Two-qubit Bell inequality for which positive operator-valued measurements are relevant

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A bipartite Bell inequality is derived which is maximally violated on the two-qubit state space if measurements describable by positive operator valued measure (POVM) elements are allowed, rather than restricting the possible measurements to projective ones. In particular, the presented Bell inequality requires POVMs in order to be maximally violated by a maximally entangled two-qubit state. This answers a question raised by N. Gisin [in *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle: Essays in Honour of Abner Shimony*, edited by W. C. Myrvold and J. Christian (Springer, The Netherlands, 2009), pp. 125–138].

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I. INTRODUCTION

All pure bipartite entangled states violate the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [1,2] by performing appropriate measurements on the subsystem's state [3]. On the other hand, any Bell inequality that can be violated in the quantum world can be maximally violated by some pure state and projective (von Neumann) measurements if no restrictions are put on the underlying Hilbert space. However, projective measurements are not the most general measurements. Still, we have found no examples in the literature where general [so-called positive operator valued measure (POVM)] measurements would provide a larger violation of some Bell inequalities by restricting them to a given dimensional state space. In fact, this concerns a more specific problem posed recently by Gisin asking for a state for which POVM measurements would perform better than projective ones, yielding a larger violation of Bell inequalities [4].

In the present paper we study these problems by restricting ourselves to the two-qubit space and to maximally entangled qubits, respectively. Note that in the case of two-outcome Bell inequalities, POVMs are not better than projective measurements with respect to the amount of Bell violation [5–8]. Further, in the case of all those multiple-outcome Bell inequalities that we are aware of in the literature, projective measurements still give maximal violation in the specific state space considered (see, e.g., Refs. [9–13]).

On a related note, we mention a recent result by Cabello and also by Nakamura on the Kochen-Specker theorem [14] proving that this theorem can be extended to a single qubit if POVM measurements can be used instead of only projective ones [15]. Taking this result together with a method of Refs. [6,16] for building a $d \times d$ pseudotelepathy game [17] from a d -dimensional Kochen-Specker construction, one may wonder whether this would entail a Bell inequality for a system of dimension 2×2 where POVMs would give a higher violation than projective ones. However, this approach turns out not to be feasible due to the proof of Brassard *et al.* [18], stating that there is no pseudotelepathy game of dimension 2×2 even if POVMs are included.

Despite the above negative results, we manage to derive a Bell inequality which proves that POVMs are relevant with respect to projective measurements for a two-qubit maximally entangled state and also for the case of a two-qubit space. First, Sec. II introduces notation, then Sec. III reviews shortly the formalism of POVM and projective measurements on qubit states. In Sec. IV, an optimization problem is presented considering positive definite matrices versus projection matrices. In Sec. V, a parametrized Bell inequality is given, for which the quantum bound is calculated either including POVM measurements or being restricted to projective measurements. We consider two cases: Either the shared state is the two-qubit singlet state (Sec. V C), or it may be any state in the two-qubit state space (Sec. V D). In Sec. VI, we conclude and pose open questions.

II. PRELIMINARIES

Let us consider a standard Bell scenario [1]. Two spacelike separated parties, Alice and Bob, share copies of a quantum state ρ in some dimension $d \times d$. The two parties can choose among N_A and N_B different measurements which are labeled by $x \in \{0,\dots,N_A-1\}$ for Alice and by $y \in \{0,\dots,N_B-1\}$ for Bob, where we denote the respective outputs by $a \in \{0,\dots,r_A-1\}$ and $b \in \{0,\dots,r_B-1\}$. In the most general description of a quantum measurement, M_a^x (M_b^y) denote the positive operator corresponding to outcome x (y) when Alice (Bob) performs measurement a (b). Then the joint conditional probabilities can be calculated in quantum theory by the formula

$$p(ab|xy) = \mathrm{Tr}\big(\rho M_a^x \otimes M_b^y\big). \quad (1)$$

The positive operators above, summing to the identity $\sum_a M_a^x = \sum_b M_b^y = \mathbb{1}$, constitute POVMs for any inputs x,y. However, in the case of projection measurements, the positive operators M_a^x and M_b^y are projectors, hence the ones belonging to the same inputs ought to be orthogonal to each other.

A Bell expression is a linear function $\vec{b}\cdot\vec{p} = \sum_{a,b,x,y} b_{abxy}\, p(ab|xy)$ of the conditional probabilities $p(ab|xy)$ defined by Eq. (1), where $\vec{b}$ has real components. In order to maximize a Bell expression, it is enough to consider pure states $\rho = |\psi\rangle\langle\psi|$. In our study, we will focus on the state space of a pair of qubits, where up to a change of local basis any

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pure state can be written as $|\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$. Now, let us take $\theta = \pi/4$, resulting in the maximally entangled two-qubit state $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Then, Gisin's problem in Ref. [4] would be resolved by exhibiting a vector $\vec{b}$, where the maximum of $\vec{b}\cdot\vec{p}$ is achieved with POVM measurements (in contrast to the more restrictive case of projective measurements) on state $|\phi^+\rangle$.

In particular, we present a Bell expression (called I_{CH3}) which consists of the CHSH expression [2] written in the Clauser-Horne form [19],

$$I_{CH} \equiv -p_A(0|0) - p_B(0|0) + p(00|00) + p(00|01) + p(00|10) - p(00|11), \tag{2}$$

and an expression involving on Alice's side a three-outcome measurement ($x = 2$),

$$I_3 \equiv -p_A(0|2) - (1 - 1/\sqrt{2})p_A(1|2) + p(00|20) + p(00|21) + p(10|20) - p(10|21), \tag{3}$$

resulting in the following Bell inequality:

$$I_{CH3} \equiv cI_{CH} + I_3 \leqslant 1, \tag{4}$$

where $c > 0$ is supposed. The local bound of 1 is obtained by examining all the deterministic strategies with factorized joint probabilities $p(ab|xy) = p_A(a|x)p_B(b|y)$, where $p_A(a|x)$, $p_B(b|y)$ denote local marginal probabilities on Alice and Bob's respective sides.

On one hand, we show by analytical means in Sec. V C that POVMs are required to obtain the optimum value of I_{CH3} on $|\phi^+\rangle$. On the other hand, it is shown (based on numerically exact computations) in Sec. V D that, if we limit the local dimension to 2, it is still beneficial to perform POVM measurements with respect to projective ones. The above results have implications in the context of dimension witnesses as well [20].

III. MEASUREMENTS ACTING ON QUBITS

A POVM is a family of positive operators $\{M_i\}$ with elements M_i, which sum to the identity $\sum M_i = \mathbb{1}$. In the case of qubits, a Bell expression is optimized by pure states and extremal POVMs, whose elements M_i are proportional to rank-1 projectors [26].

For the case of binary outcomes and qubits, the extremal POVMs are projectors parametrized by a unit vector $\vec{v} = (v_x, v_y, v_z)$,

$$M_0(\vec{v}) = \tfrac{1}{2}(\mathbb{1} + \vec{v}\cdot\vec{\sigma}), \tag{5}$$

$$M_1(\vec{v}) = \tfrac{1}{2}(\mathbb{1} - \vec{v}\cdot\vec{\sigma}), \tag{6}$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices and the probability that outcome i occurs for a state ρ is given by Born's rule, $\mathrm{Tr}[M_i(\vec{v})\rho]$. Hence for the maximally entangled state $\rho = |\phi^+\rangle\langle\phi^+|$, conditional probabilities (1) read as

$$p(ab = 00|x(\vec{a}), y(\vec{b})) = \langle\phi^+|M_{a=0}^{x(\vec{a})} \otimes M_{b=0}^{y(\vec{b})}|\phi^+\rangle = \tfrac{1}{4}(1 + \vec{a}'\cdot\vec{b}), \tag{7}$$

and the local marginal probabilities are given by

$$p_A(a = 0|x(\vec{a})) = p_B(b = 0|y(\vec{b})) = 1/2. \tag{8}$$

In the above formulas, $x(\vec{a})$ and $y(\vec{b})$ label the respective measurement settings of Alice and Bob parametrized by unit vectors $\vec{a}, \vec{b}$ according to Eq. (5). Vector $\vec{a}'$ differs from $\vec{a}$ in a sign change of a_y, that is, $\vec{a}' = (a_x, -a_y, a_z)$.

For a three-outcome generalized POVM measurement acting on qubits, each of the three extremal POVM elements M_i is proportional to rank-1 projectors, hence we have

$$M_i = \lambda_i |w_i\rangle\langle w_i|, \tag{9}$$

with $\sum_{i=0}^{2} M_i = \mathbb{1}$, where $\lambda_i > 0$ and $|w_i\rangle$ are normalized states. On the other hand, for a three-outcome projective measurement on qubits, M_i can be rank-0,−1,−2 projectors. In the case of rank 0 and rank 2, the matrix M_i is the zero and identity matrix, respectively, whereas for rank 1, M_i is defined by Eq. (5). Taking into account the constraint that for qubits the sum of the ranks of M_0 and M_1 cannot exceed 2, we have the following six possible pairs: (0,0),(0,1),(1,0),(1,1),(0,2),(2,0), where the pair (i,j) denotes the ranks of matrices M_0 and M_1, respectively.

IV. CASE STUDY

Let us consider the following optimization problem, which will turn out to play a key role in the construction of the Bell inequality I_{CH3} defined by (4). First, let us define matrices

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

and

$$F_1 = \begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix}. \tag{11}$$

Then, we wish to maximize

$$W = \mathrm{Tr}(M_0F_0) + \mathrm{Tr}(M_1F_1) \tag{12}$$

over M_0, M_1, M_2 positive 2×2 matrices subject to $M_0 + M_1 + M_2 = \mathbb{1}$. Hence M_i, $i = 0,1,2$, can be viewed as POVM elements of a three-outcome POVM. Let us denote by $\max W_{POVM}$ the maximum of (12) obtained in this way, whereas, by further constraining M_i to be projection matrices, we write the respective maximum as $\max W_{proj}$. In the following subsections these values are given explicitly.

A. Maximum with POVMs

The optimization problem in Eq. (12) with M_i being POVM elements is a typical instance of a semidefinite programming problem. Since F_0, F_1 matrices are real valued, $\max W_{POVM}$ can be obtained with real-valued matrices M_i. By solving the semidenite programming (SDP) problem using the package SEDUMI [27], we obtain

$$w \equiv \max W_{POVM} = 1.071\,419\,898\,7\cdots. \tag{13}$$

The matrices M_i corresponding to this solution obey the positivity and unity conditions up to high precision ($\sim 10^{-10}$). Below, they are written out using fewer digits:

$$M_0 = \begin{pmatrix} 0.84153 & -0.15627 \\ -0.15627 & 0.02902 \end{pmatrix}, \quad (14)$$

$$M_1 = \begin{pmatrix} 0.14061 & 0.25242 \\ 0.25242 & 0.45314 \end{pmatrix}, \quad (15)$$

and by definition $M_2 = \mathbb{1} - M_0 - M_1$. As it can be checked, each of these truncated matrices has positive eigenvalues, hence they define a valid POVM. By substituting these matrices into the expression W in (12), we regain the value of w in (13) up to five digits.

B. Maximum with projection matrices

Now, let M_i be two-dimensional projection matrices in the optimization problem (12). According to the ranks of (M_0, M_1), we have the six possibilities (0,0),(0,1),(1,0),(1,1),(0,2),(2,0), listed in Sec. III. The corresponding W_{proj} values read as follows [using the parametrization of rank-1 projectors in Eqs. (5) and (6)]:

Ranks	W_{proj}
(0,0)	0
(0,1)	$-v_x + 1 - \sqrt{2}$
(1,0)	v_z
(1,1)	$-v_x + v_z + 1 - \sqrt{2}$
(0,2)	$2 - 2\sqrt{2}$
(2,0)	0

Since $\vec{v}$ is a unit vector, $\max\{-v_x + v_z\} = \sqrt{2}$, and according to the table, we obtain $\max W_{\mathrm{proj}} = 1$. Comparing this value with the value of w in (13) shows that, indeed, POVM elements give benefit over projectors in the presented optimization problem.

V. BELL EXPRESSION I_{CH3}

We next try to explain the construction of the Bell expression I_{CH3} [introduced under (4)] building upon the results of the optimization problem in the previous section. In particular, we wish to achieve somehow that matrices F_0 and F_1 defined by Eqs. (10) and (11) would naturally arise in a Bell scenario. For this sake, suppose that Alice shares with Bob the maximally entangled quantum state $|\phi^+\rangle$ and the optimal POVM elements $M_{0,1,2}$ in Eqs. (14) and (15) correspond to Alice's three-outcome measurement, $M_a^{x=2} = M_a$ for $a = 0,1,2$. Moreover, let us assume that Bob has two binary-outcome settings, where the measurement operators corresponding to outcome 0 are described by the following projectors:

$$M_{b=0}^{y=0} = \tfrac{1}{2}(\mathbb{1} + \vec{b}_0 \cdot \vec{\sigma}),$$
$$M_{b=0}^{y=1} = \tfrac{1}{2}(\mathbb{1} + \vec{b}_1 \cdot \vec{\sigma}), \quad (16)$$

with $\vec{b}_0 = (1/\sqrt{2}, 0, 1/\sqrt{2})$ and $\vec{b}_1 = (-1/\sqrt{2}, 0, 1/\sqrt{2})$. Then F_0 and F_1 matrices of Eqs. (10) and (11) can be reproduced in the following way:

$$F_0/\sqrt{2} = -\mathbb{1} + M_{b=0}^{y=0} + M_{b=0}^{y=1},$$
$$F_1/\sqrt{2} = -(1 - 1/\sqrt{2})\mathbb{1} + M_{b=0}^{y=0} - M_{b=0}^{y=1}. \quad (17)$$

For a maximally entangled state $|\phi^+\rangle$ and for real valued 2×2 matrices A, B, we have the expectation value $\langle\phi^+|A \otimes B|\phi^+\rangle = \mathrm{Tr}(AB)/2$. Hence due to equations in (17) the optimization problem (12) can be seen as maximizing the expression $2\sqrt{2}I_3$ of Eq. (3) over Alice's measurement $\{M_0 = M_{a=0}^{x=2}, M_1 = M_{a=1}^{x=2}\}$ assuming Bob's projectors $M_{b=0}^{y=0,1}$ are defined by (16).

Let us now consider the Bell inequality I_{CH3} of (4),

$$I_{\mathrm{CH3}} = cI_{\mathrm{CH}} + I_3 \leqslant 1, \quad (18)$$

with c positive. In particular, if c is very large, then the CH expression (2) becomes dominant in I_{CH3}, entailing that the maximum quantum violation can be obtained by projection operators $M_{a=0,1}^{x=0}, M_{b=0,1}^{y=0}$ very close to the ones which maximize the CH expression. Note that for I_{CH} the maximum quantum value of $(\sqrt{2} - 1)/2$ can be obtained by the state $|\phi^+\rangle$ and by projection operators $M_{b=0}^{y=0,1}$ defined by (16) on Bob's side. Assuming the above ideal case for Bob's operators and a state $|\phi^+\rangle$, we obtain that I_{CH3} is maximal if Alice's three-outcome measurement ($x = 2$) consists of POVM elements (14) and (15) resulting from the optimization problem (12). On the other side, it is expected that if c is not extremely large, then Bob's optimal operators $M_{b=0}^{y=0,1}$ would differ somewhat from the ideal CH-violating ones, but should still be close to it, so that Alice's three-outcome measurement ($M_a^{x=2}$) would still prefer POVM elements with respect to projection-valued elements in order to get maximum violation of I_{CH3}. In the following, the validity of the above reasoning will be supported by explicit calculations. First, in Sec. V A, a lower bound is established on the violation of I_{CH3} applying POVMs on $|\phi^+\rangle$. In Sec. V B, Bell inequalities are derived from I_{CH3} restricted to projective measurements on the two-qubit space. Then, in Secs. V C and V D, the maximal violation of inequality I_{CH3} is calculated in the case of projective measurements acting on $|\phi^+\rangle$ and on the two-qubit state space, respectively. In both cases a strictly smaller violation of I_{CH3} than with the use of POVM measurements is found.

A. Lower bound with POVM

A useful lower bound on I_{CH3} can be obtained for a pair of maximally entangled qubits, and for any two-qubit state as well in the following manner. Let us take as a special choice the state $|\phi^+\rangle$ and those measurement operators which maximize cI_{CH} in (4), giving the value of $c(\sqrt{2} - 1)/2$. However, with these operators for Bob, as discussed earlier, we have the maximum value of $w/(2\sqrt{2})$ for I_3, where w comes from (13). Adding up the two values according to Eq. (4), we have the following lower bound on the expression I_{CH3} :

$$\max_{\mathrm{POVM},\phi^+} I_{\mathrm{CH3}} \geqslant c(\sqrt{2} - 1)/2 + w/(2\sqrt{2}) \quad (19)$$

for POVM measurements acting on the state $|\phi^+\rangle$ and on any two-qubit state as well. We wish to mention that any extremal POVM measurement with real coefficients on the qubit space in the form of (9) can be reproduced with rank-1 von Neumann measurements with real coefficients acting on the qutrit space. This is due to Neumark's theorem [28], stating that POVM measurements can always be seen as projective measurements acting on a larger Hilbert space. For a three-outcome measurement, an explicit construction is given by [29].

B. Deriving Bell inequalities in the case of projective measurements on qubits

Here we study the maximum quantum violation of I_{CH3} if only projective measurements are allowed on the local qubit spaces. First, let us note that in order to violate any two-party Bell inequality, each party must have at least two nondegenerate operators belonging to different measurement settings, otherwise the quantum predictions could be simulated within a local classical model. Concerning a pair of qubits and inequality I_{CH3}, this entails that both of Bob's operators must be rank-1 projectors. We now state the following lemma:

Lemma 1. Consider inequality I_{CH3} and assume that only projective measurements can be performed on the local qubit spaces. In this case, I_{CH3} can be violated only if Alice's and Bob's measurement operators corresponding to I_{CH} are rank-1 projectors.

The proof of this lemma can be found in [30]. Since we are interested in the nontrivial case that I_{CH3} can be violated, due to the above lemma, Alice's two operators $M_{a=0}^{x=0,1}$ can be considered as rank-1 projectors. Then, we are left with six possible cases, $(0,0),(0,1),(1,0),(1,1),(0,2),(2,0)$, according to the ranks of $M_{a=0}^{x=2}$ and $M_{a=1}^{x=2}$, as discussed in Sec. III. In each case above the original inequality I_{CH3} is modified as follows. If a measurement $M_a^{x=2}$ is rank 0, we set $p_A(a|2)=0$, $p(ab|2y)=0$. In the case that $M_a^{x=2}$ is rank 2, we set $p_A(a|2)=1$, $p(ab|2y)=p(b|y)$. On the other hand, if both $M_{a=0}^{x=2}$ and $M_{a=1}^{x=2}$ are rank-1 projectors, we have $p_A(0|2)+p_A(1|2)=1$ and $p(0b|2y)+p(1b|2y)=p_B(b|y)$. All the above relations hold for $a,b,y\in\{0,1\}$. In this way we derive two-outcome Bell inequalities from I_{CH3}, which look as follows:

$$I_{00}\equiv cI_{CH}\leqslant 0, \tag{20}$$

$$I_{01}\equiv cI_{CH}+p_A(0|2)+p(00|20)-p(00|21)\leqslant 1/\sqrt{2}, \tag{21}$$

$$I_{10}\equiv cI_{CH}-p_A(0|0)+p(00|00)+p(00|01)\leqslant 1, \tag{22}$$

$$I_{11}\equiv cI_{CH}-1/\sqrt{2}p_A(0|2)+p_B(0|0)-p_B(0|1)+2p(00|21)+1/\sqrt{2}-1\leqslant 1, \tag{23}$$

$$I_{02}\equiv cI_{CH}+p_B(0|0)-p_B(0|1)\leqslant 1, \tag{24}$$

$$I_{20}\equiv cI_{CH}+p_B(0|0)+p_B(0|1)-1\leqslant 1, \tag{25}$$

where I_{ij} denotes the Bell expression derived from I_{CH3} by setting $M_{a=0}^{x=2}$ to be a rank-i projector and $M_{a=1}^{x=2}$ to be rank-j projector. So, in order to get the maximum violation of I_{CH3} by von Neumann's projective measurements on qubits, we are left with calculating the maximum quantum violation of the above inequalities (20)–(25) by considering the rank-1 projectors. This is just what we will do in the following by considering the maximally entangled qubits (Sec. V C) and also the state space of two qubits (Sec. V D).

C. Maximizing I_{CH3} with projective measurements on maximally entangled qubits

First, note that the expressions I_{20} and I_{02} are equivalent up to relabeling of the outcomes. Further, for $|\phi^+\rangle$, expressions I_{00}, I_{02}, I_{20} coincide giving the quantum maximum of $c/2(\sqrt{2}-1)$. In order to calculate the quantum maximum for the remaining three cases I_{01}, I_{10}, I_{11}, we present the following lemma:

Lemma 2. Let us assume that $\vec{b}_i$, $i=1,2$ are unit vectors in the Euclidean space. Then we have $\max_{\vec{b}_1,\vec{b}_2}\{|\vec{b}_1+\vec{b}_2|+k|\vec{b}_1-\vec{b}_2|\}=2\sqrt{1+k^2}$.

Using this lemma, Eqs. (7) and (8), and the simple fact that $\vec{a}\cdot\vec{c}\leqslant|\vec{c}|$ for a unit vector $\vec{a}$, we obtain the following quantum maximum for the Bell expressions (20)–(25) with rank-1 projective measurements on $|\phi^+\rangle$:

Expression	Maximum on state $\lvert\phi^+\rangle$
I_{00}, I_{02}, I_{20}	$\frac{1}{2}c(\sqrt{2}-1)$
I_{01}	$\frac{1}{2}[1/\sqrt{2}-1-c+\sqrt{c^2+(c+1)^2}]$
I_{10}	$\frac{1}{2}[-c+\sqrt{c^2+(c+1)^2}]$
I_{11}	$\frac{1}{2}[1/\sqrt{2}+c(\sqrt{2}-1)]$.

We can observe the simple relations $I_{00}<I_{11}$ and $I_{01}<I_{10}$ between the right-hand side formulas. On the other hand, using the relationship between the quadratic and arithmetic mean, we have $I_{10}>I_{11}$ for $c>0$. Thus for $c>0$, the quantum maximum of expression I_{CH3} with projective measurements on $|\phi^+\rangle$ is provided by expression I_{10}, yielding the value of

$$\max_{\text{proj},\phi^+} I_{CH3}=\frac{-c+\sqrt{c^2+(c+1)^2}}{2}. \tag{26}$$

On the other hand, we have the lower bound (19) of expression I_{CH3} with POVM measurements on $|\phi^+\rangle$, where w is defined by (13). Note that Eq. (26) becomes bigger than 1 (i.e., the local bound on I_{CH3}) for $c>3$. Hence in the following only the interval $c>3$ will be considered. By equating the right-hand side of Eqs. (19) and (26), and solving the equation for c, we obtain the value of $(2-w^2)/(4w-4)\simeq 2.9826$. Hence for $c>3$, the right-hand side of Eq. (19) is definitely bigger than that of Eq. (26). This implies that for $c>3$, Bell inequality I_{CH3} in (4) is more strongly violated by POVMs on a pair of maximally entangled qubits than by considering only projective measurements. This answers Gisin's question [4] in the affirmative.

We may have a quantitative measure about the performance of POVMs over projective measurements by adding a fraction of p white noise to the maximally entangled two-qubit state [31],

$$\rho(p)=(1-p)|\phi^+\rangle\langle\phi^+|+p\mathbb{1}/4, \tag{27}$$

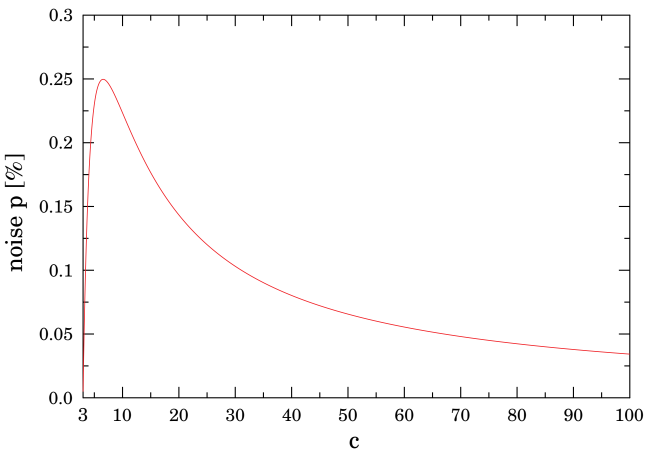

FIG. 1. (Color online) Noise thresholds in function of parameter c such that POVMs on a two-qubit Werner state [31] with mixed noise p ceases to be better than projective measurements. The curve gives an analytical lower bound for p.

such that POVMs on state $\rho(p)$ still perform better than projective ones. In the case of $\rho(p)$, a lower bound on the violation of I_{CH3} for POVM measurements is given by $(1-p)\max_{\mathrm{POVM},\phi^+} I_{CH3} + [-2c + 2(1/\sqrt{2}-1)\mathrm{Tr}M_1]p/4$, with M_1 defined in Eq. (15). Figure 1 shows in the function of c the amount of white noise which can be tolerated due to the above formula such that POVM measurements are still better than projective ones. The lower bound of $p = 0.249\,717\%$ on the maximum tolerable noise is attained by $c \simeq 6.561\,82$.

D. Maximizing I_{CH3} with von Neumann measurements on a pair of qubits

We now fix $c = 100$ and show that if the state is allowed to be any two-qubit state, POVMs can still perform better than projectors. Our task is to compute the maximum quantum violation for the derived Bell inequalities (20)–(25), assuming that rank-1 projectors act on the two-qubit space. However, we can establish an upper bound on these values by computing the SDP hierarchy of Navascués, Pironio, and Acín [10] on various levels. Incidentally, the lower bound achievable with real-valued rank-1 projectors on qubits coincides with the upper bound value coming from the SDP calculation of [10] on level 2 for each inequality in the set (20)–(25). We collected in the table below the obtained values. Note that Bell expressions I_{02} in Eq. (24) and I_{20} in Eq. (25) are equivalent to each other.

Expression	Maximum on qubits
I_{00}	20.710 68
I_{01}	20.919 28
I_{10}	21.066 90
I_{11}	21.068 01
I_{02}, I_{20}	20.717 75

This table shows that by considering projective measurements, the two-qubit maximum of the expression I_{CH3} for $c = 100$ is provided by the expression I_{11}, yielding numerically the value of 21.068 01, whereas owing to Eq. (19) the lower bound on the two-qubit maximum by considering POVMs is $100[\sqrt{(2)}-1]/2 + w/2 \simeq 21.0895$. This proves the existence of bipartite Bell inequalities for which the maximal violation on the two-qubit space can be achieved only with the use of generalized POVM measurements.

Note that any bipartite Bell inequality consisting of two measurement settings with two outcomes each on Bob's side can be maximally violated on the two-qubit space. This follows from the works of Refs. [32,33]. In particular, due to Lemma 2 of Ref. [33], the quantum maximum is achieved by a state with support on Bob's qubit. However, using the Schmidt decomposition theorem, it induces the composite space to be a pair of qubits, letting Alice's state space be a qubit as well. In light of this, I_{CH3} is a Bell inequality whose maximal violation is attained by performing POVM measurements on qubits. On the other hand, if only projective measurements are allowed, then qutrits are needed to achieve maximal violation.

VI. CONCLUSION

Though there are indications that performing POVM measurements on a given state or on a given state space may yield a benefit over projective ones, the question has not been settled yet. In the present paper we provide a bipartite Bell inequality with a small number of inputs and outputs, which answers this question in the affirmative. Moreover, we found that the improvement, which we defined in terms of noise resistance, is not marginal. It may be within the range of what is feasible experimentally nowadays.

However, one may still wonder whether it would be possible to construct even better Bell inequalities with more settings or with more parties allowing a bigger separation in the maximum of Bell values achievable with POVM versus projective measurements on a given state. A bigger gap might be suggested by the amount of communication to simulate different types of measurements on a singlet state. Whereas for projective measurements one bit of communication suffices [34], for POVM measurements the best protocol constructed so far needs on average six bits of communication [35]. Based on the best local models constructed for POVMs and for projective measurements on a mixture of d-dimensional maximally entangled states with noise [36], it is also plausible that moving from qubits to higher dimensions, POVM measurements become much more efficient than projective ones.

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[20] According to the definition introduced in Ref. [11], a two-dimensional witness is a vector $\vec{w}$ of real coefficients, such that $\vec{w}\cdot\vec{p}=\sum_{a,b,x,y} w_{abxy}\,p(ab\,xy)\leqslant w_2$ for all probabilities $p(ab\,xy)$ expressed by formula (1) with a two-qubit state ρ, such that there exist quantum correlations beyond local dimension 2 for which $\vec{w}\cdot\vec{p}>w_2$. In light of our result, in order to sensibly define the constant w_2 for any $\vec{w}$, it is not enough to carry out projective measurements in the two-qubit space, but the inclusion of the most general POVM measurements are needed too. References [11, 21–25] provide various constructions of dimension witnesses.
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[29] Let us use Eq. (9) and write $|w_i\rangle$ as real-valued normalized qubit states, $|w_i\rangle = w_{i0}|0\rangle + w_{i1}|1\rangle$. Let us write rank-1 projectors in the qutrit space as $P_i = |w_i'\rangle\langle w_i'|$, where $|w_i'\rangle = \sqrt{\lambda_i}w_{i0}|0\rangle + \sqrt{\lambda_i}w_{i1}|1\rangle + s_{jk}\sqrt{1-\lambda_i}|2\rangle$. Here $i\neq j\neq k$ and $s_{jk} = -\mathrm{sgn}(\langle w_j|w_k\rangle)$. It can be checked that in this way P_i satisfies the relation $P_iP_j = \delta_{ij}P_i$, thus P_i are rank-1 projectors corresponding to a von Neumann measurement. Moreover, for any state ρ in the qubit space, we get the equality of probabilities $\mathrm{Tr}(M_i\rho) = \mathrm{Tr}(P_i\rho)$, where M_i are defined by Eq. (9).
[30] Let us calculate separately the quantum maximum for expression I_{CH} and I_3 assuming at least one degenerate operator (i.e., rank-0 or rank-2 projectors) among $M_{a=0}^{x=0,1}$ and $M_{b=0}^{y=0,1}$. The I_3 part contains on Alice's side only one setting, which means that the quantum maximum cannot be bigger than the local maximum of 1. The I_{CH} part gives at most zero if one of the settings is degenerate. This implies the upper bound value of $1+0=1$ on the quantum maximum for expression I_{CH3} if one of Alice's operators $M_{a=0}^{x=0,1}$ or Bob's operators $M_{b=0}^{y=0,1}$ is degenerate. This entails Lemma 1.
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