of-phase, and the second sound velocity reads

$$u_2 = \sqrt{\frac{1}{m} \frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T}\right)_{N,V}} \frac{n_s}{n_n}}. \tag{31}$$

Notice that first sound and second sound are given by Eqs. (30) and (31) in the hypothesis that these two modes are decoupled. As stressed by Taylor *et al.* [18] this hypothesis is fulfilled as long as $R/(R+1) \ll (u_1^2 - u_2^2)/(4u_1^2u_2^2)$, where $R = (\bar{c}_p - \bar{c}_v)/\bar{c}_v$ is the Landau-Placzek ratio [39] with $\bar{c}_p$ the equilibrium specific heat per unit mass at constant pressure and $\bar{c}_v$ the equilibrium specific heat per unit mass at constant density $\bar{c}_v$. This inequality is met also if R is not small due to the fact that the speeds of the first and second sound of the unitary Fermi gas are never very close (see below).

By using our expression (21) for the pressure P and $(\frac{\partial P}{\partial n})_{\bar{S},V} = (5/3)P/n$ [18] the finite-temperature first sound velocity becomes

$$u_1 = v_F \sqrt{\frac{5}{9}\Phi\left(\frac{T}{T_F}\right) - \frac{5}{9}\frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)}. \tag{32}$$

From this formula and Eq. (12) it is immediate to find that for $T \to 0$ one has $u_1 \to c_1 = v_F\sqrt{\xi/3}$. By using our expression (17) for the entropy S the finite-temperature second sound velocity can be instead written as

$$u_2 = v_F \sqrt{-\frac{1}{2} \frac{\Phi'\left(\frac{T}{T_F}\right)^2}{\Phi''\left(\frac{T}{T_F}\right)} \frac{1 - \Xi\left(\frac{T}{T_F}\right)}{\Xi\left(\frac{T}{T_F}\right)}}. \tag{33}$$

From this formula, Eqs. (12) and (27) with Eq. (28), it is not difficult to show that for $T \to 0$ one has $u_2 \to c_1/\sqrt{3} = v_F\sqrt{\xi}/3$. In Fig. 6 we plot the first sound velocity u_1 and second sound velocity u_2 as a function of the scaled temperature T/T_c. These quantities are obtained by using Eqs. (32)

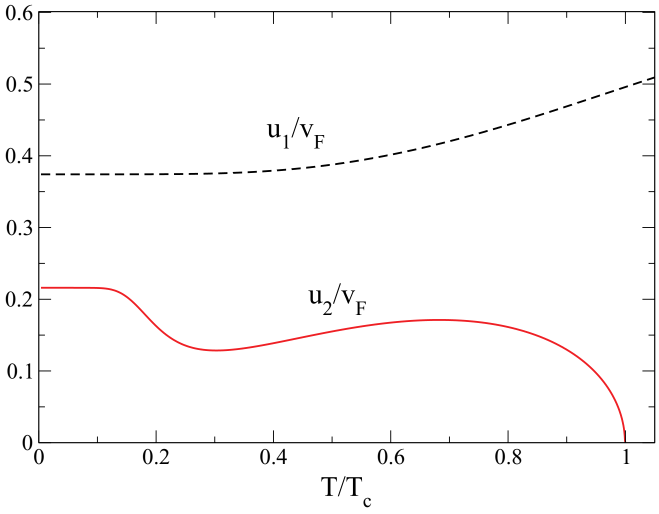

FIG. 6. (Color online) (Dashed line) Scaled first sound velocity u_1/v_F of the unitary Fermi gas as a function of the scaled temperature T/T_c, obtained using Eq. (32) with $\xi = 0.42$, $\lambda = 0.25$, $\gamma = 0.5$, and $\zeta = 0.9$. (Solid line) Scaled second sound velocity u_2/v_F of the unitary Fermi gas as a function of the scaled temperature T/T_c, obtained using Eq. (33) with $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

and (33). The figure shows that u_1 is weakly dependent on the temperature T while u_2 strongly depends on T between $T = 0$ and T_c, where it vanishes because $n_s = 0$. These results are in qualitative agreement with the recent predictions of Taylor *et al.* based on a T-matrix finite-temperature equation of state for the unitary Fermi gas [18].

VI. CONCLUSIONS

We have described the elementary excitations of the unitary Fermi gas as made of collective bosonic excitations and fermionic single-particle ones. This approach has been used many years ago by Landau with the superfluid ^{4}He [6] but it is also presently adopted to model other many-body systems, like atomic nuclei [40]. We stress that our approximation of noninteracting elementary excitations does not take into account the damping of collective modes, which becomes very important by increasing the temperature. We have obtained an analytical expression for the Helmholtz free energy and the superfluid fraction, showing that they are sound to study the thermodynamics of the unitary Fermi system, but only well below the calculated critical temperature of the superfluid phase transition. We believe that this approach to the low-temperature thermodynamics can be extended to the full BCS-BEC crossover of the Fermi gas with two equally populated spin components. In this case the model requires the knowledge of zero-temperature elementary excitations at finite values of the interaction strength $1/(k_F a_F)$.

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