Emergence of coherence in the Mott-insulator–superfluid quench of the Bose-Hubbard model

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We study the quench from the Mott-insulator to the superfluid phase in the Bose-Hubbard model and investigate the spatial-temporal growth of phase coherence, that is, phase locking between initially uncorrelated sites. To this end, we establish a hierarchy of correlations via a controlled expansion into inverse powers of the coordination number $1/Z$. It turns out that the off-diagonal long-range order spreads with a constant propagation speed, forming local condensate patches, whereas the phase correlator follows a diffusionlike growth rate.

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I. INTRODUCTION

A sweep through a symmetry-breaking quantum phase transition is one of the simplest ways to create and amplify quantum correlations, that is, entanglement. As the initial quantum state is symmetric, all directions of symmetry breaking are equally likely and seeded by quantum fluctuations. Furthermore, the diverging response time at the critical point indicates that the many-particle quantum system is driven far away from equilibrium during the sweep. While nearby points will most likely break the initial symmetry in the same direction, two very distant points may spontaneously select different directions of symmetry breaking [1]. As a result, the spatial order parameter distribution after the quench will be inhomogeneous—and its spatial correlations are directly determined by quantum correlations [2].

In contrast to the static properties of classical and quantum phase transitions, much less is known about their nonequilibrium dynamics. Motivated by recent experiments [3,4], as well as from a more fundamental point of view, this field of research is attracting increasing interest [5]. Open questions in this context include: How is the new order parameter established and how fast does it spread? Are there universal scaling laws [6] similar to those in static phase transitions close to the critical point? In this article, we address these questions for the Bose-Hubbard model, which is considered [7] one of the prototypical examples for quantum phase transitions and also relevant for experiments in optical lattices [4]. The Bose-Hubbard Hamiltonian is given by ($\hbar = 1$)

$$\hat{H} = -\frac{J}{Z}\sum_{\mu,\nu} T_{\mu\nu}\hat{a}_\mu^\dagger \hat{a}_\nu + \frac{U}{2}\sum_\mu \hat{n}_\mu(\hat{n}_\mu - 1), \qquad (1)$$

where $\hat{a}_\mu^\dagger$ and $\hat{a}_\nu$ are the creation and annihilation operators at the lattice sites μ and ν, respectively. The lattice structure is encoded in the tunneling matrix $T_{\mu\nu} \in \{0,1\}$ and J denotes the hopping rate. The coordination number $Z = \sum_\nu T_{\mu\nu} \in \mathbb{N}$ counts the number of tunneling neighbors at any given site μ. Finally, U is the on-site interaction with $\hat{n}_\mu = \hat{a}_\mu^\dagger \hat{a}_\mu$. For simplicity, we assume an average filling of one boson per site $\langle \hat{n}_\mu \rangle = 1$.

At a critical ratio of J/U (see the following), this model (1) features a quantum phase transition separating the symmetric Mott-insulator phase with a gap from the superfluid phase with long-range order (i.e., broken symmetry) and Goldstone modes [8]. We start deep in the Mott phase $J = 0$ with $|\Psi_{\rm Mott}\rangle = \prod_\mu \hat{a}_\mu^\dagger |0\rangle$ and quench to the superfluid regime by means of an instantaneous switching to a finite value of J. The theoretical description of the ensuing nonequilibrium many-particle dynamics associated with this process is a nontrivial task for the nonintegrable Bose-Hubbard Hamiltonian (1). There are two major options: numerical computations which are limited to systems of finite size or analytical calculations which require suitable approximations. In order to control the accuracy and consistency of these approximations, they should be based on the expansion in terms of a small or large parameter. For the Bose-Hubbard model, this could be a large filling $\langle \hat{n}_\mu \rangle \gg 1$ [9] or a large number of interaction partners for the extended version [10]. In this article, we assume a large coordination number $Z \gg 1$ and employ a systematic expansion in $1/Z$. A large Z can occur in a large number of spatial dimensions or a large number of tunneling partners. In the following, we focus on the second case and assume two spatial dimensions for simplicity.

II. CORRELATIONS

Let us consider the reduced density matrices for one lattice site $\hat{\rho}_\mu = {\rm Tr}_{\not\mu}\{\hat{\rho}\}$ and for two sites $\hat{\rho}_{\mu\nu} = {\rm Tr}_{\not\mu\not\nu}\{\hat{\rho}\}$, etc. Furthermore, we separate the correlated parts via $\hat{\rho}_{\mu\nu} = \hat{\rho}^{\rm c}_{\mu\nu} + \hat{\rho}_\mu\hat{\rho}_\nu$ as well as $\hat{\rho}_{\mu\nu\lambda} = \hat{\rho}^{\rm c}_{\mu\nu\lambda} + \hat{\rho}^{\rm c}_{\mu\nu}\hat{\rho}_\lambda + \hat{\rho}^{\rm c}_{\mu\lambda}\hat{\rho}_\nu + \hat{\rho}^{\rm c}_{\nu\lambda}\hat{\rho}_\mu + \hat{\rho}_\mu\hat{\rho}_\nu\hat{\rho}_\lambda$, etc. Our derivation is based on the following scaling hierarchy of correlations:

$$\hat{\rho}^{\rm c}_{\mathcal{S}} = O(Z^{1-|\mathcal{S}|}), \qquad (2)$$

where $|\mathcal{S}|$ is the number of lattice sites in the set $\mathcal{S}$, that is, $\hat{\rho}_\mu = O(Z^0)$, $\hat{\rho}^{\rm c}_{\mu\nu} = O(Z^{-1})$, $\hat{\rho}^{\rm c}_{\mu\nu\lambda} = O(Z^{-2})$, etc. This hierarchy (2) is similar to the quantum de Finetti theorem [11], the generalized cumulant expansion [12], and the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [13], but we are considering lattice sites instead of particles. In order to derive the hierarchy (2), we introduce the generating functional $\mathcal{F}(\hat{\alpha}) = \mathcal{F}(\{\hat{\alpha}_\mu\}) = \ln\,{\rm Tr}\{\hat{\rho}\prod_\mu(\mathbf{1}_\mu + \hat{\alpha}_\mu)\}$, where $\hat{\alpha}_\mu$ is an arbitrary operator acting on the lattice site μ. This functional generates all correlated density matrices

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via $\hat{\alpha}_\mu$-derivatives $\hat{\rho}_\mu = \partial\mathcal{F}/\partial\hat{\alpha}_\mu|_{\hat{\alpha}=0}$, as well as $\hat{\rho}^c_{\mu\nu} = \partial^2\mathcal{F}/\partial\hat{\alpha}_\mu\partial\hat{\alpha}_\nu|_{\hat{\alpha}=0}$, etc., where we have used the notation $\langle n_\mu|\partial\mathcal{F}/\partial\hat{\alpha}_\mu|m_\mu\rangle = \partial\mathcal{F}/\partial\langle n_\mu|\hat{\alpha}_\mu|m_\mu\rangle$. Introducing the Liouville superoperators $\widehat{\mathcal{L}}_\mu$ and $\widehat{\mathcal{L}}_{\mu\nu}$ via $i\partial_t\hat{\rho} = [\hat{H},\hat{\rho}] = \sum_\mu \widehat{\mathcal{L}}_\mu\hat{\rho} + \sum_{\mu,\nu}\widehat{\mathcal{L}}_{\mu\nu}\hat{\rho}/Z$, the temporal evolution of $\mathcal{F}$ is given by

$$i\partial_t\mathcal{F}(\hat{\alpha}) = \sum_\mu \mathrm{Tr}_\mu\left[\hat{\alpha}_\mu\,\widehat{\mathcal{L}}_\mu\frac{\partial\mathcal{F}}{\partial\hat{\alpha}_\mu}\right] + \frac{1}{Z}\sum_{\mu,\nu}\mathrm{Tr}_{\mu\nu}\left[(\hat{\alpha}_\mu+\hat{\alpha}_\nu+\hat{\alpha}_\mu\hat{\alpha}_\nu)\,\widehat{\mathcal{L}}_{\mu\nu}\left(\frac{\partial^2\mathcal{F}}{\partial\hat{\alpha}_\mu\partial\hat{\alpha}_\nu}+\frac{\partial\mathcal{F}}{\partial\hat{\alpha}_\mu}\frac{\partial\mathcal{F}}{\partial\hat{\alpha}_\nu}\right)\right]. \tag{3}$$

By taking successive derivatives, we establish the following set of equations for the correlated density matrices

$$\begin{aligned} i\partial_t\hat{\rho}^c_{\mathcal{S}} = {}& \sum_{\mu\in\mathcal{S}}\widehat{\mathcal{L}}_\mu\hat{\rho}^c_{\mathcal{S}} + \frac{1}{Z}\sum_{\mu,\nu\in\mathcal{S}}\widehat{\mathcal{L}}_{\mu\nu}\,\hat{\rho}^c_{\mathcal{S}} + \frac{1}{Z}\sum_{\kappa\notin\mathcal{S}}\sum_{\mu\in\mathcal{S}}\mathrm{Tr}_\kappa\left[\widehat{\mathcal{L}}^S_{\mu\kappa}\hat{\rho}^c_{\mathcal{S}\cup\kappa} + \sum_{\mathcal{P}\subseteq\mathcal{S}\setminus\{\mu\}}^{\mathcal{P}\cup\bar{\mathcal{P}}=\mathcal{S}\setminus\{\mu\}}\widehat{\mathcal{L}}^S_{\mu\kappa}\hat{\rho}^c_{\{\mu\}\cup\mathcal{P}}\,\hat{\rho}^c_{\{\kappa\}\cup\bar{\mathcal{P}}}\right] \\ & + \frac{1}{Z}\sum_{\mu,\nu\in\mathcal{S}}\sum_{\mathcal{P}\subseteq\mathcal{S}\setminus\{\mu,\nu\}}^{\mathcal{P}\cup\bar{\mathcal{P}}=\mathcal{S}\setminus\{\mu,\nu\}}\left\{\widehat{\mathcal{L}}_{\mu\nu}\,\hat{\rho}^c_{\{\mu\}\cup\mathcal{P}}\,\hat{\rho}^c_{\{\nu\}\cup\bar{\mathcal{P}}} - \mathrm{Tr}_\nu\left[\widehat{\mathcal{L}}^S_{\mu\nu}\left(\hat{\rho}^c_{\{\mu,\nu\}\cup\bar{\mathcal{P}}} + \sum_{\mathcal{Q}\subseteq\bar{\mathcal{P}}}^{\mathcal{Q}\cup\bar{\mathcal{Q}}=\bar{\mathcal{P}}}\hat{\rho}^c_{\{\mu\}\cup\mathcal{Q}}\,\hat{\rho}^c_{\{\nu\}\cup\bar{\mathcal{Q}}}\right)\right]\hat{\rho}^c_{\{\nu\}\cup\mathcal{P}}\right\}, \end{aligned} \tag{4}$$

where $\widehat{\mathcal{L}}^S_{\mu\nu} = \widehat{\mathcal{L}}_{\mu\nu} + \widehat{\mathcal{L}}_{\nu\mu}$. A careful inspection of this set of equations shows that the hierarchy in (2) is preserved in time. If all the correlated density matrices $\hat{\rho}^c_{\mathcal{P}}$ on the right-hand side obey the hierarchy $\hat{\rho}^c_{\mathcal{P}} = O(Z^{1-|\mathcal{P}|})$, then the time derivative on the left-hand side does also satisfy (2). This hierarchy $\hat{\rho}^c_{\mathcal{P}} = O(Z^{1-|\mathcal{P}|})$ is a very general result and can be applied to other Hamiltonians such as spin systems, etc. For the Bose-Hubbard model, starting deep in the Mott phase $|\Psi_{\mathrm{Mott}}\rangle = \prod_\mu \hat{a}^\dagger_\mu|0\rangle$ where (2) is trivially satisfied since all correlations vanish, we find that (2) remains valid for a finite time; more precisely, for a time scale of $O(\ln Z)$ limited by the instability of the growing modes, see the following. Let us consider some examples for the aforementioned evolution equation: for one lattice site $\mathcal{S} = \{\mu\}$, we get

$$i\partial_t\hat{\rho}_\mu = \widehat{\mathcal{L}}_\mu\hat{\rho}_\mu + \frac{1}{Z}\sum_\kappa \mathrm{Tr}_\kappa\big[\widehat{\mathcal{L}}^S_{\mu\kappa}\big(\hat{\rho}^c_{\mu\kappa} + \hat{\rho}_\mu\hat{\rho}_\kappa\big)\big]. \tag{5}$$

Since $\hat{\rho}^c_{\mu\kappa}$ is suppressed by $O(1/Z)$, we may neglect this term to the lowest order in $1/Z$ and thereby obtain the Gutzwiller approach [14,15]. Starting in the Mott phase $\langle\hat{a}_\kappa\rangle = 0$, we have $\mathrm{Tr}_\kappa\{\widehat{\mathcal{L}}^S_{\mu\kappa}\hat{\rho}_\kappa\} = 0$ and thus we get $\hat{\rho}_\mu = |1\rangle_\mu\langle 1| + O(1/Z) = \hat{\rho}^0_\mu + O(1/Z)$. The two-point correlation can be studied with $\mathcal{S} = \{\mu,\nu\}$:

$$\begin{aligned} i\partial_t\hat{\rho}^c_{\mu\nu} = {}& \widehat{\mathcal{L}}_\mu\hat{\rho}^c_{\mu\nu} + \frac{1}{Z}\widehat{\mathcal{L}}_{\mu\nu}\big(\hat{\rho}^c_{\mu\nu} + \hat{\rho}_\mu\hat{\rho}_\nu\big) \\ & + \frac{1}{Z}\sum_{\kappa\neq\mu,\nu}\mathrm{Tr}_\kappa\big[\widehat{\mathcal{L}}^S_{\mu\kappa}\big(\hat{\rho}^c_{\mu\nu\kappa} + \hat{\rho}^c_{\mu\nu}\hat{\rho}_\kappa + \hat{\rho}^c_{\nu\kappa}\hat{\rho}_\mu\big)\big] \\ & - \frac{\hat{\rho}_\mu}{Z}\mathrm{Tr}_\mu\big[\widehat{\mathcal{L}}^S_{\mu\nu}\big(\hat{\rho}^c_{\mu\nu} + \hat{\rho}_\mu\hat{\rho}_\nu\big)\big] + (\mu\leftrightarrow\nu). \end{aligned} \tag{6}$$

Again, to the lowest order in $1/Z$, we may neglect the three-point correlation $\hat{\rho}^c_{\mu\nu\kappa} = O(1/Z^2)$ and replace $\hat{\rho}_\mu$ by $\hat{\rho}^0_\mu$, arriving at a closed set of equations for $\hat{\rho}^c_{\mu\nu}$. Furthermore, using $\mathrm{Tr}_\nu(\widehat{\mathcal{L}}^S_{\mu\nu}\hat{\rho}_\nu) = 0$, this simplifies to

$$\begin{aligned} i\partial_t\hat{\rho}^c_{\mu\nu} = {}& (\widehat{\mathcal{L}}_\mu + \widehat{\mathcal{L}}_\nu)\hat{\rho}^c_{\mu\nu} + \frac{1}{Z}\widehat{\mathcal{L}}^S_{\mu\nu}\hat{\rho}^0_\mu\hat{\rho}^0_\nu \\ & + \frac{1}{Z}\sum_{\kappa\neq\mu,\nu}\mathrm{Tr}_\kappa\big(\widehat{\mathcal{L}}^S_{\mu\kappa}\hat{\rho}^c_{\nu\kappa}\hat{\rho}^0_\mu + \widehat{\mathcal{L}}^S_{\nu\kappa}\hat{\rho}^c_{\mu\kappa}\hat{\rho}^0_\nu\big) \\ & + O(1/Z^2). \end{aligned} \tag{7}$$

Introducing $\hat{p}_\mu = |1\rangle_\mu\langle 2|$ and $\hat{h}_\mu = |0\rangle_\mu\langle 1|$ as local particle and hole operators, we find that their correlation functions $f^{11}_{\mu\nu} = \mathrm{Tr}\{\hat{\rho}\,\hat{h}^\dagger_\mu\hat{h}_\nu\}$, $f^{12}_{\mu\nu} = \mathrm{Tr}\{\hat{\rho}\,\hat{h}^\dagger_\mu\hat{p}_\nu\}$, $f^{21}_{\mu\nu} = \mathrm{Tr}\{\hat{\rho}\,\hat{p}^\dagger_\mu\hat{h}_\nu\}$, and $f^{22}_{\mu\nu} = \mathrm{Tr}\{\hat{\rho}\,\hat{p}^\dagger_\mu\hat{p}_\nu\}$, obey a closed linear system of equations. Correlation functions $f^{nm}_{\mu\nu}$ containing higher occupation numbers $n \geqslant 3$ and $m \geqslant 3$ decouple and obey a homogeneous set of equations without the source terms stemming from $\widehat{\mathcal{L}}^S_{\mu\nu}\hat{\rho}^0_\mu\hat{\rho}^0_\nu$. Thus, they are trivially zero assuming the Mott state initially—up to the accuracy $O(1/Z^2)$ under consideration.

Assuming discrete translational invariance, we may simplify the equations via a Fourier transformation

$$\begin{aligned} (i\partial_t - U + 3JT_{\boldsymbol{k}})f^{12}_{\boldsymbol{k}} &= -\sqrt{2}JT_{\boldsymbol{k}}\big(f^{11}_{\boldsymbol{k}} + f^{22}_{\boldsymbol{k}} + 1\big), \\ i\partial_t f^{11}_{\boldsymbol{k}} = i\partial_t f^{22}_{\boldsymbol{k}} &= \sqrt{2}JT_{\boldsymbol{k}}\big(f^{12}_{\boldsymbol{k}} - f^{21}_{\boldsymbol{k}}\big), \end{aligned} \tag{8}$$

with $f^{21}_{\boldsymbol{k}} = (f^{12}_{\boldsymbol{k}})^*$. We find that $f^{11}_{\boldsymbol{k}} = f^{22}_{\boldsymbol{k}}$, indicating an effective particle-hole symmetry. The evolution is fully determined by J, U, and the Fourier transform of the tunneling matrix

$$T_{\boldsymbol{k}} = \frac{1}{Z}\sum_\mu T_{\mu\nu}e^{i\boldsymbol{k}\Delta\boldsymbol{r}_{\mu\nu}} = 1 - \frac{\boldsymbol{k}^2}{2m^*} + O(\boldsymbol{k}^4). \tag{9}$$

Assuming discrete rotational symmetry of the lattice, we obtain exact isotropy at small $\boldsymbol{k}$ with a unique effective mass m^*; otherwise one would have $m^*_x \neq m^*_y$, etc. For large Z, the sum over lattice sites μ in (9) involves many terms and thus only small wave numbers yield significant contributions, corresponding to the effective mass being small $m^* = O(1/Z)$.

III. TWO-POINT FUNCTION

The linear system in (8) can be rewritten in matrix form $i\partial_t \mathfrak{f}_{\boldsymbol{k}} = [\mathfrak{L}_{\boldsymbol{k}},\mathfrak{f}_{\boldsymbol{k}}] + \mathfrak{s}_{\boldsymbol{k}}$ with $\mathfrak{f}_{\boldsymbol{k}} = (f_{\boldsymbol{k}}^{nm})$, the source term $\mathfrak{s}_{\boldsymbol{k}}$, and the Liouvillian matrix $\mathfrak{L}_{\boldsymbol{k}}$, whose eigenvalues $\omega_{\boldsymbol{k}}^{\mathrm{p/h}}$ are associated with quasiparticle/quasihole excitations [14]. As one would expect from the particle-hole symmetry, two of the four eigenfrequencies of the full linear system in (8) vanish and the other two read

$$\pm\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}^{\mathrm{p}} - \omega_{\boldsymbol{k}}^{\mathrm{h}} = \sqrt{U^2 - 6JUT_{\boldsymbol{k}} + J^2T_{\boldsymbol{k}}^2}. \qquad (10)$$

After the quench from $J_{\mathrm{in}} \ll U$ to $J_{\mathrm{out}} = J$, the set of equations in (8) is solved for the initial conditions $\langle \hat{h}_\mu^\dagger \hat{h}_\nu\rangle_0 = \delta_{\mu\nu}$ and $\langle \hat{h}_\mu^\dagger \hat{p}_\nu\rangle_0 = \langle \hat{p}_\mu^\dagger \hat{h}_\nu\rangle_0 = \langle \hat{p}_\mu^\dagger \hat{p}_\nu\rangle_0 = 0$. Inserting $\hat{a}_\mu = \hat{h}_\mu + \sqrt{2}\,\hat{p}_\mu + \hat{r}_\mu$, where the remaining terms $\hat{r}_\mu$ containing higher occupation numbers do not contribute, we obtain

$$\langle \hat{a}_\mu^\dagger(t)\hat{a}_\nu(t)\rangle = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}(\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)} \frac{4JUT_{\boldsymbol{k}}}{N} \frac{1-\cos(\omega_{\boldsymbol{k}}t)}{\omega_{\boldsymbol{k}}^2}, \qquad (11)$$

where N denotes the total number of lattice sites. This prediction could be experimentally verified. After preparing bosonic atoms in an optical lattice deep in the Mott state $J = 0$ and quenching it to a finite $J_{\mathrm{out}} = J$, time-of-flight images for varying time intervals t after the quench yield the Fourier transform of $\langle \hat{a}_\mu^\dagger(t)\hat{a}_\nu(t)\rangle$ and allow the determination of $\omega_{\boldsymbol{k}}$ [4]. In the Mott regime $J_{\mathrm{out}} < J_{\mathrm{cr}}$, this sweeping method allows not only the measurement of the gap value $\omega_{\boldsymbol{k}=0}$, which is a signature of the insulator phase, but also the $\boldsymbol{k}$ dependence of the energy $\omega_{\boldsymbol{k}}$ required for particle-hole pair creation.

The expression under the square root in (10) has two zeros $J_\pm = U(3 \pm \sqrt{8})$ for $k = 0$. The first one marks the critical point $J_{\mathrm{cr}} = J_-$, see, for example, [14]. For $J < J_{\mathrm{cr}}$, that is, in the Mott phase, all modes are stable $\omega_{\boldsymbol{k}} \in \mathbb{R}$. For $J_{\mathrm{cr}} < J < U$ (which is in the superfluid regime), all modes with wave numbers below the value k_{cr} given by $JT_{k_{\mathrm{cr}}} = J_{\mathrm{cr}}$ become unstable and the mode $k = 0$ yields the fastest growth. For $J > U$, on the other hand, a finite wave number $k_* > 0$ yields the fastest growth and dominates the evolution of $\langle \hat{a}_\mu^\dagger(t)\hat{a}_\nu(t)\rangle$. In complete analogy to [16], one finds that the temporal growth and the spatial dependence factorize $\langle \hat{a}_\nu^\dagger(t)\hat{a}_\mu(t)\rangle \approx \mathcal{J}_0(k_*|\boldsymbol{r}_\mu - \boldsymbol{r}_\nu|)f(t)$ with the Bessel function $\mathcal{J}_0$. In this case, the correlations behave in a way which is drastically different from (13). Finally, for $J > J_+$, long-wavelength modes become stable again, $\omega_{\boldsymbol{k}=0} \in \mathbb{R}$, and only the modes within a finite k-interval grow.

In order to search for universal behavior close to the critical point, we study a quench not too far into the superfluid regime, that is, $J = J_{\mathrm{cr}}(1+\varepsilon)$ with $0 < \varepsilon \ll 1$. In this case, the dispersion curve $\omega_{\boldsymbol{k}}^2$ dips below zero for small $\boldsymbol{k}$ only, and thus the growth of correlations can be determined using the long-wavelength approximation

$$\omega_{\boldsymbol{k}} \approx i\sqrt{\gamma^2 - c^2\boldsymbol{k}^2}, \qquad (12)$$

where $c^2 = 3J(U-J)/m^* = O(Z)$ is a velocity scale. Note that the growth rate $\gamma^2 \sim J - J_{\mathrm{cr}}$ strongly depends on the distance to the critical point, whereas c^2 is nearly constant. For large lattices $N \gg 1$, the sum over $\boldsymbol{k}$ is well approximated by an integral. For large $\gamma t \gg 1$, this integral becomes dominated by the fastest growing modes and, thus, can be estimated using the saddle-point approximation

$$\langle \hat{a}_\nu^\dagger(t)\hat{a}_\mu(t)\rangle \approx \mathcal{N}(t)\exp(\gamma\sqrt{t^2 - (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)^2/c^2}), \qquad (13)$$

where the time dependence of the normalization factor $\mathcal{N}(t) = O(1/Z)$ is weak (power law) compared with the exponential growth in (13). Focusing on the dominant exponential part in (13), we find a constant propagation speed c of the correlations similar to the Lieb-Robinson bound [17]. Furthermore, there is a universal scaling behavior. Moving toward or away from the critical point through the rescaling $\gamma \to \gamma' = \lambda\gamma$, an analogous rescaling of time $t \to t' = t/\lambda$ and distance $\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r}/\lambda$ leaves the dominant behavior of (13) invariant.

As another application of (13), let us determine the condensate fraction within a compact lattice region $\mathcal{S}$ of size $|\mathcal{S}| \gg 1$. By analogy with the continuum case [18], the condensate fraction is defined via the largest eigenvalue of the two-point correlation function $\langle \hat{a}_\mu^\dagger \hat{a}_\nu\rangle$. Within $\mathcal{S}$, the largest eigenvalue corresponds to the homogeneous mode which is described by the coarse-grained operator $\hat{A}_{\mathcal{S}} = \sum_{\mu\in\mathcal{S}} \hat{a}_\mu/\sqrt{|\mathcal{S}|}$. For $|\mathcal{S}| = O(Z)$, we obtain a macroscopic occupation $N_{\mathcal{S}} = \langle \hat{A}_{\mathcal{S}}^\dagger \hat{A}_{\mathcal{S}}\rangle \gg 1$ after a finite time t. The precise scaling depends on the size $|\mathcal{S}|$. For $|\mathcal{S}| \ll c^2t^2$, the condensate fraction $N_{\mathcal{S}}/|\mathcal{S}|$ is basically independent of $|\mathcal{S}|$ and grows with $\exp\{\gamma t\}$. For $|\mathcal{S}| \gg c^2t^2$, on the other hand, the condensate fraction decays $\sim 1/|\mathcal{S}|$. This suggests that several fragmented condensate patches of size $O(ct)$ form within the region $\mathcal{S}$ which are not yet fully coherent.

These findings motivate the study of the phase correlations. To this end, we introduce a coarse-grained phase operator $\hat{\varphi}_{\mathcal{S}}$ via $\hat{A}_{\mathcal{S}} = \exp\{i\hat{\varphi}_{\mathcal{S}}\}\sqrt{\hat{N}_{\mathcal{S}}}$ with $\hat{N}_{\mathcal{S}} = \hat{A}_{\mathcal{S}}^\dagger \hat{A}_{\mathcal{S}}$. For a macroscopic occupation, $\langle \hat{N}_{\mathcal{S}}\rangle \gg 1$, the number fluctuations are negligible so that $\hat{N}_{\mathcal{S}} \approx N_{\mathcal{S}}$. Consequently, the phase between two non-overlapping regions $\mathcal{S}$ and $\mathcal{S}'$ of sizes $1 \ll |\mathcal{S}|, |\mathcal{S}'| \ll c^2t^2$ correlates according to (for large γt)

$$\langle \exp[i(\hat{\varphi}_{\mathcal{S}} - \hat{\varphi}_{\mathcal{S}'})]\rangle \approx \exp\left(-\frac{\gamma\Delta\boldsymbol{r}^2}{2c^2t}\right), \qquad (14)$$

where $\Delta\boldsymbol{r}$ is the distance between the two regions. The distance over which the phase correlation spreads obeys a diffusionlike law, $\Delta\boldsymbol{r}^2 \sim c^2t/\gamma$. For smaller distances, the relative phases become locked.

As the final example, the four-point correlation

$$\langle \hat{a}_\mu^\dagger \hat{a}_\nu^\dagger \hat{a}_\kappa \hat{a}_\lambda\rangle = \mathrm{Tr}_{\mu\nu\kappa\lambda}\{\hat{\rho}_{\mu\nu\kappa\lambda}\hat{a}_\mu^\dagger \hat{a}_\nu^\dagger \hat{a}_\kappa \hat{a}_\lambda\} = \langle \hat{a}_\mu^\dagger \hat{a}_\kappa\rangle\langle \hat{a}_\nu^\dagger \hat{a}_\lambda\rangle + \langle \hat{a}_\mu^\dagger \hat{a}_\lambda\rangle\langle \hat{a}_\nu^\dagger \hat{a}_\kappa\rangle + O(1/Z^3), \qquad (15)$$

is completely determined by (13) in leading order $1/Z^2$. Despite the similarities, this result is not based on the usual Wick expansion (we have a strongly interacting theory) but on the hierarchy (2) and the properties of the initial Mott state. For example, for $\langle \hat{a}_\kappa\rangle \neq 0$, we would get an additional contribution of the same order from $\hat{\rho}_{\mu\nu\lambda}^{\mathrm{c}} = O(1/Z^2)$. The previous equation allows us to calculate the long-range current correlation:

$$\left\langle \hat{j}^a_\mu(t)\hat{j}^b_\nu(t)\right\rangle \approx \delta_{ab}\frac{\gamma\mathcal{N}^2(t)}{2t(m^*c)^2}\exp\left(2\gamma t-\frac{\gamma\Delta\boldsymbol{r}^2_{\mu\nu}}{c^2t}\right). \quad (16)$$

IV. CONCLUSIONS

In summary, we derived a hierarchy of correlations (2) in order to describe the nonequilibrium dynamics of a lattice Bose gas (1) based on the expansion in inverse powers of the large coordination number $1/Z$. Since our method is based on the evolution equations (4) of reduced density matrices, it can easily be generalized to finite (initial) temperatures and other Hamiltonians such as the Fermi-Hubbard model or spin systems. Apart from a controlled analytic *ab initio* approach, the hierarchy (2) could also facilitate efficient numerical simulations.

The lowest order in $1/Z$ coincides with the Gutzwiller approach, cf. Eq. (5), and the higher orders describe quantum correlations, cf. Eq. (6). This method is applied to calculate the creation and amplification of quantum correlations in a quenched Mott-insulator–superfluid phase transition. We find that the off-diagonal long-range order (13) spreads with a constant velocity and obeys universal scaling laws. The correlator (14) of the phase associated with local condensate patches expands with a diffusionlike law (phase locking). As an example for higher-order correlations, we calculated the four-point function (15) which yields the current correlator (16).

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