State reconstruction of finite-dimensional compound systems via local projective measurements and one-way classical communication

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(Received 30 April 2009; revised manuscript received 10 February 2010; published 10 December 2010)

For a finite-dimensional bipartite system, the relation between local projections and post-selected state dependence on the global state submatrices is given in a general setting. As a result joint state reconstruction is derived with strict local projective measurements and one-way classical communication, reducing the number of detectors in comparison to standard procedures. Generalization to multipartite systems is given, also reducing the number of detectors for multiqubits.

DOI: 10.1103/PhysRevA.82.062308 PACS number(s): 03.67.Hk, 03.65.Wj

I. INTRODUCTION

Quantum state reconstruction relies on the ability to measure a complete set of observables and further manipulation of numerical data in a way to describe unambiguously a quantum system state. Several approaches have been given for the reconstruction of sole and joint quantum system states [1,2]. Remarkable experiments have been implemented employing these methods (see, e.g., Refs. [3,4]). For a discrete Hilbert space, quantum state reconstruction is achieved through the determination of the complete set of real parameters describing the state [5]. For a continuum Hilbert space it relies on homodyne tomographic reconstruction [2]. Unfortunately, full state reconstruction is highly demanding; for a multipartite system composed by N qubits, for example, $4^N - 1$ different real parameters (the generalized Stokes parameters), and thus an irreducible number of 3^N different settings are required. Supposing a qubit encoding over light polarization, the reconstruction is achieved through 2×3^N different measurements, with $2N$ photodetectors, most of them requiring multiple coincidences. Although it is certainly not possible to reduce the number of parameters to be determined, it is important to investigate alternative reconstruction schemes that reduce the actual experimental limitations imposed by multiple coincidence measurements, and/or the number of detectors.

In this paper we propose a state reconstruction scheme for a bipartite system of arbitrary finite dimension, shared by Alice and Bob, using only strict local projections and one-way classical communication. This scheme follows naturally from the answer to a more general question: If Alice performs projective measurements on her state, how does this affect the description of Bob's state? To answer that, we adopt an operational approach based on the global system submatrices. We obtain a general condition that an arbitrary set of projections must fulfill in order to allow complete reconstruction of the global state through active remote state preparation [6]. We analyze well-known and experimentally accessible projections, benefiting from previous derivations of single system state reconstruction [7–10], as well as the reconstruction of bipartite Gaussian continuous variable states with local measurements and classical communication [11].

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II. LOCAL PROJECTIVE MEASUREMENTS ON FINITE-DIMENSIONAL COMPOUND SYSTEMS

An arbitrary density matrix of a bipartite finite-dimensional system shared by two parts, Alice and Bob, is written in terms of well-suited $d_B \times d_B$ submatrices A_{ij} in the computational basis $\{|0,0\rangle,|0,1\rangle,\ldots,|0,d_B-1\rangle,|1,0\rangle,\ldots,|1,d_B-1\rangle,\ldots,|d_A-1,d_B-1\rangle\}$ as

$$\rho = \begin{pmatrix} A_{00} & \ldots & A_{0,d_A-1} \\ \vdots & \ddots & \vdots \\ A^{\dagger}_{0,d_A-1} & \ldots & A_{d_A-1,d_A-1} \end{pmatrix}, \quad (1)$$

where d_A (d_B) is the Hilbert space dimension of the Alice (Bob) subsystem. The ordering of the computational basis is made in such a way as to simplify algebraic manipulations. To reconstruct this arbitrary state, Alice and Bob have to perform a set of joint measurements over a large number of copies of the system. The results of those measurements randomly distribute along the observables' eigenbasis according to the state given. To optimize it, reducing the number of copies and detectors, we choose an active way, by replacing joint (coincidence) measures by local projections and classical communication. For that we derive two important results.

Lemma 1. Alice's density matrix is given by

$$\rho_A = \mathrm{Tr}_B \rho = \begin{pmatrix} \mathrm{Tr}A_{00} & \ldots & \mathrm{Tr}A_{0,d_A-1} \\ \vdots & \ddots & \vdots \\ \mathrm{Tr}A^{*}_{0,d_A-1} & \ldots & \mathrm{Tr}A_{d_A-1,d_A-1} \end{pmatrix}, \quad (2)$$

i.e., Alice's density matrix element $(\rho_A)_{ij}$ is the trace of the global state submatrix A_{ij}.

Proof. In the computational basis, $\rho = \sum_{i,k=0}^{d_A-1}\sum_{j,l=0}^{d_B-1} \rho_{ijkl}|ij\rangle\langle kl|$. Alice's state is given by the partial trace $\sum_{\nu=0}^{d_B-1} {}_B\langle\nu|\rho|\nu\rangle_B$. An arbitrary term of this sum, ${}_B\langle\nu|\rho|\nu\rangle_B$, is easily seen to be $\sum_{i,k=0}^{d_A-1}\rho_{i\nu k\nu}|i\rangle\langle k|$, and Alice's state is then $\rho_A = \sum_{\nu=0}^{d_B-1} {}_B\langle\nu|\rho|\nu\rangle_B = \sum_{\nu=0}^{d_B-1}\sum_{i,k=0}^{d_A-1}\rho_{i\nu k\nu}|i\rangle\langle k|$. An arbitrary element of ρ_A is $(\rho_A)_{\mu\eta} = \langle\mu|\rho_A|\eta\rangle = \sum_{\nu=0}^{d_B-1}\rho_{\mu\nu\eta\nu}$ and an arbitrary element of the submatrix $A_{\mu\eta}$ is $(A_{\mu\eta})_{\alpha\beta} = \rho_{\mu\alpha\eta\beta}$. Finally, the trace is simply $\mathrm{Tr}A_{\mu\eta} = \sum_{\nu=0}^{d_B-1}\rho_{\mu\nu\eta\nu} = (\rho_A)_{\mu\eta}$. ■

Proposition 1. Let $|\psi\rangle = \sum_{m=0}^{d_A-1} \alpha_m |m\rangle$ be a state in Alice's subsystem and let $P_{|\psi\rangle} = |\psi\rangle\langle\psi|$ be the projector in this state. Then we have that Bob's state after Alice's projection in state $|\psi\rangle$ is given by

$$\rho_B^{|\psi\rangle} = \frac{\sum_{m,n} \alpha_m \alpha_n^* A_{nm}}{\sum_{m,n} \alpha_m \alpha_n^* (\rho_A)_{nm}}. \quad (3)$$

Proof. Take $P_{|\psi\rangle} = \sum_{m,n=0}^{d_A-1} \alpha_m \alpha_n^* |m\rangle\langle n|$. Then it is easy to check that $(P_\psi \otimes I_B)\rho = \sum_{m,n,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \alpha_m \alpha_n^* \rho_{njkl} |mj\rangle\langle kl|$. Now, tracing out Alice's subsystem, we get $\mathrm{Tr}_A[(P_\psi \otimes I_B)\rho] = \sum_{\nu=0}^{d_A-1} {}_A\langle \nu|(\sum_{m,n,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \alpha_m \alpha_n^* \rho_{njkl} | mj\rangle\langle kl|)|\nu\rangle_A = \sum_{m,n=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \alpha_m \alpha_n^* \rho_{njml} |j\rangle\langle l|$. We see then that in Bob's subsystem, $\mathrm{Tr}_A[(P_{|\psi\rangle} \otimes I_B)\rho]$ is exactly equal to $\sum_{m,n=0}^{d_A-1} \alpha_m \alpha_n^* A_{nm}$. We conclude that Bob's state after a projection in Alice's subsystem is given by

$$\rho_B^{|\psi\rangle} = \frac{\mathrm{Tr}_A[(P_{|\psi\rangle} \otimes I_B)\rho]}{\mathrm{Tr}[(P_{|\psi\rangle} \otimes I_B)\rho]} = \frac{\sum_{m,n} \alpha_m \alpha_n^* A_{nm}}{\sum_{m,n} \alpha_m \alpha_n^* \mathrm{Tr} A_{nm}}. \quad (4)$$

With Lemma 1 we obtain Eq. (3), meaning that Alice prepares Bob's state $\rho_B^{|\psi\rangle}$ conditioned to $P_{|\psi\rangle}$. ■

III. PROTOCOL FOR JOINT STATE RECONSTRUCTION

A. General reconstruction projections

For the general procedures we notice that Eq. (3) can be solved with a series of projections on different states $|\psi^{(\nu)}\rangle = \sum_i \alpha_i^{(\nu)} |i\rangle$, where $\nu = 1,2,\ldots$. To these projections correspond the following system of equations

$$\sum_{m,n} \alpha_m^{(\nu)} \alpha_n^{(\nu)*} A_{nm} = \rho_B^{|\psi^{(\nu)}\rangle} \left(\sum_{m,n} \alpha_m^{(\nu)} \alpha_n^{(\nu)*} (\rho_A)_{nm} \right). \quad (5)$$

Performing suitable projections, so that Eq. (5) can be inverted, we can obtain the submatrices A_{mn}, and this is the same as to determine ρ. So the condition to be fulfilled by *any* set of projectors used by Alice is that the above system of equations must be invertible. In addition Bob must be able to perform local tomography of its system state. General methods for state estimation such as maximal likelihood are standardly applied in imperfect tomography and are inherent to local reconstruction methods [1,2]. Thus we simply assume that Bob is able to perform a good local tomography (with large fidelity), including all the necessary steps for estimation, with an overall of N_B copies.

B. 1 qubit ⊗ 1 qudit

Let us consider first the simple case when Alice's state is a qubit and Bob's is an arbitrary one (qudit). This is an important example, resembling the standard state reconstruction for qubit systems [5], apart from the unnecessary coincidence detections. An arbitrary density matrix in the basis $\{|0,0\rangle,\ldots,|0,d_B-1\rangle,|1,0\rangle,\ldots,|1,d_B-1\rangle\}$ is given by

$$\rho = \begin{pmatrix} A_{00} & A_{01} \\ A_{01}^\dagger & A_{11} \end{pmatrix},$$

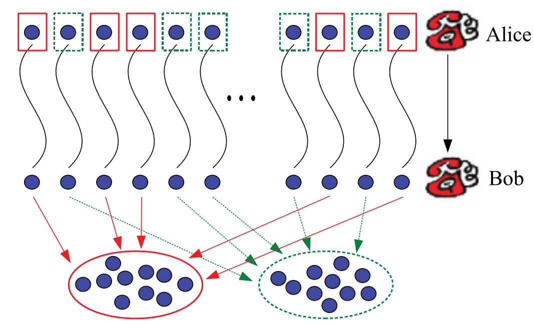

FIG. 1. (Color online) Sketch of the protocol for reconstruction of a qubit-qudit joint state. (i) Alice measures σ_z projecting on $|0\rangle$ or $|1\rangle$ depending on the outcomes ±1. (ii) Alice measures σ_x and σ_y projecting in $|0\rangle \pm |1\rangle$ and $|0\rangle \pm i|1\rangle$, respectively. Alice's results are classically communicated to Bob.

where A_{ij} are $d_B \times d_B$ submatrices. Also, Alice's density matrix can be written as

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1+z_A & x_A - iy_A \\ x_A + iy_A & 1-z_A \end{pmatrix},$$

in terms of the Stokes parameters 1, x_A, y_A, and z_A. Alice's projective measurements amount to performing the projections $P_{|0\rangle}$, $P_{|1\rangle}$, $P_{|0\rangle\pm|1\rangle}$, and $P_{|0\rangle\pm i|1\rangle}$. Equation (5) then reads

$$A_{00} = \frac{1+z_A}{2}\rho_B^{|0\rangle}, \quad A_{11} = \frac{1-z_A}{2}\rho_B^{|1\rangle}, \quad (6)$$

$$A_{00} + A_{11} \pm (A_{01} + A_{01}^\dagger) = (1 \pm x_A)\rho_B^{|0\rangle\pm|1\rangle}, \quad (7)$$

$$A_{00} + A_{11} \pm i(A_{01} - A_{01}^\dagger) = (1 \pm y_A)\rho_B^{|0\rangle\pm i|1\rangle}. \quad (8)$$

This system of equations is over determined, resulting in a redundancy, which is eliminated dealing appropriately with the measurement results. Such an issue appears naturally in the description of the protocol, which we do now. Suppose Alice and Bob share many copies of the system whose state they want to reconstruct. Suppose also that Bob needs N_B copies of an arbitrary state in his part to perform a good local tomography. Alice starts measuring σ_z on her states, performing projections $P_{|0\rangle}$ and $P_{|1\rangle}$ according to whether outcomes ±1 occur. Alice communicates her outcomes to Bob, who splits his states in two different subsets, according to these outcomes (see Fig. 1). Alice continues measuring until both outcomes occurred at least N_B times. Bob then performs local tomographies of $\rho_B^{|0\rangle}$ and $\rho_B^{|1\rangle}$. Note that Alice's outcomes are enough for Bob to obtain [12] the value z_A; by (6), Bob obtains the diagonal submatrices A_{00} and A_{11}. A subtle aspect should be noted. If Alice measures σ_z and obtains too many outcomes $+1$ (-1), Bob could not estimate ρ_B^{z-} (ρ_B^{z+}) properly. In this case, however, $z_A = \langle\sigma_z\rangle$ is very close to $+1$ (-1): we can assume then that $A_{11} = \frac{1-z_A}{2}\rho_B^{z-} \approx 0_{d_B \times d_B}$, where $0_{d_B \times d_B}$ is the null matrix of dimension $d_B \times d_B$.

The off-diagonal submatrix A_{01} is obtained with a slight modification. Alice measures σ_x on her states, obtaining outcomes ±1, corresponding respectively to projections $P_{|0\rangle\pm|1\rangle}$. Alice communicates her outcomes to Bob, who splits his states in two different subsets, according to the outcomes. The modification here is that Alice will measure *until* one of the outcomes occurred N_B times. When this situation occurs, she stops measuring. Bob then determines the density matrix corresponding to this outcome. Using (29) and the already determined diagonal submatrices A_{00} and A_{11}, Bob obtains $A_{01} + A_{01}^\dagger$. Once again Alice's outcomes are enough for Bob to obtain the value x_A; note also that he needs to use only one of the equations in (7). Repeating this procedure for σ_y, Bob obtains $A_{01} - A_{01}^\dagger$ by (8) and hence A_{01}. Alice and Bob then reconstruct the state using only strict local projective measurements and one-way classical communication. Note that x_A, y_A, and z_A are very well estimated [13].

For two-qubit implemented in optical polarization states, A_{ij} are 2×2 submatrices, which can be determined by Bob through the same kind of measurements employed by Alice. Like in the standard reconstruction [5], the number of different experimental settings is 9, but detections do not need to be in coincidence. Thus only two photodetectors are required as opposed to the four of the standard procedures [5]. The only requirement is the one-way communication of Alice's measurement results ± 1 to Bob, so that he can discriminate his states.

C. 1 qudit ⊗ 1 qudit

We present now a set of projectors suited to the case when both subsystems are of arbitrary dimensions. These projectors may be implemented in optical systems [9,10] or with Stern-Gerlach apparatuses [7]. Most importantly, they suffice to determine the elements of Alice's density matrix, which are necessary in the protocol. These projectors are $P_{|j\rangle}$ (with j varying from 0 to $d_A - 1$), $P_{|j\rangle\pm|k\rangle}$, and $P_{|j\rangle\pm i|k\rangle}$ (with $j < k$ and both varying from 0 to $d_A - 1$). Explicitly Eq. (5) for these projectors reads

$$A_{jj} = \rho_B^{|j\rangle}(\rho_A)_{jj}, \tag{9}$$

$$\Delta_{jk} = \rho_B^{|j\rangle\pm|k\rangle}\{(\rho_A)_{jj} + (\rho_A)_{kk} \pm 2\mathrm{Re}[(\rho_A)_{jk}]\}, \tag{10}$$

$$\Omega_{jk} = \rho_B^{|j\rangle\pm i|k\rangle}\{(\rho_A)_{jj} + (\rho_A)_{kk} \mp 2\mathrm{Im}[(\rho_A)_{jk}]\}, \tag{11}$$

where $\Delta_{jk} \equiv A_{jj} + A_{kk} \pm (A_{jk} + A_{jk}^\dagger)$, $\Omega_{jk} \equiv A_{jj} + A_{kk} \pm i(A_{jk} - A_{jk}^\dagger)$, and $k > j = 0, \ldots, d_A - 1$. Again the resulting system of equations is over determined.

The protocol described below resembles the qubit-qudit protocol in many steps. Suppose that Alice and Bob share many copies of the state they want to reconstruct and that Bob needs N_B copies of an arbitrary state in his subsystem to perform a good local tomography. To determine a diagonal submatrix A_{jj}, Alice starts measuring $P_{|j\rangle}$ on her states, communicating to Bob whether or not (outcomes 1 or 0, respectively) the projection actually happened. She repeats this procedure until there happened N_B outcomes 1. Bob then determines by any way the density matrix of his states corresponding to outcomes 1. As Alice's element $(\rho_A)_{jj}$ is precisely the mean value $\langle P_{|j\rangle}\rangle$, Bob can get this value with Alice's outcomes and hence, by (9), he determines A_{jj}. If $(\rho_A)_{jj}$ is too small, we can approximate A_{jj} by the null matrix, as done in the qubit-qudit case for the diagonal submatrices.

To determine an off-diagonal submatrix A_{jk} ($j \neq k$), Alice chooses first to perform one of the projections $P_{|j\rangle\pm|k\rangle}$ on her states. Due to the redundancy in Eqs. (10), only one of them is enough to determine $A_{jk} + A_{jk}^\dagger$ (it is important, however, that Alice be able to perform the other one). Alice then performs the projection chosen on her states, communicating to Bob whether or not (outcomes 1 or 0, respectively) the projection actually happened. She repeats this procedure until there happened N_B outcomes 1. Bob then determines the density matrix of his states corresponding to outcomes 1. Alice's element $\mathrm{Re}[(\rho_A)_{jk}]$ is determined if one observes that $P_{|j\rangle+|k\rangle} + P_{|j\rangle-|k\rangle} = 2(P_{|j\rangle} + P_{|k\rangle})$. Then, we have that $\langle P_{|j\rangle+|k\rangle}\rangle + \langle P_{|j\rangle-|k\rangle}\rangle = 2(\rho_A)_{jj} + 2(\rho_A)_{kk} \equiv g_{jk}$, where g_{jk} is a constant value which is already determined, since Alice's diagonal elements are known. As $\mathrm{Re}[(\rho_A)_{jk}] = \langle P_{|j\rangle+|k\rangle}\rangle - \langle P_{|j\rangle-|k\rangle}\rangle$ [7,8], we can take $\mathrm{Re}[(\rho_A)_{jk}] = 2\langle P_{|j\rangle+|k\rangle}\rangle - g_{jk}$ or $\mathrm{Re}[(\rho_A)_{jk}] = g_{jk} - 2\langle P_{|j\rangle-|k\rangle}\rangle$. This idea can also be used in the situation where the projector chosen has a too small mean value, making it impossible for Bob to have N_B copies on the corresponding subset. We can shift to the other projector, which will have a compensating larger value.

There is, however, a third extreme situation, when both mean values are small. But, in this case, $(\rho_A)_{jj}$ and $(\rho_A)_{kk}$ are necessarily small as well, and the value $(\rho_A)_{jj} + (\rho_A)_{kk} \pm 2\mathrm{Re}[(\rho_A)_{jk}]$ in (10) will be too small; then we can approximate $A_{jk} + A_{jk}^\dagger$ by the null matrix. In all situations, by (10) Bob determines $A_{jk} + A_{jk}^\dagger$. Finally, Alice chooses to perform one of the projections $P_{|j\rangle\pm i|k\rangle}$ and repeats the procedure above. The property $P_{|j\rangle+i|k\rangle} + P_{|j\rangle-i|k\rangle} = 2(P_{|j\rangle} + P_{|k\rangle})$ allows Bob to get $\mathrm{Im}[(\rho_A)_{jk}]$. Bob then finds $A_{jk} - A_{jk}^\dagger$ by (11) and hence determines A_{jk}. As all submatrices were determined, the state is reconstructed, using only local projections and one-way classical communication [13]. Also the number of detectors employed can be reduced, depending on the experimental setup employed.

IV. PHYSICAL IMPLEMENTATION

The physical implementation of the projective measurements previously discussed will obviously depend on the actual state encoding. There are many situations in practice where the reconstruction protocol would be appealing: optical systems, trapped ions, and atoms trapped in optical lattices and in Stern-Gerlach experiments. We will describe here only the possible implementation for optical systems.

A. Qubit state reconstruction

Any single qubit state may be uniquely represented by three parameters $\{S_1, S_2, S_3\}$:

$$\rho = \frac{1}{2}\sum_{i=0}^{3} S_i \sigma_i, \tag{12}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{13}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{14}$$

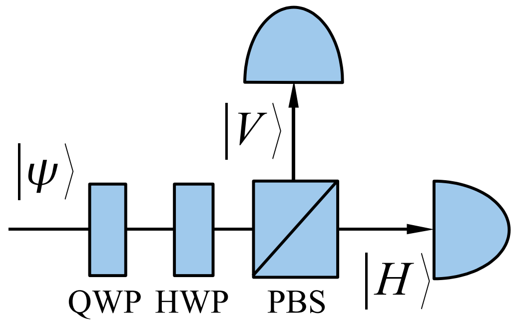

FIG. 2. (Color online) Realization of the set of projections for a qubit encoded in polarization states. Any projection and its orthogonal complement can be implemented with a quarter-wave plate (QWP), a half-wave plate (HWP), a polarized beam splitter (PBS), and two photodetectors; see Ref. [5] for details.

and $S_i = \mathrm{Tr}\{\sigma_i \rho\}$; $\sum_{i=0}^{3} S_i^2 \geqslant 1$. Physically, each of these parameters directly corresponds to the outcome of a specific pair of projective measurements:

$$S_0 = P_{|0\rangle} + P_{|1\rangle}, \tag{15}$$

$$S_1 = P_{\frac{1}{2}(|0\rangle+|1\rangle)} - P_{\frac{1}{2}(|0\rangle-|1\rangle)}, \tag{16}$$

$$S_2 = P_{\frac{1}{2}(|0\rangle+i|1\rangle)} - P_{\frac{1}{2}(|0\rangle-i|1\rangle)}, \tag{17}$$

$$S_3 = P_{|0\rangle} - P_{|1\rangle}, \tag{18}$$

where $P_{|\psi\rangle}$ is the probability to measure state $|\psi\rangle$ and follow the property $P_{|\psi\rangle} + P_{|\psi^\perp\rangle} = 1 \rightarrow P_{|\psi\rangle} - P_{|\psi^\perp\rangle} = 2P_{|\psi\rangle} - 1$. For photon polarization qubit encoding these are usually called Stokes parameters describing the coordinate of the state in the Poincaré sphere. The necessary set of orthogonal states are horizontal, vertical, diagonal, antidiagonal, right-circular, and left-circular light polarization, respectively, given by

$$|H\rangle = |0\rangle, \tag{19}$$

$$|V\rangle = |1\rangle, \tag{20}$$

$$|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}, \tag{21}$$

$$|A\rangle = (|H\rangle - |V\rangle)/\sqrt{2}, \tag{22}$$

$$|R\rangle = (|H\rangle + i|V\rangle)/\sqrt{2}, \tag{23}$$

$$|L\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}, \tag{24}$$

and their implementation can be simply given in a specific arrangement of the setup indicated in Fig. 2 as has been discussed in Ref. [5]. A total number of three distinct arrangements are necessary. Also the extension of reconstruction for two qubits can be obtained with two similar sets, one for each qubit. That gives a total number of 15 distinct experimental arrangements.

B. Qudit state reconstruction

For qudits, an encoding over light polarization is not possible, so one needs a set of light states that accounts for the larger Hilbert space. One possible linear optical implementation would be through optical paths that a single photon would take in an arrangement of beam splitters such as in Fig. 3 for the simplest case of a qutrit, as described in Ref. [10]. In that case, depending on the phases ϕ_1 and ϕ_2 imprinted in the light by the two elements in the figure, one

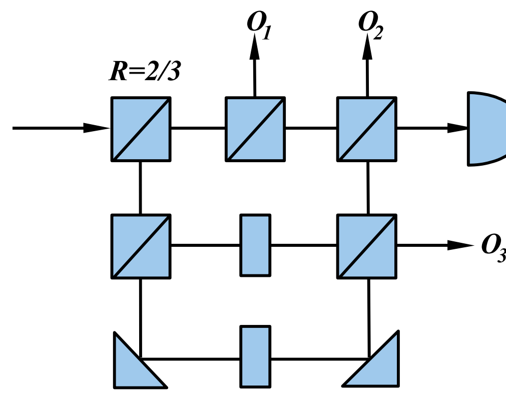

FIG. 3. (Color online) Encoding of a qutrit through an interferometric path taken by a single photon as given by the three registers O_1, O_2, and O_3 representing the qutrit basis states $|0\rangle$, $|1\rangle$, and $|2\rangle$, respectively. All the beam splitters, except the one indicated, have reflectance $R = 1/2$. Whenever the detector does not click, a qutrit state is generated depending on the phases ϕ_1 and ϕ_2 imprinted. The same setup allows the set of state projections. See Ref. [10] for details.

would be able to generate a full set of orthogonal states for a qutrit. This same setup allows the implementation of the projections on those states (see Ref. [10] for details) and generalizes for qudits. So the projectors, $P_{|j\rangle}$, $P_{|j\rangle\pm|k\rangle}$, $P_{|j\rangle\pm i|k\rangle}$, necessary for deriving Eqs. (9)–(11), are, in principle, implementable with present technology. Also important to mention is the remarkable recent achievement in Ref. [14] for the reconstruction of a qutrit-qutrit state involving the orbital angular momentum of an atomic ensemble and a photon. The same scheme could be used to implement the projections outlined in the paper and could be extended for higher dimensional Hilbert spaces.

V. MULTIPLE QUDITS STATE RECONSTRUCTION

When dealing with a system composed of N subsystems, the state space is composed of the tensor product of the individual Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N. \tag{25}$$

We can view this system as a bipartite one, considering

$$\mathcal{H} = \underbrace{\mathcal{H}_1}_{\mathcal{H}_{A^{(1)}}} \otimes \underbrace{\mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N}_{\mathcal{H}_{B^{(1)}}} = \mathcal{H}_{A^{(1)}} \otimes \mathcal{H}_{B^{(1)}}. \tag{26}$$

We can then apply the protocol of bipartite state reconstruction to this bipartite system. Local projective measurements are performed on $\mathcal{H}_1$ and the outcomes are sent to $\mathcal{H}_{B^{(1)}}$ via a one-way classical channel. The relation between the submatrices of the global density matrix and the local projections on $\mathcal{H}_1$ is given by (5), with the relabeling of $A \rightarrow A^{(1)}$ and $B \rightarrow B^{(1)}$. As the measurements are projective, after measuring the subsystem $\mathcal{H}_1$ will be left in pure states and so will not affect subsequent operations made on $H_{B^{(1)}}$. We can thus consider only $H_{B^{(1)}}$ and use the same reasoning for this system. If, for example, the first subsystem uses the local projections $P_{|i\rangle}$, $P_{|j\rangle\pm|k\rangle}$, $P_{|j\rangle\pm i|k\rangle}$, tomography of the various remotely prepared states $\rho_{B^{(1)}}^{|j\rangle}$, $\rho_{B^{(1)}}^{|j\rangle\pm|k\rangle}$, $\rho_{B^{(1)}}^{|j\rangle\pm i|k\rangle}$ must be realized in order to solve (5). But these states are defined on $\mathcal{H}_{B^{(1)}} = \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$.

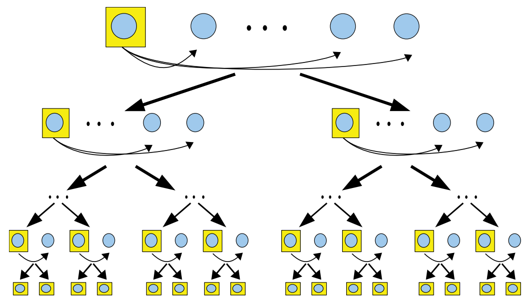

FIG. 4. (Color online) Reconstruction of a N-partite qubit state through local measurements and one-way classical communication and sequential bipartition. Local measurements in $\mathcal{H}_{A^{(i)}}$ classically communicated conditionally split the subensemble for $\mathcal{H}_{B^{(i)}}$, $i = 1,\ldots,N-1$. The final local measurement allow the reconstruction of the joint state.

We can then repeat the procedure for each state. We call

$$\mathcal{H}_{B^{(1)}} = \underbrace{\mathcal{H}_2}_{\mathcal{H}_{A^{(2)}}} \otimes \underbrace{\mathcal{H}_3 \otimes \cdots \otimes \mathcal{H}_N}_{\mathcal{H}_{B^{(2)}}} = \mathcal{H}_{A^{(2)}} \otimes \mathcal{H}_{B^{(2)}}. \quad (27)$$

Local projective measurements, $P_{|i\rangle}$, $P_{|j\rangle\pm|k\rangle}$, $P_{|j\rangle\pm i|k\rangle}$, are performed on the states defined on $\mathcal{H}_2$ and the outcomes are sent to $\mathcal{H}_3,\mathcal{H}_4,\ldots,\mathcal{H}_N$ via a one-way classical channel acting as $\rho_{B^{(2)}}^{|j\rangle}$, $\rho_{B^{(2)}}^{|j\rangle\pm|k\rangle}$, $\rho_{B^{(2)}}^{|j\rangle\pm i|k\rangle}$. The submatrices of each state will obey (5), with relabeling of $A \to A^{(2)}$ and $B \to B^{(2)}$. Tomography of the various states prepared remotely by the local projective measurements made on $\mathcal{H}_2$ must be done, in order to solve (5) and so on (see Fig. 4). We repeat then for each state the same reasoning, and after $N-1$ repetitions of this procedure, the last one being over subsystems $A^{(N-1)}$ and $B^{(N-1)}$, we can reconstruct the global state density matrix (see example in the next section). Note that if the N subsystems are delocalized in space, since the communication is one way, only the last station will have complete information to reconstruct the whole joint state at the end. For N qubits, one concludes that the reconstruction task requires N detectors, in contrast to the usual $2N$ with coincidence.

VI. EXAMPLE: THREE-QUBIT STATE RECONSTRUCTION

As an example of multipartite state reconstruction, we consider the three-qubit case. The state space is composed of the tensor product $H = H_A \otimes H_B \otimes H_C$ of Alice, Bob, and Charlie. Calling $H_{A^{(1)}} = H_A$ and $H_{B^{(1)}} = H_B \otimes H_C$, the density matrix of the system $H = H_{A^{(1)}} \otimes H_{B^{(1)}}$ will be

$$\rho = \begin{pmatrix} A_{00}^{(1)} & A_{01}^{(1)} \\ A_{01}^{(1)\dagger} & A_{11}^{(1)} \end{pmatrix},$$

with $A_{00}^{(1)}$, $A_{01}^{(1)}$, and $A_{11}^{(1)}$ being 4×4 submatrices. As shown in Sec. III B, we have the following system of equations for the submatrices:

$$A_{00}^{(1)} = \frac{1+z_A}{2}\rho_{B^{(1)}}^{|0\rangle}, \quad A_{11}^{(1)} = \frac{1-z_A}{2}\rho_{B^{(1)}}^{|1\rangle}, \quad (28)$$

$$A_{00}^{(1)} + A_{11}^{(1)} \pm \left(A_{01}^{(1)} + A_{01}^{(1)\dagger}\right) = (1\pm x_A)\rho_{B^{(1)}}^{|0\rangle\pm|1\rangle}, \quad (29)$$

$$A_{00}^{(1)} + A_{11}^{(1)} \pm i\left(A_{01}^{(1)} - A_{01}^{(1)\dagger}\right) = (1\pm y_A)\rho_{B^{(1)}}^{|0\rangle\pm i|1\rangle}. \quad (30)$$

Alice performs measurements of σ_i, $i = x,y,z$ on her states and sends the outcomes to Bob and Charlie. After each measurement, the joint states of Bob and Charlie will be left in states $\rho_{B^{(1)}}^{|0\rangle}$, $\rho_{B^{(1)}}^{|1\rangle}$, $\rho_{B^{(1)}}^{|0\rangle\pm|1\rangle}$, and $\rho_{B^{(1)}}^{|0\rangle\pm i|1\rangle}$, according to Alice's results. For Eq. (29), only one of the states $\rho_{B^{(1)}}^{|0\rangle\pm|1\rangle}$ needs to be remotely prepared, since the equation is redundant. Without loss of generality, we assume that during the measurements of σ_x, Alice realizes that $\rho_{B^{(1)}}^{|0\rangle+|1\rangle}$ is easier to prepare (needs less copies). Thus she will conditionally prepare this state consuming some number of copies—a number such that Bob and Charlie are able to reconstruct the state just prepared. In the same fashion, let us assume that during the measurement of σ_y, Alice chooses to prepare $\rho_{B^{(1)}}^{|0\rangle+i|1\rangle}$, consuming some number of copies.

For Eqs. (28), both $\rho_{B^{(1)}}^{|0\rangle}$ and $\rho_{B^{(1)}}^{|1\rangle}$ are needed to determine the diagonal submatrices, unless one of them is too difficult to prepare, so that we can approximate the corresponding submatrix by the null matrix, as explained in Sec. III B. So, Alice prepares these two states performing measurements of σ_z.

Bob and Charlie have to reconstruct states $\rho_{B^{(1)}}^{|0\rangle}$, $\rho_{B^{(1)}}^{|1\rangle}$, $\rho_{B^{(1)}}^{|0\rangle+|1\rangle}$, and $\rho_{B^{(1)}}^{|0\rangle+i|1\rangle}$. Let us label them, in this order, as ρ_{BC}^k, with $k = 1,\ldots,6$. We now call $H_{A^{(2)}} = H_B$ and $H_{B^{(2)}} = H_C$. So, we have that the density matrix of the system is given by

$$\rho_{BC}^k = \begin{pmatrix} A_{00}^{k(2)} & A_{01}^{k(2)} \\ A_{01}^{k(2)\dagger} & A_{11}^{k(2)} \end{pmatrix},$$

with $A_{00}^{k(2)}$, $A_{01}^{k(2)}$, and $A_{11}^{k(2)}$ being 2×2 submatrices. Then we have the following set of equations:

$$A_{00}^{k(2)} = \frac{1+z_B^k}{2}\rho_C^{|0\rangle(k)}, \quad A_{11}^{k(2)} = \frac{1-z_B^k}{2}\rho_C^{|1\rangle(k)},$$

$$A_{00}^{k(2)} + A_{11}^{k(2)} \pm \left(A_{01}^{k(2)} + A_{01}^{k(2)\dagger}\right) = \left(1\pm x_B^k\right)\rho_C^{|0\rangle\pm|1\rangle(k)},$$

$$A_{00}^{k(2)} + A_{11}^{k(2)} \pm i\left(A_{01}^{k(2)} - A_{01}^{k(2)\dagger}\right) = \left(1\pm y_B^k\right)\rho_C^{|0\rangle\pm i|1\rangle(k)}.$$

Bob measures σ_i $(i = x,y,z)$ on his part of the bipartite state ρ_{BC}^k and sends the outcomes to Charlie. He then remotely prepares $\rho_C^{|0\rangle(k)}$ and $\rho_C^{|1\rangle(k)}$ during the measurement of σ_z, obtaining the diagonal submatrices above. For the nondiagonal submatrices, for each k he chooses one of the states $\rho_C^{|0\rangle\pm|1\rangle(k)}$ to prepare, during the measurement of σ_x. The same reasoning applies for the measurement of σ_y. Finally, Charlie reconstructs the states prepared by Bob, and with the classical information given by Alice and Bob, he can determine the global state ρ.

VII. NUMBER OF COPIES

The number of copies that we estimate here is obtained considering that both Alice and Bob use projective measurements with standard projectors $P_{|j\rangle}$, $P_{|j\rangle\pm|k\rangle}$, $P_{|j\rangle\pm i|k\rangle}$. Let us first consider single systems. For a d-level system, let N_P be the number of copies needed to estimate the mean value of an arbitrary projector $P_{|\psi\rangle}$ with some desired accuracy. This number will depend on the physical system considered, the experimental setup used, and the accuracy desired and is obtained through numerical analysis. When we are dealing with a single system, the mean values of d^2-1 different projectors are needed, as is well known. Note that this is less than the total number $2d^2 - d$ of standard projectors. We stress that this is typical of a single system tomography. So, we conclude that the overall number of copies needed will be $(d^2 - 1)N_P + \epsilon$. Here ϵ represents the extra copies needed due to errors not implicit in the number N_P, such as the change of basis of measurements; this number is assumed to be low, compared to N_P.

Now let us consider a bipartite system with arbitrary dimensions d_A and d_B. In standard reconstruction protocols, we have to obtain the mean values of all the products $\langle P_A \otimes P_B\rangle$. It is clear then that we need $(2d_A^2 - d_A) \times (2d_B^2 - d_B)$ different measurements to be performed: due to coincidence requirements, all different projectors of both systems must be considered. Now let $N_{P'}$ be the number of copies needed to estimate with some accuracy the mean value of an arbitrary product of the form $P_{|\psi\rangle} \otimes P_{|\psi'\rangle}$. We conclude that $\gamma = [(2d_A^2 - d_A)(2d_B^2 - d_B) - 1]N_{P'} + \epsilon$ copies [15] are needed in standard reconstruction protocols based on projective measurements (the -1 is due to the normalization condition).

For a multipartite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, it is straightforward that $\{[\prod_{i=1}^N (2d_i^2 - d_i)] - 1\}N_{P'}$ is the number of copies consumed in standard protocols, where $d_i = \dim\mathcal{H}_i$ and $N_{P'}$ is the number of copies required to obtain the mean value of an arbitrary projector $P_1 \otimes P_2 \otimes \cdots \otimes P_N$. In the multiqubit case, for example, the number of copies consumed in standard protocols will be $(6^N - 1)N_{P'}$.

Let us consider now a bipartite state of two qubits. As explained previously, Alice has to remotely prepare one of the states $\rho_B^{|0\rangle\pm|1\rangle}$, one of the states $\rho_B^{|0\rangle\pm i|1\rangle}$, and both of the states $\rho_B^{|0\rangle}$ and $\rho_B^{|1\rangle}$. Bob's task is to reconstruct these states in his subsystem. Since he performs only local tomographies, he will need $N_B = (2^2 - 1)N_P = 3N_P$ copies for each of these states. But each of these four states needs also some number of copies to be remotely prepared by Alice. For example, if Alice obtains 80% of $+1$ outcomes during the measurement of σ_x, she will need $3N_P/0.8 = 3.75N_P$ copies to remotely prepare $3N_P$ copies of the state $\rho_B^{|0\rangle+|1\rangle}$ for Bob. In general, if Alice obtains $n_{x\sigma}$ (σ represents the outcome which appeared most) outcomes from a total of n_x measurements of σ_x, she will need $(n_x/n_{x\sigma})3N_P$ copies to remotely prepare $3N_P$ copies of the state corresponding to this outcome [16]. Thus the number of copies ranges from $3N_P$ ($n_{x\sigma} = n_x$) to $6N_P$ ($n_{x\sigma} = n_x/2$). The same reasoning is valid for the states originated from the measurement of σ_y: $(n_y/n_{y\sigma})3N_P$ copies are consumed, where n_y is the total number of measurements of σ_y and $n_{y\sigma}$ is the number of outcomes which appeared most.

In the measurement of σ_z, however, Alice needs to prepare both states $\rho_B^{|0\rangle}$ and $\rho_B^{|1\rangle}$. If, for example, Alice obtains 80% of $+1$ outcomes during the measurement of σ_z, she will need $3N_P/0.2 = 15N_P$ copies to remotely prepare $3N_P$ copies of the state $\rho_B^{|1\rangle}$, which is generated 20% of the time. The other $12N_P$ states will be of course in the $\rho_B^{|0\rangle}$ state. In general, if Alice obtains a majority of $n_{z\sigma}$ outcomes from a total of n_z measurements of σ_z, she will need $[n_z/(n_z - n_{z\sigma})]3N_P$. In a careless analysis, the number of copies would range from $6N_P$, when $n_{z\sigma} = n_z/2$ to ∞, when $n_{z\sigma} = n_z$. But the last situation corresponds to a $z_A = +1$ or $z_A = -1$, which restricts the corresponding submatrix to be the null matrix. So, the real number of copies needed in the extreme situations of $z_A = \pm 1$ is $3N_P$, since we will need to determine only one of the states $\rho_B^{|0\rangle}, \rho_B^{|1\rangle}$ (the submatrix corresponding to the other state will be the null matrix). However, there could be situations where the number $n_{z\sigma}$ is too close to n_z, but the fidelity of reconstruction required (see discussion in the next section) is so high that the approximation by the null matrix would be inappropriate. In this case, our protocol could consume more copies than standard reconstruction schemes. Let T_V designate a *threshold value* for the percentage $n_{z\sigma}/n_z$, in the sense that above this value we can consider z_A as nearly $+1$ or -1 and one of the diagonal submatrices can be approximated by the null matrix. It is clear that T_V should be a function of the fidelity of reconstruction. The overall number of copies for a two-qubit system is then

$$\left(\frac{n_x}{n_{x\sigma}} + \frac{n_y}{n_{y\sigma}} + \frac{n_z}{n_z - n_{z\sigma}\Theta(T_V - n_{z\sigma}/n_z)}\right)3N_P,$$

where $\Theta(x) = 1$ if $x \geqslant 0$, and 0 if $x < 0$. The generalization for a qubit-qudit system is straightforward:

$$\left(\frac{n_x}{n_{x\sigma}} + \frac{n_y}{n_{y\sigma}} + \frac{n_z}{n_z - n_{z\sigma}\Theta(T_V - n_{z\sigma}/n_z)}\right)\left(d_B^2 - 1\right)N_P,$$

since $(d_B^2 - 1)N_P$ is the number of copies required by Bob to perform a local tomography. Finally, it is easy to see that the number of copies required for the reconstruction of the important case of an N-qubit system is given by

$$\prod_{i=1}^{N-1}\left(\frac{n_x^i}{n_{x\sigma}^i} + \frac{n_y^i}{n_{y\sigma}^i} + \frac{n_z^i}{n_z^i - n_{z\sigma}^i\Theta\left(T_V^i - n_{z\sigma}^i/n_z^i\right)}\right)3N_P,$$

with obvious notation. This should be compared with the number of copies required by standard protocols: $(6^N - 1)N_{P'}$, where $N_{P'}$ is the number of copies required to obtain the mean value of an arbitrary projector $P_1 \otimes P_2 \otimes \cdots \otimes P_N$. We see that plenty of situations can occur, depending on the state to be reconstructed and the fidelity required. The most economic situation of our protocol occurs when $n_{x\sigma}^i/n_x^i = n_{y\sigma}^i/n_y^i = n_{z\sigma}^i/n_z = 1$, where we will have an overall number of $3^{N-1} \times 3N_P = 3^N N_P$ copies needed. Thus, we would have a huge decrease in the number of copies for increasing N, in comparison with the number $(6^N - 1)N_{P'}$ of standard protocols. But it should be clear that this situation is highly unlikely in general. The opposite situation should be a huge increase in the number of copies due to a high fidelity requirement, since all the qubits should have $n_{z\sigma}^i/n_z^i$ near the threshold value T_V. In fact, it is easy to see that in the first situation the reduced states are all pure and thus we have a multipartite system in a state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_{N-1}\rangle \otimes \rho_N$. In this situation each subsystem could perform local tomographies of their states, with no need of conditional operations at all. A deeper analysis will be addressed in future research.

VIII. FIDELITY OF RECONSTRUCTION

How close the inferred state is to the original state depends on several parameters, such as the mixing degree of the state, and more or less copies have to be used accordingly. There will be, however, a maximum number of copies which have to be used to obtain a state with some prefixed accuracy, and that was named N_B: the number of copies Bob needs to reconstruct his state with arbitrary fidelity. The global state fidelity of reconstruction is not covered here, since this kind of calculation goes beyond the scope of the present work. However, we gave a rough estimation of the total number of copies needed to reconstruct the global state when both parts use projective measurements, and we implicitly assume that this number is the one to obtain the state with a good fidelity of reconstruction. We believe that the numbers given can help in future research in order to calculate fidelity calculations.

IX. CONCLUDING REMARKS

Quantum state reconstruction is an extremely important task in any implementation of quantum computation or quantum communication protocol. It is supposed to determine unambiguously a single or joint quantum system state. Its implementation for composed systems is severely compromised by the requirement of joint measurements, which increases in number for increasing individual Hilbert space dimension and/or number of subsystems. We have given an alternative procedure for reconstruction based on local measurements and classical communication that enables bipartite or multipartite quantum systems of arbitrary Hilbert space dimension to be reconstructed without the need of joint or coincidence measurements, saving on the number of detectors. This saving of resources at the cost of classical information may indicate deep consequences for quantum communication. Indeed, the splitting into subsets due to specific projective measurements at one part may be related to *steering*, a one-way nonlocal action [6]. When one performs local projective measurements on a bi- or multipartite system it is actually acting remotely (nonlocally) at the other parties, which can then with a convenient protocol optimize the use of resources. Also, in a practical situation, the fidelity of the reconstructed state should be dealt with. All those subjects are demanding on their own and need profound considerations. We leave it for future research.

ACKNOWLEDGMENTS

This work is supported by CAPES, CNPq, and FAPESP through the National Institute for Quantum Information.

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[12] This value is given by $z_A = \langle \sigma_z \rangle = (n_+ - n_-)/(n_+ + n_-)$, where $n_\pm$ is the number of ± 1 outcomes, respectively.

[13] In practice, both parts can combine previously the way of doing measurements in order to avoid storing or delaying of particles, in a *prepare and measure protocol*. Before the experiment, Alice tells Bob how she will organize her measurements. He will make his measurements suited to her information, after Alice's projections. Bob can organize his results according to Alice's classical information, organizing the subsets in order to do the calculations.

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[15] We suppose $N_{P'} \geqslant N_P$ in general, since the estimation of the mean value of a projector acting on an individual system does not require coincidence requirements and also the Hilbert space dimension is lower. However, for a small number N of subsystems, it is fair to say that $N_{P'} \approx N_P$, since coincidence requirements will not consume many copies.

[16] In practice, in the first N_P measurements she will have a good estimate of the percentage $n_{x\sigma}/n_x$. But this estimate could change during subsequent measurements. However, by the end of the first $3N_P$ measurements (the minimal number of copies required) Alice will have a very good estimate of the number of additional measurements needed.