Multipartite entanglement and quantum phase transitions in the one-, two-, and three-dimensional transverse-field Ising model

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In this paper we consider the quantum phase transition in the Ising model in the presence of a transverse field in one, two, and three dimensions from a multipartite entanglement point of view. Using *exact* numerical solutions, we are able to study such systems up to 25 qubits. The Meyer-Wallach measure of global entanglement is used to study the critical behavior of this model. The transition we consider is between a symmetric Greenberger-Horne-Zeilinger–like state to a paramagnetic product state. We find that global entanglement serves as a good indicator of quantum phase transition with interesting scaling behavior. We use finite-size scaling to extract the critical point as well as some critical exponents for the one- and two-dimensional models. Our results indicate that such multipartite measure of global entanglement shows universal features regardless of dimension d. Our results also provide evidence that multipartite entanglement is better suited for the study of quantum phase transitions than the much-studied bipartite measures.

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I. INTRODUCTION

There has been much work on entanglement in the past 20 years [1–8]. Entanglement is a purely quantum phenomenon with no classical counterpart. It is thought to hold the key to a deeper understanding of the theoretical aspects of quantum mechanics. From a more practical aspect, entanglement is the key ingredient in many information processing applications, including quantum computation and quantum cryptography [9]. On the other hand, there has been much cross-fertilization in the fields of condensed-matter physics and quantum information theory in recent years [10–18]. Here many traditional condensed-matter systems including fermionic and bosonic gases, and in particular lattice spin models, have been investigated in the light of new developments in quantum information theory and entanglement in particular [10]. It has been found that entanglement plays a crucial role in the low-temperature physics of many of these systems, particularly in their ground (zero-temperature) states [16–19]. A very fruitful avenue along these lines has been the relation of entanglement and phase transitions in general, and quantum (ground-state) phase transitions in particular. This is a bit surprising since entanglement was originally thought to be somewhat fragile and thus easily destroyed by fluctuations.

In a quantum phase transition (QPT) [20], a thermodynamic system described by a Hamiltonian $H(\lambda)$ changes its macroscopic phase at the critical value of the control parameter λ_c. In recent studies of many thermodynamic systems exhibiting QPT, in particular quantum spin models [16–19], it has become clear that the onset of transition is accompanied by a marked change in the entanglement. Depending on the model, entanglement could peak, show discontinuous behavior, or show diverging derivatives with scaling behavior at the critical point [10,21]. What is less clear is the exact role (or the general mechanism) through which entanglement and QPT are related. In such studies, various quantitatively different measures of entanglement have been used. Therefore, for example, one would like to know if there are universal features in the entanglement of various spin models exhibiting QPT.

Another important feature which is emerging out of recent studies of condensed-matter systems from quantum information perspectives is the need for multipartite measures of entanglement [22–26]. This, by the way, is an example of the cross-fertilization referred to earlier. Since the root of quantum theory [27] is originally in bipartite systems like Bell states, it has been natural to study macroscopic systems using bipartite measures such as the von Neumann entropy or concurrence. In fact, with very few exceptions, the general body of the current literature has used such bipartite measures to study many-particle systems. Although this has been so because of a matter of tradition and/or convenience, there is increasing evidence that such measures are generally inadequate to study QPT in condensed-matter systems [28,29]. After all, it is natural to use multipartite entanglement if one is to study the role of entanglement in multipartite (many-particle) systems, because important types of entanglement in such systems (e.g., various n-tangles) may not be captured by a bipartite measure but would be included in a (ideal) multipartite measure. Additionally, some multipartite measures (as discussed in Sec. III) have thermodynamic properties (e.g., extensivity) that make them more suitable for studies of such thermodynamic phenomena as QPT. Another equally important shortcoming is that most such studies have been carried out for one-dimensional (1D) models. Although this is perhaps because of computational difficulties, it is certainly not well justified. As is well known, spatial dimension (d) plays an important role in the physics of thermodynamic systems and in phase transitions in particular [30].

Here we propose to study QPT in a prototypical transverse-field quantum Ising model using the Meyer-Wallach [31] measure of global entanglement in one, two, as well as three dimensions. Such a global entanglement measure seems to be well suited for studies of many-particle systems [22,32]. Since analytic results are usually difficult to come up with, numerical results with finite-size systems are typically the way to proceed. However, solving quantum lattice spin systems numerically is also computationally expensive as only a

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few qubits (spins) can be solved exactly and approximation techniques have limited success in one dimension and are more limited in higher dimensions [33]. Very recently, however, such systems have been studied using efficient numerics [34].

In this article, we solve the transverse-field quantum Ising model numerically (exact) for up to 25 qubits in one, two, and three dimensions. Our main result is that global entanglement is a measure well suited to study QPT with some universal features in any dimension. We show that global entanglement has interesting scaling properties near the critical point. Using finite-size scaling arguments, we extract critical points as well as some critical exponents for the 1D and 2D models consistent with previous studies. Owing to system-size limits, we are only able to study the smallest 3D system and thus cannot perform finite-size studies. However, the general shape of global entanglement in the 3D model (see later discussion of Fig. 7) indicates that our 1D and 2D results easily generalize to 3D systems. More important, our results provide a general framework for computation of an accessible measure of entanglement and its relevance to QPTs in many-particle thermodynamic systems.

This paper is structured as follows: in Sec. II, we discuss the multidimensional quantum Ising model in the presence of a transverse field and its ground-state properties relevant to our study here. In Sec. III, we discuss some key concepts regarding the Meyer-Wallach measure of global entanglement, while our main results are presented in Sec. IV. Our concluding remarks, including suggestions for further work, are presented in Sec. V.

II. TRANSVERSE-FIELD ISING MODEL

The system under consideration here is the ferromagnetic Ising model in a transverse field given by the Hamiltonian

$$H = -\lambda \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - \sum_{i=1}^{N} \sigma_i^z, \tag{1}$$

where $\lambda = \frac{J}{B}$ and J is the ferromagnetic coupling constant, B is the magnetic field, N is the total number of spins (qubits), and σ_i^x and σ_i^z are the Pauli spin matrices in x and z directions at the site i, respectively; $\langle ij \rangle$ means sites i and j are nearest neighbors on a regular d-dimensional lattice. We use periodic boundary conditions. This model has been extensively studied in one dimension, but less is known about its properties in two and three dimensions. Relevant to our study here is the QPT this model exhibits regardless of dimension. At zero field this model exhibits ferromagnetic behavior with net magnetization in the x direction while, in the large-field limit, it exhibits a paramagnetic behavior where all spins point in the field direction z. The transition between these two phases occurs at the critical value of $\lambda = \lambda_c$, in the thermodynamic limit. It is well known that the ground state is a product state in both these limits [35]. In the first limit, the ground state is twofold degenerate, one being a product state of spins pointing in the positive x direction, $|+\rangle = |x;0\rangle_1 |x;0\rangle_2 \cdots |x;0\rangle_N$; the other is $|-\rangle = |x;1\rangle_1 |x;1\rangle_2 \cdots |x;1\rangle_N$, which is the global phase flip of the first one. In the second limit, the ground state is a product state of spins pointing in the positive z direction $|0\rangle$. Both limits of the ground state are product states which are disentangled, but there is another possibility for the ground state in the first limit which arises from linearity of the Schrödinger equation. As the $|+\rangle$ and $|-\rangle$ are solutions for the ground state in this limit, the superposition of these degenerate states is also another acceptable solution for the ground state when the applied field (B) tends to zero. This possibility is a Greenberger-Horne-Zeilinger (GHZ)-like state which has genuine multiqubit entanglement [36],

$$|\mathrm{GHZ}\rangle_N = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle). \tag{2}$$

The possibility of a GHZ-like ground state is fascinating from a fundamental theoretical point of view because it represents a coherent superposition of two macroscopically distinct states and, hence, is often called a cat state. It is proven in Ref. [37] that in this limit the ground state of the Ising model is a locally unitary equivalent to an N-partite GHZ state. Multipartite correlations and non-locality of a GHZ-like ground state under magnetic effects has been studied in Ref. [38], and Ref. [39] discusses how to prepare an Ising chain in a GHZ state using a single global control field. It is important to note that such a ground state would show zero net magnetization (i.e., $\langle M_x \rangle = 0$). Therefore, such a quantity could not be used as an order parameter to signal the phase transition under consideration here. We also note that such ground states have recently attracted attention from a symmetry-breaking point of view [40]. Here we study the multipartite entanglement properties of such a ground state and its subsequent transition to a (paramagnetic) product state as a function of λ.

III. GLOBAL ENTANGLEMENT

Global entanglement, defined by the Meyer-Wallach entanglement measure of pure states [31] and henceforth denoted by E_{gl}, is a monotone [41] and a very useful measure of multipartite entanglement. As we show briefly, E_{gl} is a measure of total nonlocal information per particle in a general multipartite system. Therefore, E_{gl} gives an intuitive meaning to multipartite entanglement as well as being an experimentally accessible measure [41–43].

A finite amount of information can be attributed to an N-qubit pure state which is N bits of information according to the Brukner-Zeilinger operationally invariant information measure [44]. This information can be distributed in local as well as nonlocal form, which is associated with entanglement [45]. This information has a complimentary relation:

$$I_{\mathrm{total}} = I_{\mathrm{local}} + I_{\mathrm{nonlocal}}. \tag{3}$$

The total information is conserved unless transferred to the environment through decoherence. The amount of information in local form is $I_{\mathrm{local}} = \sum_{i=1}^{N} I_i$, where $I_i = 2\mathrm{Tr}\rho_i^2 - 1$ is the operationally invariant information measure of a qubit [44], and ρ_i is the single-particle reduced density matrix obtained by tracing over the other particles' degrees of freedom. Therefore, according to Eq. (3), $I_{\mathrm{nonlocal}} = \sum_{i=1}^{N} 2(1 - \mathrm{Tr}\rho_i^2)$, which is distributed in different kinds of quantum correlations, the tangles, among the system

$$I_{\mathrm{nonlocal}} = 2 \sum_{i_1 < i_2} \tau_{i_1 i_2} + \cdots + N \sum_{i_1 < \cdots < i_N} \tau_{i_1 \cdots i_N}, \tag{4}$$

where the first term is referred to as a 2-tangle, the next being a 3-tangle, and the last term the N-tangle of the system. One can view these tangles as different types of nonlocal information distribution. Therefore, since E_{gl} is the sum of single-particle linear entropies per unit particle in a multipartite system [31], it can be written as

$$E_{gl} = \frac{1}{N}\left[2\sum_{i_1<i_2}\tau_{i_1 i_2} + \cdots + N\sum_{i_1<\cdots<i_N}\tau_{i_1\cdots i_N}\right]. \quad (5)$$

Therefore, E_{gl} is the average of tangles per particles ($\frac{\langle\tau\rangle}{N}$), without giving detailed knowledge of tangle distribution among the individual particles. This is much like the average energy per particle in an interacting many-particle system. The Meyer-Wallach measure was originally introduced as a multipartite entanglement to distinguish it from bipartite entanglement measures like entropy of entanglement. But, as shown above, E_{gl} is an average quantity and therefore cannot distinguish between entangled states which have equal $\langle\tau\rangle$ yet different distributions of tangles, like $|GHZ\rangle_N$ and $|EPR\rangle^{\otimes\frac{N}{2}}$. However, E_{gl} can distinguish between GHZ and W states since they have different values of $\langle\tau\rangle$, so E_{gl}, like a thermodynamical variable, determines the general amount of a property in a quantum system without giving detailed knowledge of its sharing among the constituents. One expects this property of global entanglement to play an important role in studying macroscopic properties of multipartite quantum systems [22,32].

To obtain E_{gl} in these systems, we have to calculate the single-particle reduced density matrix, ρ_i. Since we use periodic boundary conditions, the reduced density matrix is the same for all particles. So E_{gl} reduced to linear entropy of a single-particle density matrix, ρ_i, is

$$E_{gl} = 2\left(1 - \mathrm{Tr}\rho_i^2\right). \quad (6)$$

IV. RESULTS

Using Eq. (6), we can therefore easily calculate E_{gl} exactly for any dimension d, up to the limitations set by computational limits of our numerics. We start by showing our results for the 1D model. Figure 1 shows E_{gl} versus λ and the inset shows its derivative for various system sizes up to $N = 24$. The general behavior shown here is that of E_{gl} increasing slowly from its zero value at $\lambda = 0$ with a sharp transition to its large λ value of 1 around $\lambda_c = 1$. The critical point is better seen in the derivative (inset), which peaks at the maximal value $\lambda_m(N)$. As the system size increases, the peak of the derivative sharpens and moves closer to the critical point $\lambda_c = 1$.

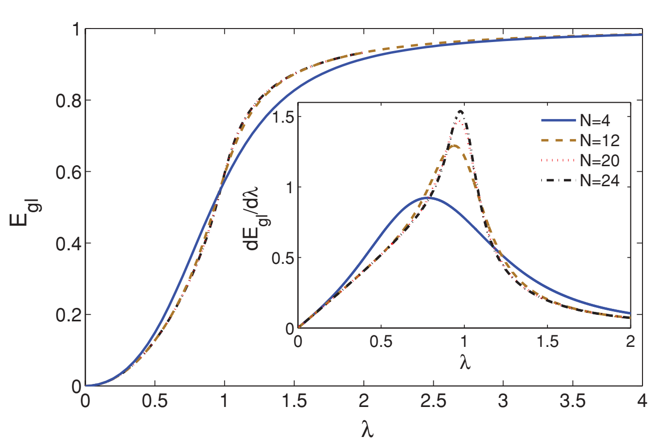

FIG. 1. (Color online) Global entanglement as a function of λ for the 1D transverse Ising model. The inset shows the derivative and the system sizes used. Increasing N sharpens the peak and moves it closer to the critical point.

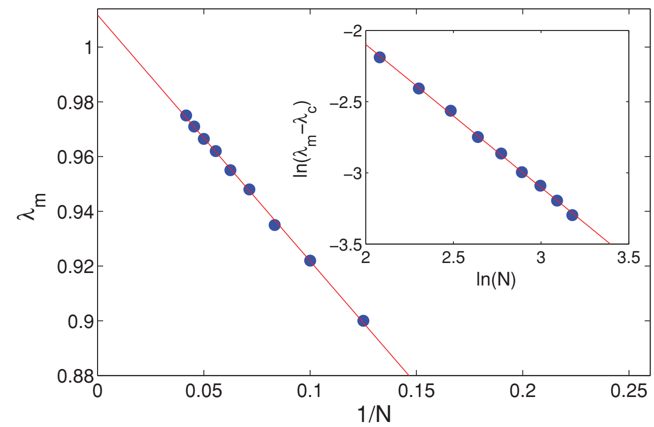

FIG. 2. (Color online) Convergence of λ_m to the critical point as $N \to \infty$, for the 1D transverse Ising model. The y intercept is 1.01. The inset shows the relation $|\lambda_c - \lambda_m| \sim N^{-1.00}$.

The extrapolation to the infinite system size along with the convergence to the critical point (inset) is shown in Fig. 2. As one can see, $\lambda_m(\infty) = 1.01$, which is very close to the well-known result of $\lambda_c = 1$, showing that E_{gl} is a good indicator of the critical behavior of this model. The inset shows that the convergence to the critical point is in accordance with $|\lambda_c - \lambda_m| \sim N^{-\alpha}$ with exponent $\alpha = 1.00$.

We next examine the scaling behavior of E_{gl} near the critical point. According to scaling ansatz [46], we have $dE_{gl}/d\lambda \sim Q(N^{\frac{1}{\nu}}(\lambda - \lambda_m))$, where ν is the correlation length critical exponent, and $Q(x) \sim \ln(x)$ is generally assumed. As is seen in Fig. 3, an acceptable collapse occurs for various values of N using the scaling ansatz with the critical exponent $\nu = 1.06$, in line with previous studies [17] and close to the exact result

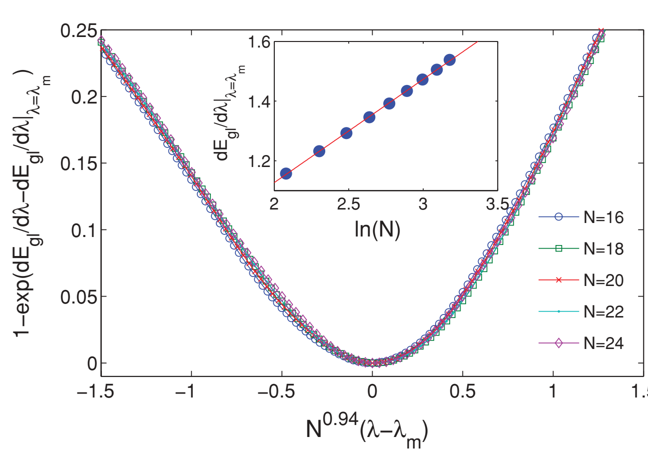

FIG. 3. (Color online) Finite-size scaling of global entanglement for the 1D transverse Ising model. The inset shows the logarithmic divergence of the value of the derivative at the maximal point λ_m.

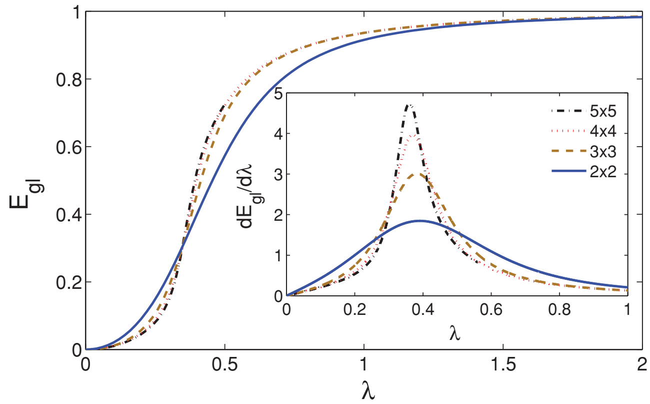

FIG. 4. (Color online) Global entanglement as a function of λ for the 2D transverse Ising model. The inset shows the derivative and the system sizes used. Increasing system sizes sharpens the peak and moves it toward the critical point.

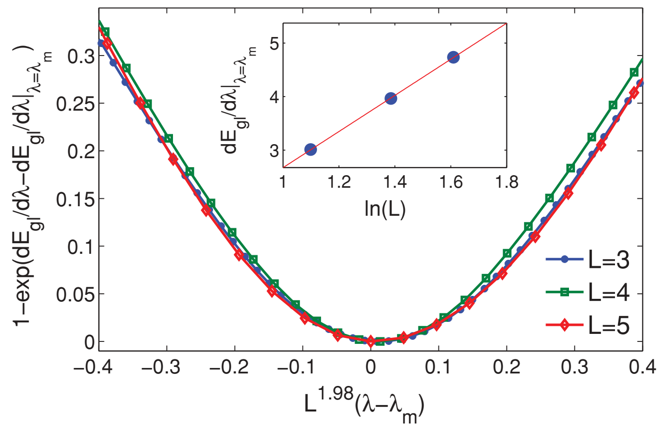

FIG. 6. (Color online) Finite-size scaling of global entanglement for the 2D transverse Ising model. The inset shows the logarithmic divergence of the value of the derivative at the maximal point λ_m.

of $\nu = 1$. The inset shows the logarithmic divergence of the maximum (peak) of the derivative of $E_{\rm gl}$. Hence, the general shape of $E_{\rm gl}$, the logarithmic divergence of its derivative at the critical point, along with its consistency with finite-size scaling ansatz provides strong evidence for the well-suitedness of such a measure for the 1D Ising model. The question now is if such features also hold for higher-dimensional models.

We next turn to the 2D model. Using periodic boundary conditions, we have been able to study such a model for up to $L^2 = 5^2 = 25 = N$ qubits. Figures 4, 5, and 6 show results similar to that of Figs. 1, 2, and 3. We note the following: The general shape of the $E_{\rm gl}$ still remains (Fig. 4), with a (logarithmic) divergence of the derivative at the critical point (insets of Figs. 4 and 6). The critical point is now identified as $\lambda_m(\infty) = 0.329$ (Fig. 5), consistent with recent studies using infinite projected entangled-pair states ($\lambda_c = 0.3268$) [34], as well as quantum Monte Carlo simulations ($\lambda_c = 0.3285$) [47]. Interestingly, our simple method obtains a more acceptable result than the recent similar multipartite entanglement study based on matrix and tensor product states, which obtained $\lambda_c = 0.308$ [24]. We note that the convergence to the critical point (inset of Fig. 5) is in accordance with $|\lambda_c - \lambda_m| \sim L^{-\alpha}$ with exponent $\alpha = 1.00$ being exactly the same as in the 1D case. The difference here is that this convergence occurs from above the critical point as opposed to the 1D case. The finite-size scaling ansatz is also valid (Fig. 6), giving the correlation length exponent $\nu = 0.51$. The inset of Fig. 6 shows the logarithmic divergence of the derivative at the critical point.

In Fig. 7, we show our result for the 3D version of this system for the only system size we are able to study. The general behavior of $E_{\rm gl}$ seen in the 1D and 2D models is clearly seen here for the 3D case as well. While we are not able to perform a scaling analysis similar to the 1D and 2D models, it seems reasonable to assume that the same general behavior carries over to the 3D model. We note that $\lambda_m(L = 2) = 0.26$ here, which would understandably be different from the infinite-size limit but is in the right ballpark of $\lambda_c \approx 0.2$ [47].

Finally, it is worth considering another important form of multipartite entanglement, namely, genuine entanglement. Genuine entanglement in a many-particle system represents the amount of entanglement shared by all particles. Therefore, genuine entanglement is equal to the N-tangle, the last term

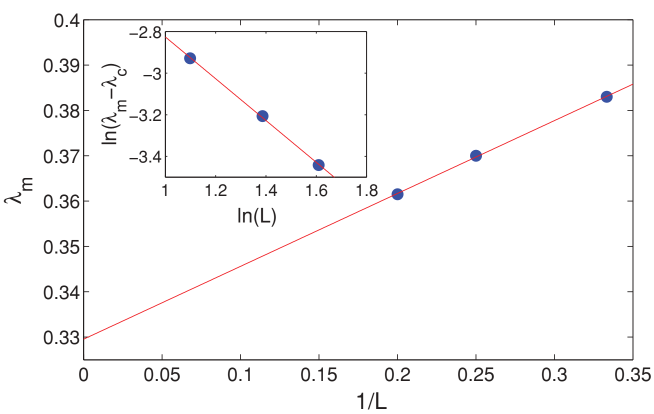

FIG. 5. (Color online) Convergence of λ_m to the critical point as $L \to \infty$, for the 2D transverse Ising model. The y intercept is 0.329. The inset shows the relation $|\lambda_c - \lambda_m| \sim L^{-1.00}$.

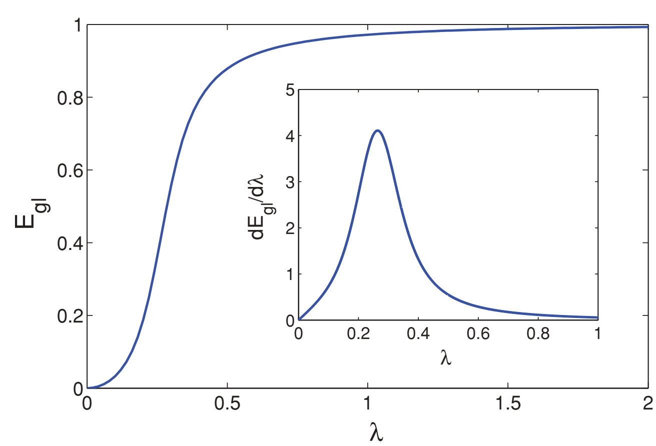

FIG. 7. (Color online) Global entanglement and its derivative (inset) for the $2 \times 2 \times 2$ transverse Ising system, the only 3D system we have been able to study.

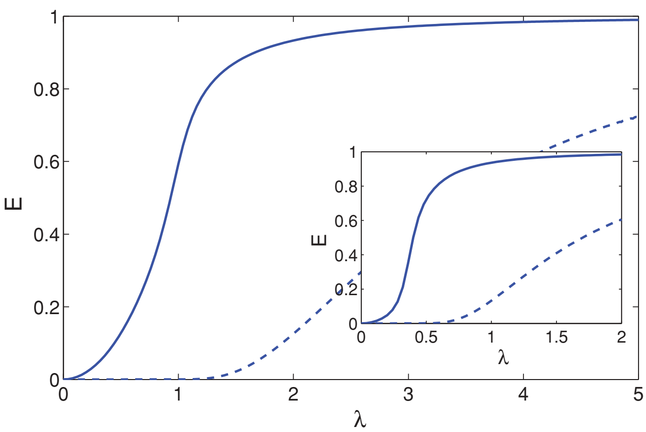

FIG. 8. (Color online) Global entanglement (continuous curve) and genuine entanglement (dashed curve) for the 1D (main figure) and 2D (inset) transverse Ising model. The 1D result is for $N = 16$ and the 2D result is for a 4×4 system.

in Eq. (5). One might expect that such a term would gradually lose its significance as N increases. However, due to the GHZ nature of our ground state, this term (N-tangle) is the dominant term in E_{gl} and its dominance increases with increasing λ. This is shown in Fig. 8 for both the 1D and 2D models, where one can easily see that increasing λ increases the share of N-tangle in E_{gl}. It is also worth noting that the structure of genuine entanglement is very similar to an order parameter. It is zero on one side of the transition and becomes nonzero around the critical point, rising to its maximum at $1/\lambda = 0$. This behavior becomes more pronounced as system sizes get larger; however, we note that the transition is not sharp and is in fact "rounded." However, since net magnetization cannot be used as an order parameter here, such behavior deserves further attention.

V. CONCLUDING REMARKS

In this paper we have studied the quantum phase transition in the transverse-field Ising model from a multipartite entanglement point of view on one-, two-, and three-dimensional square lattices. Our work is interesting from various points of view. First, we use a multipartite global entanglement as a measure. Second, we study the symmetric GHZ-like ground state and its transition to the paramagnetic product state. Third, by studying QPT in various dimensions, we are able to establish common features of such a transition in different universality classes. We find that global entanglement is a good indicator of such transitions with universal aspects, including scaling, in any dimension. The well-suitedness of such a measure is displayed in the nice fits obtained in Figs. 2 or 5, for example. As a by-product, we find critical points and various exponents for the 1D and 2D models consistent with previous studies. We note that our estimation of the critical points for the 1D and 2D models are to within 1% of the generally accepted values, an impressive result given the limited size of the systems studied here, providing further evidence for well-suitedness of our measure when compared with similar studies using bipartite measures. Our estimation of ν, although acceptable, is understandably less impressive because finite-size scaling collapses require larger system sizes to obtain better estimates for ν [48]. We note that our main goal is to investigate the (universal) features of global entanglement in quantum phase transitions, not to produce reliable exponents for such models. Since the parameter d determines the universality class of the systems considered here, the fact that we see similar behavior of global entanglement at the QPT regardless of d shows what we have thus far referred to as universal features of global entanglement.

We close by mentioning that similar studies could be carried out for more general spin models exhibiting more complicated quantum phase transitions. It would be interesting to see if such universal features of global entanglement carry over to other models.

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[1] A. Peres, Found. Phys. **29**, 589 (1999).
[2] R. F. Werner and M. M. Wolf, Quantum Inf. Comput. **1**, 1 (2002).
[3] M. B. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007).
[4] B. M. Terhal, J. Theor. Comp. Sci. **287**, 313 (2002).
[5] S. J. van Enk, N. Lütkenhaus, and H. J. Kimble, Phys. Rev. A **75**, 052318 (2007).
[6] J. Audretsch, *Entangled Systems* (Wiley, Weinheim, Germany, 2007).
[7] D. Bruß, J. Math. Phys. **43**, 4237 (2002).
[8] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum information* (Cambridge University, Cambridge, UK, 2000).
[10] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[11] J. I. Latorre and A. Riera, J. Phys. A: Math. Theor. **42**, 504002 (2009).
[12] M. C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001).
[13] W. Dür, L. Hartmann, M. Hein, M. Lewenstein, and H. J. Briegel, Phys. Rev. Lett. **94**, 097203 (2005).
[14] O. Gühne, G. Toth, and H. J. Briegel, New J. Phys. **7**, 229 (2005).
[15] X. Wang, Phys. Rev. A **64**, 012313 (2001).
[16] T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).
[17] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
[18] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
[19] P. Lou and J. Y. Lee, Phys. Rev. B **74**, 134402 (2006).
[20] S. Sachdev, *Quantum Phase Transitions* (Cambridge University, Cambridge, UK, 2000).

[21] S. Gu, G. Tian, and H. Lin, Chin. Phys. Lett. **24**, 2737 (2007).
[22] T. R. de Oliveira, G. Rigolin, M. C. de Oliveira, and E. Miranda, Phys. Rev. Lett. **97**, 170401 (2006).
[23] T. C. Wei, D. Das, S. Mukhopadyay, S. Vishveshwara, and P. M. Goldbart, Phys. Rev. A **71**, 060305(R) (2005).
[24] C. Y. Huang, and F. L. Lin, Phy. Rev. A **81**, 032304 (2010).
[25] A. Anfossi, P. Giorda, A. Montorsi, and F. Traversa, Phys. Rev. Lett. **95**, 056402 (2005).
[26] T. R. de Oliveira, G. Rigolin, and M. C. de Oliveira, Phys. Rev. A **73**, 010305(R) (2006).
[27] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[28] M. F. Yang, Phys. Rev. A **71**, 030302(R) (2005).
[29] L. Amico, F. Baroni, A. Fubini, D. Patane, V. Tognetti, and P. Verrucchi, Phys. Rev. A **74**, 022322 (2006).
[30] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, New York, 1992).
[31] D. A. Meyer and N. R. Wallach, J. Math. Phys. **43**, 4273 (2002).
[32] A. Montakhab and A. Asadian, Phys. Rev. A **77**, 062322 (2008).
[33] The Lanczos diagonalization technique does not significantly improve our system-size limitations (A. Langari, private communication), and density-matrix renormalization group methods are powerful but mostly for 1D systems.
[34] J. Jordan, R. Orus, G. Vidal, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. **101**, 250602 (2008).
[35] P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).
[36] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Mechanics, and Conceptions of the Universe* (Kluwer, Dordrecht, Netherlands, 1989), pp. 73–76.
[37] P. Stelmachovic and V. Buzek, Phys. Rev. A **70**, 032313 (2004).
[38] S. Campbell and M. Paternostro, Phys. Rev. A **82**, 042324 (2010).
[39] S. Yang, A. Bayat, and S. Bose, e-print arXiv:1005.2571 [quant-ph].
[40] T. R. de Oliveira, G. Rigolin, M. C. de Oliveira, and E. Miranda, Phys. Rev. A **77**, 032325 (2008).
[41] G. K. Brennen, Quantum Inf. Comput. **3**, 619 (2003).
[42] P. J. Love, A. M. van den Brink, A. Yu. Smirnov, M. H. S. Amin, M. Grajcar, E. Il'ichev, A. Izmalkov, and A. M. Zagoskin, Quantum Inf. Process. **6**, 187 (2007).
[43] S. Boixo and A. Monras, Phys. Rev. Lett. **100**, 100503 (2008).
[44] C. Brukner and A. Zeilinger, Phys. Rev. Lett. **83**, 3354 (1999).
[45] M. Horodecki and R. Horodecki, Phys. Lett. A **244**, 473 (1998); M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, Phys. Rev. Lett. **90**, 100402 (2003); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A **71**, 062307 (2005).
[46] M. N. Barber, in *Phase Transitions and Critical Phenomena* (Academic, London, 1983), Vol. 8, pp. 146–259.
[47] H. W. J. Blote and Y. Deng, Phys. Rev. E **66**, 066110 (2002).
[48] The d-dimensional transverse-field quantum Ising model is in the same universality class as that of the $(d+1)$-dimensional classical Ising model [19]. Therefore, our obtained values of ν should correspond to the 2D classical Ising ($\nu = 1$, exact) and the 3D classical Ising ($\nu \approx 0.6$) models, respectively.