

Scattering amplitude of ultracold atoms near the p -wave magnetic Feshbach resonance

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Most of the current theories on the p -wave superfluid in cold atomic gases are based on the effective-range theory for the two-body scattering, where the low-energy p -wave scattering amplitude $f_1(k)$ is given by $f_1(k) = -1/[ik + 1/(\mathcal{V}k^2) + 1/\mathcal{R}]$. Here k is the incident momentum, \mathcal{V} and \mathcal{R} are the k -independent scattering volume and effective range, respectively. However, due to the long-range nature of the van der Waals interaction between two colliding ultracold atoms, the p -wave scattering amplitude of the two atoms is not described by the effective-range theory [J. Math. Phys. **4**, 54 (1963); Phys. Rev. A **58**, 4222 (1998)]. In this paper we provide an explicit calculation for the p -wave scattering of two ultracold atoms near the p -wave magnetic Feshbach resonance. We show that in this case the low-energy p -wave scattering amplitude $f_1(k) = -1/[ik + 1/(\mathcal{V}^{\text{eff}}k^2) + 1/(\mathcal{S}^{\text{eff}}k) + 1/\mathcal{R}^{\text{eff}}]$ where \mathcal{V}^{eff} , \mathcal{S}^{eff} , and \mathcal{R}^{eff} are k -dependent parameters. Based on this result, we identify sufficient conditions for the effective-range theory to be a good approximation of the exact scattering amplitude. Using these conditions we show that the effective-range theory is a good approximation for the p -wave scattering in the ultracold gases of ⁶Li and ⁴⁰K when the scattering volume is enhanced by the resonance.

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I. INTRODUCTION

Recently, ultracold atomic gases with strong p -wave interaction have attracted broad interest both experimentally [1–13], and theoretically [14–63]. The p -wave magnetic Feshbach resonances, which can generate tunable p -wave interatomic interactions, have been observed in the cold gases of ⁴⁰K [1–3], ⁶Li [6,7], ⁶Li-⁸⁷Rb mixture [11], ⁶Li-⁴⁰K mixture [12], and ⁴⁰K-⁸⁷Rb mixture [13]. The p -wave Feshbach molecules have also been created and studied in the gases of ⁴⁰K [4,5] and ⁶Li [6,8–10]. These experimental achievements stimulate theoretical researches on quantum superfluids of ultracold atomic gases with strong p -wave interactions [14–50], as well as the relevant few-body problems [50–63].

Until now, most theories of ultracold atomic gases with strong p -wave interactions [33–36,38–40,50–55] are based on the low-energy expansion of the p -wave scattering amplitude $f_1(k)$ given by the effective-range theory [64]:

$$f_1(k) = -\frac{1}{ik + \frac{1}{\mathcal{V}k^2} + \frac{1}{\mathcal{R}}}. \quad (1)$$

Here \vec{k} is the relative momentum of the two atoms; \mathcal{V} is the scattering volume and \mathcal{R} is the effective range. The effective-range theory for the scattering amplitude is used in both theories of p -wave atomic superfluids [33–36,38–40,50] and related few-body problems [50–55].

However, the effective-range theory is correct only for the *short-range* potentials (e.g., Yukawa potential) [64] which decays faster than any power-law function $r^{-\gamma}$ in the large interatomic distance limit $r \rightarrow \infty$. Here \vec{r} is the relative coordinate between two atoms. However, a realistic interaction between two cold atoms is described by a *long-range* potential dominated by the van der Waals term $-\hbar^2\beta_6^4/(r^6m)$ in the limit $r \rightarrow \infty$. Here m is the atomic mass, \hbar is the Planck constant, and β_6 is the van der Waals length. Due to the long-range nature of the van der Waals potential, the effective-range theory, and the

low-energy expansion (1) of the p -wave scattering amplitude are not applicable any longer [65,66].

In the presence of a p -wave magnetic Feshbach resonance (PMFR) in the ultracold gases of polarized fermionic atoms, the p -wave scattering amplitude of the atomic collision is contributed by both the background potential in the open channel and the bound state in the closed channel. It seems that the long-range nature of the background potential makes the final scattering amplitude inconsistent with the effective-range theory.

Therefore, it is essential to investigate the condition under which the effective-range theory (1) can be used as an approximation of the exact p -wave scattering amplitude under a PMFR. If the effective-range theory provides a good approximation of the scattering amplitude, then previous theories on p -wave superfluid would be applicable; on the other hand, if the exact scattering amplitude is found to deviate significantly from the one in Eq. (1), then previous theories should be modified.

The low-energy p -wave scattering amplitude near a PMFR has been investigated in Refs. [51,53]. However, these studies are based on simplified models of the atomic interaction (e.g., zero background potential [51] or a separable background potential that decays exponentially in the momentum space [53]). The long-range van der Waals potential is not taken into account in either case. Due to these simplifications, the scattering amplitudes given in Refs. [51,53] automatically have the form of Eq. (1) and cannot be used to judge the applicability of the effective-range theory.

In this paper, based on the realistic long-range interatomic potential, we provide an explicit calculation for the low-energy p -wave scattering amplitude of two spin-polarized fermionic atoms near a PMFR, and then discuss the condition under which the effective-range theory can be used as a good approximation. We find sufficient conditions for the effective-range theory to be applicable, and show that for the ultracold gases

of ${}^6\text{Li}$ and ${}^{40}\text{K}$ with the Fermi temperature on the order of and below $1\ \mu\text{K}$, the effective-range theory can be used as a good approximation in the resonance regime where the scattering volume is enhanced.

A. Main results

The main results of this paper are summarized as follows.

In this work we first calculate the exact expression of the low-energy p -wave scattering amplitude with a PMFR. We prove that the scattering amplitude can be expressed as

$$f_{1m_z}(k) = -\frac{1}{ik + \frac{1}{\mathcal{V}^{\text{eff}}(k; B; m_z)k^2} + \frac{1}{\mathcal{S}^{\text{eff}}(k; B; m_z)k} + \frac{1}{\mathcal{R}^{\text{eff}}(k; B; m_z)}}, \quad (2)$$

with B the strength of the magnetic field applied along the z axis. Here m_z is the z component of the angular momentum; \mathcal{V}^{eff} , \mathcal{S}^{eff} , and \mathcal{R}^{eff} are k -dependent scattering parameters. We obtain the general expressions for them. It is pointed out that, the denominator of $f_{1m_z}(k)$ cannot be expressed as a Laurent series with k -independent coefficients because \mathcal{V}^{eff} , \mathcal{S}^{eff} , and \mathcal{R}^{eff} are not analytical functions of k .

Equation (2) shows that, in the presence of a PMFR, the inconsistency of the scattering amplitude with the effective-range theory is displayed in a complicated manner. The low-energy p -wave scattering amplitude in Eq. (2) is different from the one obtained from the standard effective-range theory in the following two senses:

- (1) The scattering parameters (\mathcal{V}^{eff} , \mathcal{S}^{eff} , \mathcal{R}^{eff}) depend on the incident momentum k .
- (2) The term $1/[\mathcal{S}^{\text{eff}}(k; B; m_z)k]$ cannot be included in the effective-range theory.

After obtaining the p -wave scattering amplitude under a PMFR, we discuss the applicability of the effective-range theory as an approximation of the scattering amplitude (2). We find that, in the Bose-Einstein condensate (BEC) side of the PMFR where \mathcal{V}^{eff} and \mathcal{R}^{eff} have the same sign, sufficient conditions for the validity of the effective-range theory are $r_1, r_2 \ll 1$. In the BCS side of the resonance, the sufficient conditions become $r_1, r_2, r_3 \ll 1$. Here r_1, r_2 , and r_3 are defined in Eqs. (38), (33), and (42). If these conditions are satisfied, the scattering amplitude of our system can be approximated as

$$f_{1m_z}(k) \approx -\frac{1}{ik + \frac{1}{\mathcal{V}^{\text{eff}}(0; B; m_z)k^2} + \frac{1}{\mathcal{R}^{\text{eff}}(0; B; m_z)}}, \quad (3)$$

which has the same form as Eq. (1) derived by the effective-range theory.

Qualitatively speaking, the previous sufficient conditions mean that we can use the effective-range theory if the fermionic momentum of the cold gas is sufficiently low, the magnetic field is tuned close enough to the resonance point, and the background scattering potential in the open channel is far away from the zero-energy shape resonance point. For realistic cold gases of fermionic atoms, if the background scattering is far away from the shape resonance, the effective-range theory can usually be used in the entire region where the p -wave interaction is negligible.

The paper is organized as follows. In Sec. II, we calculate the p -wave scattering amplitude near a PMFR, and obtain the

low-energy expansion in (2). The parameters (\mathcal{V}^{eff} , \mathcal{S}^{eff} , \mathcal{R}^{eff}) are expressed in terms of the background scattering parameters and the magnetic field. In Sec. III we discuss and give the sufficient conditions for applicability of the effective-range theory. We further show that these conditions are well satisfied in the cold gases of ${}^{40}\text{K}$ and ${}^6\text{Li}$ when the scattering volume is enhanced by a PMFR, and then the previous results based on the effective-range theory are applicable for these systems. In Sec. IV we conclude this paper. We describe some details of our calculations in the appendices to avoid digressing from the main subjects.

II. LOW-ENERGY SCATTERING AMPLITUDE NEAR THE P -WAVE FESHBACH RESONANCE

A. p -wave phase shifts with PMFR

In this section we calculate the p -wave scattering amplitude in the presence of a PMFR with a magnetic field along the z direction. We begin with the two-channel Hamiltonian for the relative motion of two atoms (Fig. 1):

$$H = \begin{pmatrix} T + V^{\text{(bg)}}(r) & W(r) \\ W(r) & T + V^{\text{(cl)}}(\vec{r}) + \varepsilon(B) \end{pmatrix}, \quad (4)$$

where T is the kinetic energy of relative motion, $V^{\text{(bg)}}(r)$ is the background scattering potential in the open channel, $W(r)$ is the coupling between the open and closed channel, and $V^{\text{(cl)}}(\vec{r})$ is the interaction potential in the closed channel, that has a B -dependent positive threshold $\varepsilon(B)$. In this paper, for simplicity, we assume that the background potential $V^{\text{(bg)}}(r)$ is independent of the direction of \vec{r} and invariant under the $\text{SO}(3)$ rotation. We further assume that, in the closed channel there are only three bare p -wave bound states $|\phi_{\text{res}}^{(m_z)}\rangle$ which are near resonance with the threshold of the open channel. Here $m_z = 0, \pm 1$ is the projected angular momentum along the z axis. The energy difference $E_{m_z}^{\text{(cl)}}(B) = \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)})$ of $|\phi_{\text{res}}^{(m_z)}\rangle$ relative to the open channel is determined by the strength of the magnetic field. The difference between $B_{\text{res}}^{(m_z)}$ with $m_z = 0, \pm 1$ depends on the atomic magnetic dipole. For ${}^6\text{Li}$ which has small magnetic dipole, we have $|B_{\text{res}}^{(0)} - B_{\text{res}}^{(\pm 1)}| \sim 10\ \text{mG}$ [9], while for ${}^{40}\text{K}$ with large magnetic dipole, the gap $|B_{\text{res}}^{(0)} - B_{\text{res}}^{(\pm 1)}|$ becomes as large as $0.5\ \text{G}$ [5].

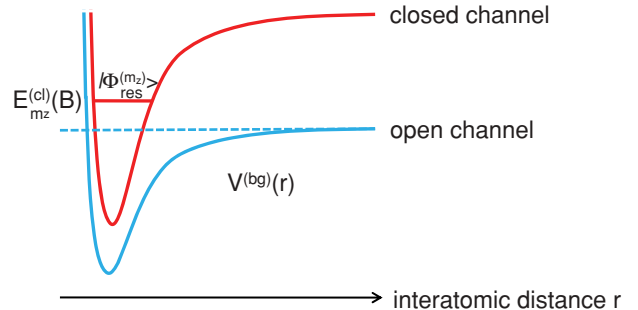


FIG. 1. (Color online) Two-channel model of the p -wave Feshbach resonance.

The p -wave scattering amplitude in the open channel can be defined with the standard scattering theory [64]. To this end we first introduce the two-component stationary scattering state,

$$|\Phi_{\vec{k}}^{(+)}\rangle \equiv \begin{pmatrix} |\phi_{\vec{k}}^{(\text{op}+)}\rangle \\ |\phi_{\vec{k}}^{(\text{cl}+)}\rangle \end{pmatrix} = \Omega_+ \begin{pmatrix} |\vec{k}\rangle \\ 0 \end{pmatrix}, \quad (5)$$

where $|\vec{k}\rangle$ is the eigenstate of the atomic relative momentum with eigenvalue $\hbar\vec{k}$, and Ω_+ is the Møller operator defined as

$$\Omega_+ = \lim_{\tau \rightarrow \infty} e^{-iH\tau/\hbar} e^{iT\tau/\hbar}. \quad (6)$$

In the large interatomic distance limit $r \rightarrow \infty$, the asymptotic behavior of the state $|\Phi_{\vec{k}}^{(+)}\rangle$ can be expressed as

$$\langle \vec{r} | \Phi_{\vec{k}}^{(+)} \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \begin{pmatrix} e^{i\vec{k}\cdot\vec{r}} + f(\hat{r}, \vec{k}) \frac{e^{ikr}}{r} \\ 0 \end{pmatrix}, \quad (7)$$

with $\hat{r} = \vec{r}/r$ and $f(\hat{r}, \vec{k})$ the scattering amplitude which can be further expanded in terms of different partial waves:

$$f(\hat{r}, \vec{k}) = 4\pi \sum_{l m_z} f_{l m_z}(k) Y_l^{m_z}(\hat{r}) Y_l^{m_z}(\hat{k})^*. \quad (8)$$

Here $Y_l^{m_z}(\hat{r})$ is the spherical harmonic function. For a scattering potential with the SO(3) symmetry, the partial wave scattering amplitude only depends on the quantum number of the angular momentum l . In our case, the SO(3) symmetry is broken by the interaction between the atomic magnetic dipole and the magnetic field. Then we have an m_z -dependent scattering amplitude $f_{l m_z}(k)$.

In the case of low-energy scattering between two spin-polarized fermionic atoms, one can ignore all the high partial wave scattering amplitudes $f_{l m_z}(k)$ with $l \geq 2$, and only consider the p -wave amplitudes $f_{1 m_z}(k)$, which can be further expressed in terms of the p -wave phase shifts $\delta_{1 m_z}(k)$:

$$f_{1 m_z}(k) = -\frac{1}{ik - k \cot \delta_{1 m_z}(k)}. \quad (9)$$

During the scattering process, the bare bound state $|\phi_{\text{res}}^{(m_z)}\rangle$ is coupled with the p -wave background scattering states in the open channel and significantly changes the p -wave scattering amplitude $f_{1 m_z}(k)$. This effect can be directly treated via the Feshbach resonance theory (e.g., the methods in Refs. [67,69]).

After a straightforward calculation in Appendix A, we find that the final phase shift $\delta_{1 m_z}(k)$ is the sum of the background phase shift $\delta_1^{(\text{bg})}(k)$ for the background potential $V^{(\text{bg})}(r)$ and a correction $\Delta_{1 m_z}(k)$ given by the closed channel:

$$\delta_1(k) = \delta_1^{(\text{bg})}(k) + \Delta_{1 m_z}(k). \quad (10)$$

Here $\Delta_{1 m_z}(k)$ satisfies

$$-k \cot \Delta_{1 m_z}(k) = \frac{k \hbar^2 k^2 / m - E_{m_z}^{(\text{cl})}(B) - g_{m_z}(k^2)}{\pi \left| \langle \phi_{\text{res}}^{(m_z)} | W | \psi_{k 1 m_z}^{(\text{bg}+)} \rangle \right|^2}. \quad (11)$$

with $|\psi_{k 1 m_z}^{(\text{bg}+)}\rangle$ and $g_{m_z}(k^2)$ given by

$$|\phi_{\vec{k}}^{(\text{bg}+)}\rangle = \left(\frac{2}{m\hbar k} \right)^{\frac{1}{2}} \sum_{l, m_z} |\psi_{k l m_z}^{(\text{bg}+)}\rangle Y_l^{m_z}(\hat{k})^*, \quad (12)$$

$$g_{m_z}(k^2) = \text{Re} \langle \phi_{\text{res}}^{(m_z)} | W G_+^{(\text{bg})}(k^2) W | \phi_{\text{res}}^{(m_z)} \rangle. \quad (13)$$

In the previously mentioned equations we have used the background Green's function $G_+^{(\text{bg})}(k^2)$,

$$G_+^{(\text{bg})}(k^2) = \frac{1}{\hbar^2 k^2 / m - T - V^{(\text{bg})} + i0^+}, \quad (14)$$

and the background scattering state $|\phi_{\vec{k}}^{(\text{bg}+)}\rangle$ defined as

$$|\phi_{\vec{k}}^{(\text{bg}+)}\rangle = |\vec{k}\rangle + G_+^{(\text{bg})}(k^2) V^{(\text{bg})} |\vec{k}\rangle. \quad (15)$$

In the following subsections, we evaluate the low-energy expression of the scattering amplitude $f_{1 m_z}(k)$ by expanding the term $-k \cot \delta_{1 m_z}(k)$ in Eq. (9) in the limit $k \rightarrow 0$. As shown in Eq. (10), the phase shift $\delta_{1 m_z}(k)$ is the sum of $\delta_1^{(\text{bg})}(k)$ and $\Delta_{1 m_z}(k)$. The low-energy behavior of background phase shift $\delta_1^{(\text{bg})}(k)$ is already known to be [65,66]

$$-k \cot \delta_1^{(\text{bg})}(k) = \frac{1}{\mathcal{V}^{(\text{bg})}} \frac{1}{k^2} + \frac{1}{\mathcal{S}^{(\text{bg})}} \frac{1}{k} + \frac{1}{\mathcal{R}^{(\text{bg})}}. \quad (16)$$

Therefore, if we can further obtain the low-energy expansion of term $-k \cot \Delta_{1 m_z}(k)$, then the corresponding expressions of $-k \cot \delta_{1 m_z}(k)$ and $f_{1 m_z}(k)$ can be calculated straightforwardly.

B. The low-energy expansion of $-k \cot \Delta_{1 m_z}(k)$

In this subsection we investigate the expression of $-k \cot \Delta_{1 m_z}(k)$ in the limit $k \rightarrow 0$. To this end, we need to expand both the numerator and the denominator of (11) in the low-energy limit.

In this paper we assume the background scattering volume in the open channel is finite. It can be proved that (Appendix C), in this case the function $g_{m_z}(k^2)$ can be expanded as

$$g_{m_z}(k^2) = g_{m_z}^{(0)} + g_{m_z}^{(2)} k^2 + O(k^3), \quad (17)$$

with $g_{m_z}^{(2)} \leq 0$.

On the other hand, due to the long-range nature of the van der Waals potential, the partial wave scattering state $|\psi_{k 1 m_z}^{(\text{bg}+)}\rangle$ is not an analytical function of the incident momentum k in the neighborhood of $k = 0$ [64]. To investigate the low-energy behavior of $|\psi_{k 1 m_z}^{(\text{bg}+)}\rangle$ and then the denominator of (11), we separate the nonanalytical part of $|\psi_{k 1 m_z}^{(\text{bg}+)}\rangle$ by introducing the background Jost function $\mathcal{J}(k)$ [64] defined by

$$\langle \vec{r} | \psi_{k 1 m_z}^{(\text{bg}+)} \rangle = i^l \frac{1}{\hbar \mathcal{J}(k)} \left(\frac{m}{\pi k} \right)^{\frac{1}{2}} \frac{1}{r} Y_1^{m_z}(\hat{r}) \tilde{F}_{k 1 m_z}^{(\text{bg})}(r). \quad (18)$$

Here $\tilde{F}_{k 1 m_z}^{(\text{bg})}(r)$ is the canonical solution of the radial equation,

$$\left(-\frac{d^2}{dr^2} + V^{(\text{bg})}(r) + \frac{2}{r^2} \right) \tilde{F}_{k 1 m_z}^{(\text{bg})}(r) = k^2 \tilde{F}_{k 1 m_z}^{(\text{bg})}(r), \quad (19)$$

with boundary condition,

$$\tilde{F}_{k 1 m_z}^{(\text{bg})}(r \rightarrow 0) \rightarrow \hat{j}_1(kr), \quad (20)$$

where

$$\hat{j}_1(x) = \frac{\sin x}{x} - \cos x \quad (21)$$

is the first-order regular Riccati-Bessel function [64]. According to the standard scattering theory [64], $\tilde{F}_{k1m_z}^{(\text{bg})}(r)$ is an analytical function of k , and can be expanded as a Talyor series of k :

$$\tilde{F}_{k1m_z}^{(\text{bg})}(r) = \frac{1}{(2n)!} \sum_{n=1}^{\infty} \frac{d^{2n}}{dk^{2n}} \tilde{F}_{k1m_z}^{(\text{bg})}(r) \Big|_{k=0} k^{2n}. \quad (22)$$

It can be shown that all the odd-order terms in the previously mentioned Taylor series vanish. Thus the nonanalytical part of $|\psi_{k1m_z}^{(\text{bg}+)}$ is included in the term with the Jost function $\mathcal{J}(k)$.

Substituting Eqs. (22), (18), and (17) into Eq. (11), we find that in the low-energy limit the factor $-k \cot \Delta_{1m_z}(k)$ takes the form,

$$-k \cot \Delta_{1m_z}(k) = \frac{1}{\mathcal{V}^{(\Delta)}(B; k; m_z)} \frac{1}{k^2} + \frac{1}{\mathcal{R}^{(\Delta)}(B; k; m_z)}. \quad (23)$$

Here we have the k -dependent parameters:

$$\mathcal{V}^{(\Delta)}(B; k; m_z) = -|\mathcal{J}(k)|^{-2} \frac{\pi w_{m_z}}{\mu_{\text{res}}} \frac{1}{B - B_0}; \quad (24)$$

$$\begin{aligned} \mathcal{R}^{(\Delta)}(B; k; m_z) &= |\mathcal{J}(k)|^{-2} \pi w_{m_z} \\ &\times \left[\left(\frac{\hbar^2}{m} - g_{m_z}^{(2)} \right) - \frac{w'_{m_z}}{w_{m_z}} \mu_{\text{res}} (B - B_0) \right]^{-1}, \end{aligned} \quad (25)$$

with the parameters B_0 and w_{m_z} defined by

$$\begin{aligned} B_0 &= B_{\text{res}} - g_{m_z}^{(0)}/\mu_{\text{res}}; \quad (26) \\ w_{m_z} &= \frac{1}{6} \frac{d^3}{dk^3} [|\phi_{\text{res}}^{(m_z)}| W |\psi_{k1m_z}^{(\text{bg}+)}|^2 |\mathcal{J}(k)|^2]_{k=0}; \quad (27) \\ w'_{m_z} &= \frac{1}{120} \frac{d^5}{dk^5} [|\phi_{\text{res}}^{(m_z)}| W |\psi_{k1m_z}^{(\text{bg}+)}|^2 |\mathcal{J}(k)|^2]_{k=0}. \end{aligned}$$

C. The low-energy p -wave scattering amplitude

In the previous subsection we get the expansion (23) of the factor $-k \cot \Delta_{1m_z}(k)$. Substituting Eqs. (23), (16), and (10) into (9), we finally obtain the low-energy behavior of the p -wave scattering amplitude $f_{1m_z}(k)$ in the presence of a PMFR:

$$f_{1m_z}(k) = -\frac{1}{ik + \frac{1}{\mathcal{V}^{\text{eff}}(k; B; m_z)k^2} + \frac{1}{\mathcal{S}^{\text{eff}}(k; B; m_z)k} + \frac{1}{\mathcal{R}^{\text{eff}}(k; B; m_z)}}, \quad (28)$$

where the k -dependent scattering parameters are given by

$$\mathcal{V}^{\text{eff}} = \mathcal{V}^{(\text{bg})} \left(1 - \frac{b_{m_z}}{B - B_0} |\mathcal{J}(k)|^{-2} \right); \quad (29)$$

$$\mathcal{S}^{\text{eff}} = \frac{\mathcal{S}^{(\text{bg})}}{\mathcal{V}^{(\text{bg})^2}} \mathcal{V}^{\text{eff}2}; \quad (30)$$

$$\frac{1}{\mathcal{R}^{\text{eff}}} = \frac{1}{\mathcal{R}^{(\Delta)}} (1 - 2x + x^2) + \frac{1}{\mathcal{R}^{(\text{bg})}} x^2 + \frac{\mathcal{V}^{(\text{bg})}}{\mathcal{S}^{(\text{bg})^2}} (x^2 - x^3). \quad (31)$$

Here the parameters b_{m_z} and x are defined as

$$b_{m_z} = \frac{\pi w_{m_z}}{\mathcal{V}^{(\text{bg})} \mu_{\text{res}}}; \quad x = \frac{\mathcal{V}^{(\text{bg})}}{\mathcal{V}^{\text{eff}}}.$$

So far we have obtained the low-energy expression of the p -wave scattering amplitude $f_{1m_z}(k)$ in the presence of PMFR. With the help of the scattering theory, we obtain the general expressions (28) and (29) for the scattering amplitude $f_{1m_z}(k)$ as well as the scattering parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$), which are formulated in terms of the background scattering parameters ($\mathcal{V}^{(\text{bg})}, \mathcal{S}^{(\text{bg})}, \mathcal{R}^{(\text{bg})}$) and the magnetic field. Although due to the long-range nature of the van der Waals potential we cannot express the denominator of $f_{1m_z}(k)$ as a Laurent series with k -independent coefficients, we successfully include all the k -dependence of the parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$) into the Jost function $\mathcal{J}(k)$.

Equation (29) clearly shows the effect of the PMFA, namely, the scattering volume \mathcal{V}^{eff} diverges under the magnetic field $B = B_0$. In the cold atom gases, in order to have the observable effects with p -wave interaction, the scattering volume $|\mathcal{V}^{\text{eff}}|$ should be large enough. Particularly, $|\mathcal{V}^{\text{eff}}|^{1/3}$ should be much larger than the van der Waals length so that in the BCS region, the superfluid transition temperature $T_c \sim (E_F/k_B) \exp[-\pi/(2k_B^3 \mathcal{V}^{\text{eff}})]$ [71] is achievable and in the BEC region the binding energy of the p -wave Feshbach molecule is robust with respect to the details of the atom-atom interaction potential.

Finally, we consider the dependence of the effective range \mathcal{R}^{eff} on the magnetic field B . According to Eq. (29), \mathcal{R}^{eff} depends on B through the ratio x between $\mathcal{V}^{(\text{bg})}$ and \mathcal{V}^{eff} , and the quantity $\mathcal{R}^{(\Delta)}(B; k; m_z)$. In the cold gases of ${}^6\text{Li}$ and ${}^{40}\text{K}$, the background scattering volumes $\mathcal{V}^{(\text{bg})}$ are on the order of $(10^5 - 10^6) a_0^3$. According to the previous discussion, they are too small to create p -wave superfluids [71]. Therefore in these systems the strong enough p -wave interactions can only be obtained in the resonance region with $|\mathcal{V}^{\text{eff}}| \gg |\mathcal{V}^{(\text{bg})}|$ or $x \ll 1$, which implies $\mathcal{R}^{\text{eff}} \approx \mathcal{R}^{(\Delta)}(B; k; m_z)$. On the other hand, according to Eq. (25), the dependence of $\mathcal{R}^{(\Delta)}(B; k; m_z)$ on B is significant when the magnetic field is sufficiently far away from the resonant point B_0 so that the factor $|w'_{m_z} \mu_{\text{res}} (B - B_0)/w_{m_z}|$ is comparable to or larger than $|\hbar^2/m - g_{m_z}^{(2)}|$. The values of w'_{m_z} and w_{m_z} are not available for ${}^6\text{Li}$ and ${}^{40}\text{K}$. Nevertheless, the binding energies of the p -wave Feshbach molecules were measured and found to be linear functions of the magnetic field [5,9] in the region with large enough p -wave scattering volumes ($\mathcal{V}^{\text{eff}} \gtrsim 10^7 a_0^3$). This observation shows that in these regions the term $|w'_{m_z} \mu_{\text{res}} (B - B_0)/w_{m_z}|$ is negligible and the effective range \mathcal{R}^{eff} can be approximated as a B -independent constant $\mathcal{R}^{(\Delta)}(0; k; m_z)$.

III. THE APPLICABILITY OF THE EFFECTIVE-RANGE THEORY

In the previous section, we have obtained the expression (28) of the p -wave scattering amplitude $f_{1m_z}(k)$ in the region near the point of a PMFR. It is apparent that this expression is different from Eq. (1) given by the effective-range theory in the following two senses:

(1) In the standard effective-range theory, the scattering volume \mathcal{V} and effective-range \mathcal{R} are independent of the incident momentum k . Nevertheless, in the expression (28) the scattering parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$) depend on k through the Jost function $\mathcal{J}(k)$.

(2) The term $1/[\mathcal{S}^{\text{eff}}(k; B; m_z)k]$ cannot be included in the effective-range theory.

It is apparent that, if under some condition, the scattering amplitude (28) can be approximated as

$$f_{1m_z}(k) \approx -\frac{1}{ik + \frac{1}{\mathcal{V}^{\text{eff}}(0; B; m_z)k^2} + \frac{1}{\mathcal{R}^{\text{eff}}(0; B; m_z)}}, \quad (32)$$

that is, both the k dependence of $|\mathcal{J}(k)|^2$ and the term $1/[\mathcal{S}^{\text{eff}}(k; B; m_z)k]$ can be neglected, the behavior of the system would be approximately described by the effective-range theory. In this section, we investigate the conditions for the approximation (32) or (3), or the applicability of effective-range theory. We will consider the importance of the term $1/[\mathcal{S}^{\text{eff}}(k; B; m_z)k]$ and the k dependence of the Jost function, respectively.

A. The k dependence of the scattering parameters

In this subsection we investigate the sufficient condition for the k dependence of the scattering parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$) to be neglected. As we have discussed earlier, the k dependence of the scattering parameters arises from the modulus square of the Jost function $\mathcal{J}(k)$. In the ultracold gases of the fermionic atoms, the maximum value of the relative momentum of two atoms is on the order of the Fermi momentum k_F . Therefore, the importance of the k dependence of the parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$) can be described by the factor,

$$r_1 = \frac{|\mathcal{J}(k_F)|^{-2} - |\mathcal{J}(0)|^{-2}}{|\mathcal{J}(0)|^{-2}}. \quad (33)$$

Obviously, when $r_1 \ll 1$, we can replace $|\mathcal{J}(k)|^{-2}$ with $|\mathcal{J}(0)|^{-2}$ and neglect the k dependence of the parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$).

To investigate the behavior of the ratio r_1 , we first calculate the Jost function $\mathcal{J}(k)$. By means of the quantum defect theory [74], we can obtain the expression of $|\mathcal{J}(k)|^{-2}$ (see Appendix D):

$$|\mathcal{J}(k)|^{-2} = \alpha^{-2} \beta_6^3 \frac{\pi}{2} \left\{ [D_{ff}(k) - K_{l=1}^0 D_{gf}(k)]^2 + [D_{fg}(k) - K_{l=1}^0 D_{gg}(k)]^2 \right\}^{-1}, \quad (34)$$

where α is a k -independent coefficient and $D_{ij}(k) = (k\beta_6)^{3/2} Z_{ij}(k)$ ($i, j = f, g$) with $Z_{ij}(k)$ defined in [68]. In Fig. 2 we plot the functions $D_{ij}(k)$ in the low-energy case.

The parameter $K_{l=1}^0$ denoted as $K_{l=1}^0$ is related to the background scattering parameters [66]. Expanding the p -wave phase shift in Eq. (7) of Ref. [66], we can express ($\mathcal{V}^{(\text{bg})}, \mathcal{S}^{(\text{bg})}$) in terms of $K_{l=1}^0$:

$$\mathcal{V}^{(\text{bg})} = -\frac{(1 + K_{l=1}^0)\pi}{18K_{l=1}^0 \Gamma[3/4]^2} \beta_6^3, \quad (35)$$

$$\mathcal{S}^{(\text{bg})} = -\frac{35(1 + K_{l=1}^0)^2 \pi}{324(K_{l=1}^0)^2 \Gamma[3/4]^4} \beta_6^2. \quad (36)$$

The previous expression shows that when $K_{l=1}^0 \gtrsim \pi/(18\Gamma[3/4]^2) \sim 0.1$, we have $\mathcal{V}^{(\text{bg})} \sim \beta_6^3$ and the background scattering potential is far away from the shape resonance; when $K_{l=1}^0$ is much smaller than 0.1, the background potential is in the shape resonance region which gives $\mathcal{V}^{(\text{bg})} \gg \beta_6^3$.

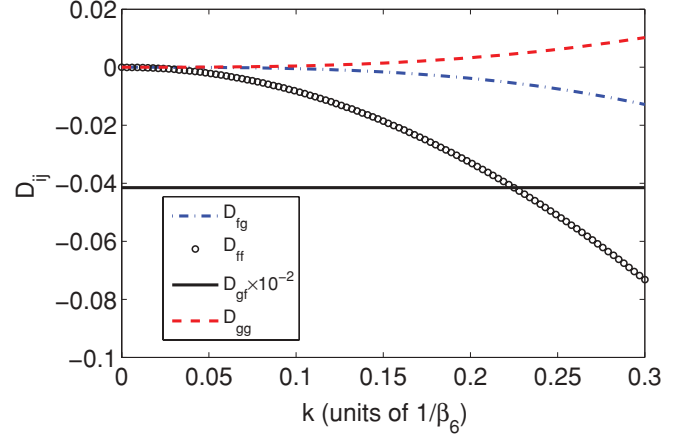


FIG. 2. (Color online) The k dependence of $D_{fg}(k)$ (blue dash-dotted curve), $D_{ff}(k)$ (open circle), $D_{gf}(k)$ (black solid curve), and $D_{gg}(k)$ (red dashed curve) in Eq. (34).

Now we consider the features of the ratio r_1 , which is determined by the parameter $K_{l=1}^0$. Figure 2 shows that in the low-energy case with $k\beta_6 \ll 1$, the function $D_{gf}(k)$ is almost a k -independent constant and much larger than the other three D functions. Therefore, if the parameter $K_{l=1}^0$ is large or the background scattering potential in the open channel is far away from the shape resonance, then according to Eq. (34), the Jost function $|\mathcal{J}(k)|$ is dominated by the term with $K_{l=1}^0 D_{gf}(k)$. In this case the variation of $|\mathcal{J}(k)|^{-2}$ with respect to k is negligible and we have $r_1 \ll 1$. On the other hand, if $K_{l=1}^0$ is close to zero and the background scattering potential is close to the shape resonance, then $|\mathcal{J}(k)|$ becomes a rapidly changing function of k and the ratio r_1 would be significant.

The previous argument is quantitatively verified in Fig. 3, where the ratio r_1 is plotted as functions of k with respect to different values of $K_{l=1}^0$ or $\mathcal{V}^{(\text{bg})}$. It is clearly demonstrated that, if the fermionic momentum $k_F \lesssim 0.1\beta_6^{-1}$, then the ratio

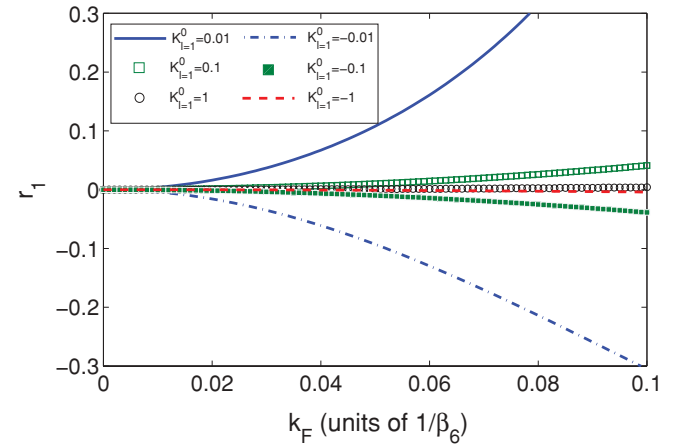


FIG. 3. (Color online) Ratio r_1 defined in Eq. (33) as a function of the Fermi momentum k_F with $K_{l=1}^0 = 0.01$ ($\mathcal{V}^{(\text{bg})} = -11.79\beta_6^3$, blue solid curve), $K_{l=1}^0 = 0.1$ ($\mathcal{V}^{(\text{bg})} = -1.27\beta_6^3$, green empty square), $K_{l=1}^0 = 1$ ($\mathcal{V}^{(\text{bg})} = -0.23\beta_6^3$, black empty circle), $K_{l=1}^0 = -0.01$ ($\mathcal{V}^{(\text{bg})} = 11.51\beta_6^3$, blue dash-dotted curve), $K_{l=1}^0 = -0.1$ ($\mathcal{V}^{(\text{bg})} = 10.4\beta_6^3$, green solid square), $K_{l=1}^0 = -1$ ($\mathcal{V}^{(\text{bg})} = 0$, red dashed curve).

r_1 and the k dependence of the scattering parameters can be neglected when the background potential is far off from the shape resonance so that $\mathcal{V}^{(\text{bg})} \lesssim \beta_6^3$. If $k_F \lesssim 0.01\beta_6^{-1}$, this restriction can be further relaxed to $\mathcal{V}^{(\text{bg})} \lesssim 10\beta_6^3$.

B. The importance of the term $1/[\mathcal{S}^{\text{eff}}(k; B; m_z)k]$

Now we discuss the importance of the term $1/[\mathcal{S}^{\text{eff}}(k; B; m_z)k]$. Since the purpose of this paper is to obtain the sufficient condition for the effective-range theory, or the approximation (32), for simplicity, in this subsection we assume the condition $r_1 \ll 1$ obtained in the previous section is already met, and the k dependence of the coefficients ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$) can be neglected.

In the BEC side of the PMFR where $B < B_0$, since $g_{m_z}^{(2)} < 0$, the parameters \mathcal{R}^{eff} and \mathcal{V}^{eff} have the same sign. In that case, if the absolute value of $1/(\mathcal{S}^{\text{eff}}k)$ is much smaller than the one of $1/(\mathcal{V}^{\text{eff}}k^2)$, then it would be also much smaller than $1/(\mathcal{V}^{\text{eff}}k^2) + 1/\mathcal{R}^{\text{eff}}$, and can be neglected. It is obvious that, in the limit $k \rightarrow 0$, the term $1/(\mathcal{S}^{\text{eff}}k)$ would be much smaller than $1/(\mathcal{V}^{\text{eff}}k^2)$. Therefore the importance of the term $1/(\mathcal{S}^{\text{eff}}k)$ is actually determined by the ratio r_2 between the two terms for the upper limit of the relative momentum k_F :

$$r_2 = \left| \frac{1/[\mathcal{S}^{\text{eff}}(0; B; m_z)k_F]}{1/[\mathcal{V}^{\text{eff}}(0; B; m_z)k_F^2]} \right|. \quad (37)$$

When r_2 is much smaller than unity, we can neglect the term $1/(\mathcal{S}^{\text{eff}}k)$. If r_2 is comparable to or larger than unity, the term $1/(\mathcal{S}^{\text{eff}}k)$ would be necessary for the theory.

The straightforward calculation with Eqs. (29), (30), (35), and (36) yields

$$r_2 = \frac{\pi}{35} \frac{\beta_6^3}{|\mathcal{V}^{\text{eff}}(k_F; B; m_z)|} (\beta_6 k_F). \quad (38)$$

In practice, for cold atom systems we have $\beta_6 k_F \ll 1$. Therefore, in the resonance region with $\mathcal{V}^{\text{eff}} \gtrsim \beta_6^3$, the ratio r_2 in Eq. (38) is much smaller than unity, and then the term $1/(\mathcal{S}^{\text{eff}}k)$, is negligible.

In the BEC side of the resonance with $B > B_0$, the parameters \mathcal{V}^{eff} and \mathcal{R}^{eff} have different signs. In that case, there is a special momentum,

$$k_* = \sqrt{-\frac{\mathcal{R}^{\text{eff}}(0; B; m_z)}{\mathcal{V}^{\text{eff}}(0; B; m_z)}}, \quad (39)$$

which makes the terms $1/[\mathcal{V}^{\text{eff}}k_*^2]$ and $1/[\mathcal{R}^{\text{eff}}]$ cancel with each other or

$$\frac{1}{\mathcal{V}^{\text{eff}}(0; B; m_z)k_*^2} + \frac{1}{\mathcal{R}^{\text{eff}}(0; B; m_z)} = 0. \quad (40)$$

Therefore, if the atomic relative momentum k is far away from k_* , the absolute value of $1/(\mathcal{R}^{\text{eff}}k^2)$ would be quite different from the one of $1/\mathcal{V}^{\text{eff}}$. In that case we can still neglect the term $1/(\mathcal{S}^{\text{eff}}k)$ under the condition $r_2 \ll 1$ or $|1/(\mathcal{S}^{\text{eff}}k)| \ll |1/(\mathcal{R}^{\text{eff}}k^2)|$.

If the atomic relative momentum k is in the neighborhood of k_* and the term $1/(\mathcal{R}^{\text{eff}}k^2)$ is canceled with $1/\mathcal{V}^{\text{eff}}$, the scattering amplitude (28) can be expressed as

$$f_{1m_z}(k) = -\frac{1}{ik_* + \frac{1}{\mathcal{S}^{\text{eff}}(0; B; m_z)k_*}}. \quad (41)$$

In that case, if the absolute value of $1/[\mathcal{S}^{\text{eff}}(0; B; m_z)k_*]$ is much smaller than k_* , we can also neglect the term with \mathcal{S}^{eff} , even in the neighborhood of k_* . We define a parameter r_3 as

$$r_3 = \frac{1}{|\mathcal{S}^{\text{eff}}(0; B; m_z)k_*^2|}.$$

Then the term with \mathcal{S}^{eff} can be neglected when $r_{2,3} \ll 1$. A further calculation with Eqs. (39), (29) (30), (35), and (36) implies that

$$r_3 = \frac{\pi\beta_6^4}{35|\mathcal{V}^{\text{eff}}(0; B; m_z)\mathcal{R}^{\text{eff}}(0; B; m_z)|}. \quad (42)$$

As shown earlier, the condition $r_3 \ll 1$ is obtained for the momentum region $k \sim k_*$. Since the realistic momentum of the atomic relative motion takes the value between zero and k_F , in the cases with $k_F < k_*$, we can disregard the restriction of the ratio r_3 , and use effective-range theory under the condition $r_1, r_2 \ll 1$ in both BEC and BCS sides of the resonance.

C. Summary

In summary, the general sufficient conditions for the effective-range theory in the BCS side of the resonance are given by

$$r_1, r_2, r_3 \ll 1, \quad (43)$$

while the ones for the BEC side are

$$r_1, r_2 \ll 1. \quad (44)$$

From the definition of the ratios r_1, r_2 , and r_3 , we notice that for the typical cold gases of Fermi atoms, the crucial factors for the use of effective range theory are the background p -wave scattering volume $\mathcal{V}^{(\text{bg})}$ and the B dependence of the factor $\mathcal{R}^{(\Delta)}(0; B; m_z)$. If the background p -wave scattering is far away from the shape resonance so that $\mathcal{V}^{(\text{bg})} \sim \beta_6^3 \sim (100a_0)^3$ then according to our previous discussions and Eq. (38), the conditions $r_1, r_2 \ll 1$ can be satisfied in the region $\mathcal{V}^{\text{eff}} \gtrsim 10\beta_6^3 \sim 10^7 a_0^3$ where the p -wave interaction is strong enough for the creation of p -wave superfluids. In that region we also have $\mathcal{R}^{\text{eff}}(0; B; m_z) = \mathcal{R}^{(\Delta)}(0; B; m_z)$. If $\mathcal{R}^{(\Delta)}(0; B; m_z)$ can be further approximated as a B -independent constant which is of the order β_6 , then the condition $r_3 \ll 1$ can also be satisfied, and the effective range theory gives a good approximation of the real scattering amplitude. In the following subsection we show that the PMFRs in the cold gases of ^{40}K and ^6Li are precisely in this case.

D. Discussion for the cold gases of ^{40}K and ^6Li

In the previous subsections we have obtained the sufficient conditions (43) and (44) of the effective-range theory for the

p -wave scattering amplitudes of polarized fermionic atoms near a PMFR. In this subsection, with the help of the conditions, we discuss the use of the effective-range theory in the ultracold gases of ^{40}K and ^6Li .

For the ultracold gas with ^{40}K atoms in the state $|9/2, -7/2\rangle$, we have $C_6 = 3897$ (a.u.) [75] and $\mathcal{V}^{(\text{bg})} = -10^6 a_0^3$ [3]. These parameters lead to $\beta_6 = 130a_0$ and $K_{l=1}^0 = -0.16$. If the Fermi temperature $T_F = 1 \mu\text{K}$, then we have $k_F \beta_6 = 0.06$. The straightforward calculation shows that $r_1 = 0.01$. Therefore the k dependence of the scattering parameters can be safely neglected. The p -wave Feshbach resonance for the states with $m_l = \pm 1$ occurs at $B_0 = 198.37$ G with width $\Delta B = 25$ G and the effective range $\mathcal{R}^{\text{eff}} = 47.2a_0$. The resonance for the states with $m_l = 0$ occurs at $B_0 = 198.85$ G with a width $\Delta B = 22$ G and effective range $\mathcal{R}^{\text{eff}} = 46.2a_0$ [3]. According to these data we have $r_3 < 0.02$ when $k_* < k_F$. Then the effect from the ratio r_3 is also negligible. Therefore the sufficient condition for the use of the effective-range theory simply becomes $r_2 \ll 1$. Further calculation shows that $r_2 \leq 0.013$ when $|\mathcal{V}^{\text{eff}}| \geq |\mathcal{V}^{(\text{bg})}|$. Then the effective-range approximation (32) is applicable for ^{40}K atoms in the state $|9/2, -7/2\rangle$ in the whole region of PMFR with $|\mathcal{V}^{\text{eff}}| \geq |\mathcal{V}^{(\text{bg})}|$. The condition for the effective-range approximation is violated only in the small region $220.5 \text{ G} < B < 221 \text{ G}$ ($m_l = 0$) or $223 \text{ G} < B < 223.7 \text{ G}$ ($m_l = \pm 1$) where we have $|\mathcal{V}^{\text{eff}}| \leq 0.005 \beta_6^3$ or $r_2 \geq 1$.

Now we consider the gas with ^6Li atoms in the ground hyperfine state $|F = 1; m_F = 1\rangle$. In that case we have $C_6 = 1393$ (a.u.) [70,76] and $\mathcal{V}^{(\text{bg})} = -(35.3a_0)^3$. These parameters lead to $\beta_6 = 62a_0$ and $K_{l=1}^0 = -0.38$. If the Fermi temperature $T_F = 1 \mu\text{K}$, we have $k_F \beta_6 = 0.01$ which implies $r_1 = 2 \times 10^{-4}$. Then similarly to the earlier case, the effective-range approximation (32) is also applicable for ^6Li atoms in the whole region of a PMFR with $|\mathcal{V}^{\text{eff}}| \geq |\mathcal{V}^{(\text{bg})}|$.

IV. CONCLUSION

In this work we obtain the explicit expression of the p -wave scattering amplitude of two ultracold spin-polarized fermionic atoms near the p -wave Feshbach resonance. We show that due to the long-range nature of the van der Waals potential, the scattering amplitude is explicitly described by Eq. (2) in the low-energy case. With the help of the quantum defect theory, we formulate all the scattering parameters ($\mathcal{V}^{\text{eff}}, \mathcal{S}^{\text{eff}}, \mathcal{R}^{\text{eff}}$) in terms of the background parameters and the interchannel coupling.

Based on this result, we discuss the applicability of the effective-range theory, or Eq. (3) as an approximation of the exact scattering amplitude. We show that, in the BEC side of the resonance, the sufficient conditions of the effective-range theory can be quantitatively described as $r_1, r_2 \ll 1$ while in the BCS side the conditions become $r_1, r_2, r_3 \ll 1$, where r_1, r_2, r_3 are defined in Eqs. (38), (33), and (42). The applicability of the effective-range theory for the ultracold gases of ^{40}K and ^6Li are examined with our results. The effective-range theory is shown to be a good approximation in both cases in the resonance regime where the absolute value of the scattering volume is equal to or larger than the background one.

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APPENDIX A: THE p -WAVE PHASE SHIFT WITH PMFR

In this Appendix we show the derivation of the p -wave phase shift $\delta_{1m_z}(k)$ in Eqs. (10) and (11). Our calculation is based on the method in Ref. [69]. We start from the scattering state $|\Phi_k^{(+)}\rangle$ in Eq. (5). According to the scattering theory [64], the open-channel component $|\phi_k^{(\text{op}+)}\rangle$ and closed-channel component $|\phi_k^{(\text{cl})}\rangle$ of $|\Phi_k^{(+)}\rangle$ satisfy the two-channel Lippman-Schwinger equation,

$$|\phi_k^{(\text{op}+)}\rangle = |\vec{k}\rangle + G_0^{(+)}(k^2)V^{(\text{bg})}|\phi_k^{(\text{op}+)}\rangle + G_0^{(+)}(k^2)W|\phi_k^{(\text{cl}+)}\rangle; \quad (\text{A1})$$

$$|\phi_k^{(\text{cl}+)}\rangle = G_0^{(\text{cl})}(k^2)V^{(\text{cl})}|\phi_k^{(\text{cl}+)}\rangle + G_0^{(\text{cl})}(k^2)W|\phi_k^{(\text{op}+)}\rangle, \quad (\text{A2})$$

with the free Green's functions,

$$G_0^{(+)}(k^2) = \frac{1}{\hbar^2 k^2/m - T + i0^+}; \quad (\text{A3})$$

$$G_0^{(\text{cl})}(k^2) = \frac{1}{\hbar^2 k^2/m - T - \varepsilon(B)}. \quad (\text{A4})$$

We further define the background Green's function $G_{\text{bg}}^{(+)}(k^2)$ and the closed-channel Green's function G_{cl} as

$$G_{\text{bg}}^{(+)}(k^2) = \frac{1}{\hbar^2 k^2/m + i0^+ - T - V^{(\text{bg})}}; \quad (\text{A5})$$

$$G_{\text{cl}}(k^2) = \frac{1}{\hbar^2 k^2/m - T - V^{(\text{cl})} - \varepsilon(B)}. \quad (\text{A6})$$

Then we have the relations,

$$G_0^{(+)}(k^2) = G_{\text{bg}}^{(+)}(k^2) - G_{\text{bg}}^{(+)}(k^2)V^{(\text{bg})}G_0^{(+)}(k^2); \quad (\text{A7})$$

$$G_0^{(\text{cl})}(k^2) = G_{\text{cl}}(k^2) - G_{\text{cl}}(k^2)V^{(\text{cl})}G_0^{(\text{cl})}(k^2). \quad (\text{A8})$$

Substituting Eqs. (A7) and (A8) into the last terms of the right-hand side (r.h.s.) of Eqs. (A1) and (A2), and using the Lippman Schwinger equation (15) for the background scattering state $|\phi_k^{(\text{bg}+)}\rangle$, we get the equation which relates $|\Phi_k^{(+)}\rangle$ to the background scattering state $|\phi_k^{(\text{bg}+)}\rangle$ [69] as

$$|\phi_k^{(\text{op}+)}\rangle = |\phi_k^{(\text{bg}+)}\rangle + G_{\text{bg}}^{(+)}(k^2)W|\phi_k^{(\text{cl}+)}\rangle, \quad (\text{A9})$$

$$|\phi_k^{(\text{cl}+)}\rangle = G_{\text{cl}}(k^2)W|\phi_k^{(\text{op}+)}\rangle. \quad (\text{A10})$$

To calculate the p -wave phase shifts $\delta_{1m_z}(k)$, we operate the projection operator \mathcal{P}_{m_z} for the manifold ($l = 1, L_z = m_z$) on both of the two sides of Eqs. (A9) and (A10). Then we have

$$\mathcal{P}_{m_z}|\phi_k^{(\text{op}+)}\rangle = \mathcal{P}_{m_z}|\phi_k^{(\text{bg}+)}\rangle + \mathcal{P}_{m_z}G_{\text{bg}}^{(+)}(k^2)W\mathcal{P}_{m_z}|\phi_k^{(\text{cl}+)}\rangle, \quad (\text{A11})$$

$$\mathcal{P}_{m_z}|\phi_k^{(\text{cl}+)}\rangle = |\phi_{\text{res}}^{(m_z)}\rangle \frac{\langle \phi_{\text{res}}^{(m_z)} | W \mathcal{P}_{m_z} | \phi_k^{(\text{op}+)} \rangle}{\hbar^2 k^2/m - \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)})}. \quad (\text{A12})$$

Here we have used

$$\mathcal{P}_{m_z} G_{\text{bg}}^{(+)}(k^2) = \mathcal{P}_{m_z} G_{\text{bg}}^{(+)}(k^2) \mathcal{P}_{m_z}; \quad (\text{A13})$$

$$\mathcal{P}_{m_z} G_{\text{cl}}(k^2) = \mathcal{P}_{m_z} G_{\text{cl}}(k^2) \mathcal{P}_{m_z}, \quad (\text{A14})$$

which are guaranteed by the rotational symmetry around the z axis of the system. We also made the approximation,

$$G_{\text{cl}}(k^2) \approx \sum_{m_z} \frac{|\phi_{\text{res}}^{(m_z)}\rangle \langle \phi_{\text{res}}^{(m_z)}|}{\hbar^2 k^2/m - \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)})}, \quad (\text{A15})$$

that is, we only take into account the contribution from the near-resonance bound state $|\phi_{\text{res}}^{(m_z)}\rangle$ in the closed channel.

Substituting Eq. (A12) into Eq. (A11), we get

$$\begin{aligned} \mathcal{P}_{m_z} |\phi_{\vec{k}}^{(\text{op}+)}\rangle &= \mathcal{P}_{m_z} |\phi_{\vec{k}}^{(\text{bg}+)}\rangle \\ &+ G_{\text{bg}}^{(+)}(k^2) W |\phi_{\text{res}}^{(m_z)}\rangle A^{(m_z)}(B, k^2), \end{aligned} \quad (\text{A16})$$

with

$$A^{(m_z)}(B, k^2) = \frac{\langle \phi_{\text{res}}^{(m_z)} | W \mathcal{P}_{m_z} | \phi_{\vec{k}}^{(\text{op}+)} \rangle}{\hbar^2 k^2/m - \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)})}. \quad (\text{A17})$$

Replacing the $\mathcal{P}_{m_z} |\phi_{\vec{k}}^{(\text{op}+)}\rangle$ in the r.h.s. of Eq. (A17) with the r.h.s. of (A16), we get

$$\begin{aligned} A^{(m_z)}(B, k^2) &= \frac{\langle \phi_{\text{res}}^{(m_z)} | W | \phi_{\vec{k}}^{(\text{bg}+)} \rangle}{\hbar^2 k^2/m - \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)}) - \langle \phi_{\text{res}}^{(m_z)} | W G_{\text{bg}}^{(+)}(E) W | \phi_{\text{res}}^{(m_z)} \rangle}. \end{aligned} \quad (\text{A18})$$

Substituting Eq. (A18) into Eq. (A16), and using the asymptotic expression (7) of the scattering state and the definition (8) of the partial wave scattering amplitude, we can obtain the p -wave scattering amplitude,

$$f_{1m_z}(k) = f_{1m_z}^{(\text{bg})}(k) - \frac{\pi}{k} \frac{\langle \psi_{k1m_z}^{(\text{bg}-)} | W | \phi_{\text{res}}^{(m_z)} \rangle \langle \phi_{\text{res}}^{(m_z)} | W | \psi_{k1m_z}^{(\text{bg}+)} \rangle}{\hbar^2 k^2/m - \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)}) - \langle \phi_{\text{res}}^{(m_z)} | W G_{\text{bg}}^{(+)}(k^2) W | \phi_{\text{res}}^{(m_z)} \rangle}. \quad (\text{A19})$$

Here $|\psi_{k1m_z}^{(\text{bg}+)}\rangle$ is defined in Eq. (12). $|\psi_{k1m_z}^{(\text{bg}-)}\rangle$ is defined as

$$|\phi_{\vec{k}}^{(\text{bg}-)}\rangle = \left(\frac{2}{m\hbar k} \right)^{\frac{1}{2}} \sum_{l, m_z} |\psi_{klm_z}^{(\text{bg}-)}\rangle Y_l^{m_z}(\hat{k})^*, \quad (\text{A20})$$

with

$$|\phi_{\vec{k}}^{(\text{bg}-)}\rangle = |\vec{k}\rangle + \frac{1}{\hbar^2 k^2/m - T - V^{(\text{bg})} + i0^-} V^{(\text{bg})} |\vec{k}\rangle \quad (\text{A21})$$

the background state with an ingoing boundary condition. In the previous calculation we also used the asymptotic behavior of the background Green's function:

$$\lim_{r \rightarrow \infty} \langle \vec{r} | G_{\text{bg}}^{(+)}(k^2) | \vec{r}' \rangle = -m \sqrt{\frac{\pi}{2\hbar}} \frac{e^{ikr}}{r} \langle \phi_{k\hat{r}}^{(\text{bg}-)} | \vec{r} \rangle, \quad (\text{A22})$$

with $\hat{r} = \vec{r}/r$.

With straightforward calculation, we can further rewrite the scattering amplitude $f_{1m_z}(k)$ in Eq. (A19) as

$$f_{1m_z}(k) = f_{1m_z}^{(\text{bg})}(k) - e^{2i\delta_1^{(\text{bg})}(k)} \frac{1}{ik + \mathcal{C}(k)}, \quad (\text{A23})$$

with

$$\begin{aligned} \mathcal{C}(k) &= \frac{k}{\pi} \frac{1}{|\langle \phi_{\text{res}}^{(m_z)} | W | \psi_{k10}^{(+)} \rangle|^2} \\ &\times [\hbar^2 k^2/m - \mu_{\text{res}}(B - B_{\text{res}}^{(m_z)}) - g_{m_z}(k^2)]. \end{aligned} \quad (\text{A24})$$

In the derivation of Eq. (A23) we have used the relation (Appendix B),

$$|\psi_{k1m_z}^{(-)}\rangle = e^{-2i\delta_1^{(\text{bg})}(k)} |\psi_{k1m_z}^{(+)}\rangle, \quad (\text{A25})$$

and

$$\begin{aligned} G_{\text{bg}}^{(+)}(k^2) &= (m\hbar) \int d\vec{k}' \frac{|\phi_{\vec{k}'}^{(\text{bg}+)}\rangle \langle \phi_{\vec{k}'}^{(\text{bg}+)}|}{k^2 + i0^+ - k'^2} \\ &= -(m^2/\hbar) \pi i \int d\vec{k}' \delta(k^2 - k'^2) |\phi_{\vec{k}'}^{(\text{bg}+)}\rangle \langle \phi_{\vec{k}'}^{(\text{bg}+)}| \\ &+ (m\hbar) \text{P} \int d\vec{k}' \frac{|\phi_{\vec{k}'}^{(\text{bg}+)}\rangle \langle \phi_{\vec{k}'}^{(\text{bg}+)}|}{k^2 - k'^2}. \end{aligned} \quad (\text{A26})$$

Here P refers to the principle value of the integral.

Considering the relation (9) between the scattering amplitude $f_{1m_z}(k)$ and the phase shift $\delta_{1m_z}(k)$, it is easy to prove that the phase shift $\delta_{1m_z}(k)$ corresponding to the scattering amplitude (A23) is the one given in Eqs. (10) and (11). This result can also be proved with the method in Ref. [67].

APPENDIX B: THE SCATTERING STATES WITH INGOING AND OUTGOING BOUNDARY CONDITIONS

In this Appendix we prove the formula (A25) in Appendix A. We start from the relationship [69] between the three-dimensional scattering states with ingoing and outgoing boundary conditions:

$$\langle \vec{r} | \phi_{\vec{k}}^{(\text{bg}-)} \rangle = \langle \phi_{-\vec{k}}^{(\text{bg}+)} | \vec{r} \rangle. \quad (\text{B1})$$

Considering the definitions (12) and (A20) of $|\psi_{klm_z}^{(\text{bg}\pm)}\rangle$, we can obtain

$$\sum_{l, m_z} \langle \vec{r} | \psi_{klm_z}^{(\text{bg}+)} \rangle Y_l^{m_z}(\hat{k})^* = \sum_{l, m_z} \langle \psi_{klm_z}^{(\text{bg}-)} | \vec{r} \rangle Y_l^{m_z}(-\hat{k}). \quad (\text{B2})$$

We further define the one-dimensional functions $F_{k_1}^{(\text{bg}\pm)}(r)$ as [64]

$$\langle \vec{r} | \psi_{k_1 m_z}^{(\text{bg}\pm)} \rangle = i^l \frac{1}{\hbar} \left(\frac{m}{\pi k} \right)^{\frac{1}{2}} \frac{1}{r} F_{k_1}^{(\text{bg}\pm)}(r) Y_1^{m_z}(\hat{r}). \quad (\text{B3})$$

Using the relationships,

$$Y_1^{m_z}(-\hat{k}) = (-1)^{m_z+1} Y_1^{-m_z}(\hat{k})^*, \quad (\text{B4})$$

$$Y_1^{m_z}(\hat{r}) = (-1)^{m_z} Y_1^{-m_z}(\hat{r})^*, \quad (\text{B5})$$

we get

$$F_{k_1}^{(\text{bg}+)}(r) = F_{k_1}^{(\text{bg}-)}(r)^*. \quad (\text{B6})$$

On the other hand, we know that $F_{k_1}^{(\text{bg}+)}(r)$ and $F_{k_1}^{(\text{bg}-)}(r)$ satisfy the same differential equation,

$$\left(-\frac{d^2}{dr^2} + V^{(\text{bg})} + \frac{2}{r^2} \right) F_{k_1}^{(\text{bg}\pm)}(r) = k^2 F_{k_1}^{(\text{bg}\pm)}(r), \quad (\text{B7})$$

with the same boundary condition $F_{k_1}^{(\text{bg}\pm)}(0) = 0$. Then $F_{k_1}^{(\text{bg}-)}(r)$ is proportional to $F_{k_1}^{(\text{bg}+)}(r)$. To calculate the ratio between $F_{k_1}^{(\text{bg}\pm)}(r)$, we consider their asymptotic behaviors in the limit $r \rightarrow \infty$:

$$F_{k_1}^{(\text{bg}+)}(r) = \hat{j}_1(kr) + k f_{k_1}^{(\text{bg})}(k) [\hat{n}_1(kr) + i \hat{j}_1(kr)]; \quad (\text{B8})$$

$$F_{k_1}^{(\text{bg}-)}(r) = \hat{j}_1(kr) + k f_{k_1}^{(\text{bg})}(k)^* [\hat{n}_1(kr) - i \hat{j}_1(kr)], \quad (\text{B9})$$

with $\hat{j}_1(x)$ defined in (21) and

$$\hat{n}_1(x) = -\frac{\cos x}{x} - \sin x \quad (\text{B10})$$

is the irregular first-order Riccati-Bessel function. Comparing the coefficients of $\hat{n}_1(kr)$, we have

$$F_{k_1}^{(\text{bg}-)}(r) = e^{-2i\delta_1^{(\text{bg})}(k)} F_{k_1}^{(\text{bg}+)}(r). \quad (\text{B11})$$

Here we have used

$$\begin{aligned} f_{k_1}^{(\text{bg})}(k) &= -\frac{1}{ik - k \cot \delta_1^{(\text{bg})}(k)} \\ &= \frac{1}{k} \sin \delta_1^{(\text{bg})}(k) e^{i\delta_1^{(\text{bg})}(k)}. \end{aligned} \quad (\text{B12})$$

Substituting Eq. (B11) into Eq. (B3), we obtain Eq. (A25).

APPENDIX C: THE EXPANSION OF THE FACTOR $g_{m_z}(k^2)$

In this Appendix we prove the Eq. (17). We firstly rewrite the factor $g_{m_z}(k^2)$ as

$$\begin{aligned} g_{m_z}(k^2) &= \text{Re} \langle \phi_{\text{res}}^{(m_z)} | W G_+^{(\text{bg})}(0) W | \phi_{\text{res}}^{(m_z)} \rangle \\ &\quad - k^2 \text{Re} \langle \phi_{\text{res}}^{(m_z)} | W G_+^{(\text{bg})}(0) G_+^{(\text{bg})}(k^2) W | \phi_{\text{res}}^{(m_z)} \rangle. \end{aligned} \quad (\text{C1})$$

Here we have used the identity

$$G_+^{(\text{bg})}(k^2) = G_+^{(\text{bg})}(0) - k^2 G_+^{(\text{bg})}(0) G_+^{(\text{bg})}(k^2). \quad (\text{C2})$$

The first term in the r.h.s. of (C1) is independent of k . It contributes to the constant term $g_{m_z}^{(0)}$ in Eq. (17).

On the other hand, the second term in Eq. (C1) can be rewritten as

$$\begin{aligned} &k^2 \text{Re} \langle \phi_{\text{res}}^{(m_z)} | W G_+^{(\text{bg})}(0) G_+^{(\text{bg})}(k^2) W | \phi_{\text{res}}^{(m_z)} \rangle \\ &= \frac{m^2 k^2}{\hbar^4} \text{Re} \lim_{\varsigma_1, \varsigma_2 \rightarrow 0^+} \int d\vec{k}' \frac{|\langle \phi_{\text{res}}^{(m_z)} | W | \phi_{\vec{k}'}^{(\text{bg}+)} \rangle|^2}{(k'^2 - i\varsigma_1)(k'^2 - k^2 - i\varsigma_2)}. \end{aligned} \quad (\text{C3})$$

In the limit $k = 0$, we have

$$\begin{aligned} &\lim_{k \rightarrow 0} \text{Re} \langle \phi_{\text{res}}^{(m_z)} | W G_+^{(\text{bg})}(0) G_+^{(\text{bg})}(k^2) W | \phi_{\text{res}}^{(m_z)} \rangle \\ &\propto \int dp \frac{|\langle \phi_{\text{res}}^{(m_z)} | W | \psi_{p 1 m_z}^{(\text{bg}+)} \rangle|^2}{p^3}. \end{aligned} \quad (\text{C4})$$

We know that the function $|\langle \phi_{\text{res}}^{(m_z)} | W | \psi_{p 1 m_z}^{(\text{bg}+)} \rangle|^2$ decays to zero when $p \rightarrow \infty$. On the other hand, as we have shown in Sec. IIIB, the factor $|\mathcal{J}(k)|^2$ tends to a nonzero constant in the low-energy limit if the background scattering volume in the open channel is finite. Using the relationship (18) between $\mathcal{J}(k)$, $\tilde{F}_{k_1}^{(\text{bg}\pm)}(r)$ and $|\psi_{p 1 m_z}^{(\text{bg}+)}\rangle$, and the low-energy behavior (22) of $\tilde{F}_{k_1}^{(\text{bg}\pm)}(r)$, it is easy to prove that the factor $|\langle \phi_{\text{res}}^{(m_z)} | W | \psi_{p 1 m_z}^{(\text{bg}+)} \rangle|^2$ is proportional to p^3 in the limit $p \rightarrow 0$. Therefore, the previous integration in Eq. (C4) converges to a finite constant in the limit $k^2 \rightarrow 0$. Then the expansion in Eq. (17) is proved and we have

$$g_{m_z}^{(2)} = -\langle \phi_{\text{res}}^{(m_z)} | W G_+^{(\text{bg})}(0) G_+^{(\text{bg})}(0) W | \phi_{\text{res}}^{(m_z)} \rangle \leq 0. \quad (\text{C5})$$

APPENDIX D: THE BACKGROUND JOST FUNCTION

In this Appendix we calculate the Jost function $\mathcal{J}(k)$ of the background scattering state. To this end, we introduce a function $\bar{F}_{k_1}^{(\text{bg}\pm)}(r) = \tilde{F}_{k_1}^{(\text{bg}\pm)}(r)/k^2$ where $\tilde{F}_{k_1}^{(\text{bg}\pm)}(r)$ is defined in Eq. (18). We first note that $\bar{F}_{k_1}^{(\text{bg}\pm)}(r)$ is a solution of the radial equation,

$$\left(-\frac{d^2}{dr^2} + V^{(\text{bg})}(r) + \frac{2}{r^2} \right) \bar{F}_{k_1}^{(\text{bg})}(r) = k^2 \bar{F}_{k_1}^{(\text{bg})}(r), \quad (\text{D1})$$

with a k -independent boundary condition,

$$\bar{F}_{k_1}^{(\text{bg})}(r \rightarrow 0) \rightarrow r^2. \quad (\text{D2})$$

Following the spirit of quantum defect theory [74], we assume the scattering potential $\mathcal{V}^{(\text{bg})}(r)$ can be approximated as the van der Waals potential $-\hbar^2 \beta_6^4 / (r^6 m)$ when r is larger than a critical distance r_0 which is much smaller than β_6 . In the region with $r < r_0$, $\mathcal{V}^{(\text{bg})}(r)$ is assumed to be so large that the atomic kinetic energy k^2 is negligible in comparison with $\mathcal{V}^{(\text{bg})}(r)$, and then $\bar{F}_{k_1}^{(\text{bg})}(r)$ is independent of k .

In the region $r > r_0$, $\bar{F}_{k_1}^{(\text{bg})}(r)$ is the superposition of the two independent solutions $\chi_{\epsilon 1}^{(0)}(r)$ and $\kappa_{\epsilon 1}^{(0)}(r)$ of Eq. (D1) [68] [in Ref. [68], $\chi_{\epsilon 1}^{(0)}(r)$ and $\kappa_{\epsilon 1}^{(0)}(r)$ are denoted as $f_{\epsilon 1}^{(0)}(r)$ and $g_{\epsilon 1}^{(0)}(r)$]:

$$\bar{F}_{k_1}^{(\text{bg})}(r) = \alpha_k \chi_{k^2 1}^{(0)}(r) + \beta_k \kappa_{k^2 1}^{(0)}(r). \quad (\text{D3})$$

In the short distance region with $r \ll \beta_6$, $\chi_{k^2 1}^{(0)}(r)$, and $\kappa_{k^2 1}^{(0)}(r)$ are almost independent of k [74].

The wave function $\tilde{F}_{k1}^{(\text{bg})}(r)$ in the two regions are connected at point $r = r_0$, where $\tilde{F}_{k1}^{(\text{bg})}(r)$, $\chi_{\epsilon_1}^{(0)}(r)$, and $\kappa_{\epsilon_1}^{(0)}(r)$ are approximately independent of k . Then α_k and β_k are independent of k and we have

$$\tilde{F}_{k1}^{(\text{bg})}(r) = \alpha \chi_{\epsilon_1}^{(0)}(r) + \beta \kappa_{\epsilon_1}^{(0)}(r), \quad (\text{D4})$$

with $\epsilon_k = \hbar^2 k^2 / m$. The previous equation leads to

$$\tilde{F}_{k1}^{(\text{bg})}(r) = k^2 \alpha \chi_{\epsilon_1}^{(0)}(r) + k^2 \beta \kappa_{\epsilon_1}^{(0)}(r). \quad (\text{D5})$$

In the region $r \rightarrow \infty$, the asymptotic behaviors of $\chi_{\epsilon_1}^{(0)}(r)$ and $\kappa_{\epsilon_1}^{(0)}(r)$ can be expressed in terms of $Z_{ij}(k)$ ($i, j = f, g$) defined in [68]

$$\chi_{\epsilon_1}^{(0)}(r) \rightarrow \sqrt{\frac{2}{\pi k}} \left[Z_{ff} \sin\left(kr - \frac{\pi}{2}\right) - Z_{fg} \cos\left(kr - \frac{\pi}{2}\right) \right]; \quad (\text{D6})$$

$$\kappa_{\epsilon_1}^{(0)}(r) \rightarrow \sqrt{\frac{2}{\pi k}} \left[Z_{gf} \sin\left(kr - \frac{\pi}{2}\right) - Z_{gg} \cos\left(kr - \frac{\pi}{2}\right) \right]. \quad (\text{D7})$$

On the other hand we know that in the same limit we have [64]

$$\tilde{F}_{k1}^{(\text{bg})}(r) = \mathcal{J}(k) \left[e^{i\delta_1^{(\text{bg})}(k)} \sin\left(kr - \frac{\pi}{2} + \delta_1^{(\text{bg})}(k)\right) \right]. \quad (\text{D8})$$

Together with Eqs. (D5), (D6), and (D7) as well as (D8), we obtain the expression of (34) of $|\mathcal{J}(k)|^2$ and the background phase shift:

$$\tan \delta_1^{(\text{bg})} = -\frac{K_{l=1}^0 Z_{gg} - Z_{fg}}{K_{l=1}^0 Z_{gf} - Z_{ff}}, \quad (\text{D9})$$

where $D_{ij}(k) = (k\beta_6)^{3/2} Z_{ij}(k)$ and $K_{l=1}^0 = -\beta/\alpha$ is the one in Sec. III. The result (D9) is also given in [66].

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