Detection of genuinely entangled and nonseparable n-partite quantum states

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We investigate the detection of entanglement in n-partite quantum states. We obtain practical separability criteria to identify genuinely entangled and nonseparable mixed quantum states. No numerical optimization or eigenvalue evaluation is needed, and our criteria can be evaluated by simple computations involving components of the density matrix. We provide examples in which our criteria perform better than all known separability criteria. Specifically, we are able to detect genuine n-partite entanglement which has previously not been identified. In addition, our criteria can be used in today's experiments.

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I. INTRODUCTION

Entanglement plays a fundamental role in quantum information processing and is responsible for many quantum tasks such as quantum cryptography with Bell's theorem [1], quantum dense coding [2], quantum teleportation [3], quantum communication [1–7], quantum computation [8,9], etc. Thus, entanglement is not only the subject of philosophical debates, but also a resource for tasks that cannot be performed by means of classical resources [10,11].

Deciding whether or not a state is entangled has proven to be a very challenging problem that currently lacks a full computable solution. In the bipartite setting, there are some well-known (necessary) criteria for separability, such as the Bell inequalities [12], positive partial transposition (PPT) [13] (which is also sufficient for two-qubit or one-qubit and one-qutrit systems [14]), reduction [15,16], range [17], majority [18], realignment [19–21], generalized realignment [22], etc., which work very well in many cases, but are far from perfect [10]. For multipartite entanglement (more than two parties), the situation is even more complicated as there exist states that are inseparable under any fixed partition, but are still not considered genuinely multipartite entangled (which we will define) [23]. Likewise, there exist states that are biseparable with respect to each fixed partition; however, they are not fully separable (for some examples, see Refs. [24–26]). Vast areas of multipartite-state spaces are still unexplored because of the lack of suitable tools for detecting and characterizing entanglement.

Recently, Gühne and Seevinck [23] presented a method for deriving separability criteria within different classes of three-qubit and four-qubit entanglement using density-matrix elements. Huber *et al.* [27] developed a general framework to identify genuinely multipartite-entangled mixed quantum states in arbitrary-dimensional systems. From the framework introduced in Ref. [27], a k-separability criterion was derived in Ref. [28]. In addition, we studied the separability of n-partite quantum states and obtained practical separability criteria for different classes of n-qubit and n-qudit quantum states [29].

In this paper, we derive separability criteria to identify genuinely entangled and nonseparable n-partite mixed quantum states. The resulting criteria are easily computable from the density matrix, and no optimization or eigenvalue evaluation is needed. We first describe our criteria and then provide examples in which we can detect genuine n-partite entanglement beyond all previously studied criteria. Finally, we briefly comment on the ability of our criteria to be implemented in today's experiments without the need for quantum-state tomography.

II. DEFINITIONS

An n-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ (dim $\mathcal{H} = d_i \geqslant 2$) is called biseparable if there is a bipartition $j_1 j_2 \cdots j_k | j_{k+1} \cdots j_n$ such that

$$|\psi\rangle = |\psi_1\rangle_{j_1 j_2 \cdots j_k} |\psi_2\rangle_{j_{k+1}\cdots j_n}, \tag{1}$$

where $|\psi_1\rangle_{j_1 j_2 \cdots j_k}$ is the state of particles $j_1, j_2, \ldots, j_k$, $|\psi_2\rangle_{j_{k+1}\cdots j_n}$ is the state of particles $j_{k+1}, \ldots, j_n$, and $\{j_1, j_2, \ldots, j_n\} = \{1,2,\ldots,n\}$. An n-partite mixed state ρ is biseparable if it can be written as a convex combination of biseparable pure states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2}$$

where $|\psi_i\rangle$ might be biseparable under different partitions. If an n-partite state is not biseparable, then it is called genuinely n-partite entangled. An n-partite pure state is fully separable if it is of the form

$$|\psi\rangle = |\psi\rangle_1 |\psi\rangle_2 \cdots |\psi\rangle_n, \tag{3}$$

and an n-partite mixed state is fully separable if it is a mixture of fully separable pure states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{4}$$

where p_i forms a probability distribution, and $|\psi_i\rangle$ is fully separable. If an n-partite state is not fully separable, then we call it nonseparable. We consider separability criteria of biseparable and fully separable n-qubit and n-qudit states.

Throughout this paper, let ρ be a density matrix describing an n-particle system whose state space is Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n$, where $\dim\mathcal{H}_l = d_l$, $l = 1,2,\ldots,n$.

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We denote its entries by $\rho_{i,j}$, where $1 \leqslant i, j \leqslant d_1 d_2 \cdots d_n$. We introduce the further notation of $|\Phi_{ij}\rangle = |\phi_i\rangle|\phi_j\rangle$ with $|\phi_i\rangle = |x\cdots xyx\cdots x\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where the local state of $\mathcal{H}_k$ is $|x\rangle$ for $k \neq i$ and $|y\rangle$ for $k = i$. Furthermore, let P denote the operator that performs a simultaneous local permutation on all subsystems in $(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n)^{\otimes 2}$, while P_i just performs a permutation on $\mathcal{H}_i^{\otimes 2}$ and leaves all other subsystems unchanged.

III. DETECTION OF GENUINELY ENTANGLED *n*-PARTITE QUANTUM STATES

Theorem 1. Let ρ be a biseparable n-partite density matrix acting on Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\dim\mathcal{H}_l = d_l$, $l = 1,2,\ldots,n$. Then

$$\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} \leqslant \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ij}\rangle} + (n-2)\sum_i\sqrt{\langle\Phi_{ii}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ii}\rangle}. \tag{5}$$

If an n-partite state ρ does not satisfy inequality (5), then ρ is genuinely n-partite entangled.

Proof. To prove that inequality (5) is indeed satisfied by all biseparable states ρ, let us first verify that this holds for any pure state ρ that is biseparable under some partition.

Suppose that $\rho = |\psi\rangle\langle\psi|$ is a biseparable pure state under the partition of $\{1,2,\ldots,n\}$ into two disjoint subsets $\{1,2,\ldots,n\} = A \cup B$ with $A = \{j_1,j_2,\ldots,j_k\}$ and $B = \{j_{k+1},\ldots,j_n\}$, and

$$\begin{aligned}|\psi\rangle &= |\psi_1\rangle_{j_1j_2\cdots j_k}|\psi_2\rangle_{j_{k+1}\cdots j_n}\\ &= \left(\sum_{i_1,i_2,\ldots,i_k} a_{i_1i_2\cdots i_k}|i_1i_2\cdots i_k\rangle\right)_{j_1j_2\cdots j_k}\\ &\quad\times\left(\sum_{i_{k+1},\ldots,i_n} b_{i_{k+1}\cdots i_n}|i_{k+1}\cdots i_n\rangle\right)_{j_{k+1}\cdots j_n}\\ &= \sum_{i_1,i_2,\ldots,i_n} a_{i_1i_2\cdots i_k}b_{i_{k+1}\cdots i_n}|i_1i_2\cdots i_n\rangle_{j_1j_2\cdots j_n},\end{aligned} \tag{6}$$

then

$$\rho_{\sum_{l=1}^n i_l d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1,\ \sum_{l=1}^n i'_l d_{j_l+1}d_{j_l+2}\cdots d_nd_{n+1}+1} = a_{i_1i_2\cdots i_k}b_{i_{k+1}\cdots i_n}a^*_{i'_1i'_2\cdots i'_k}b^*_{i'_{k+1}\cdots i'_n}. \tag{7}$$

Here the sum is over all possible values of $i_1,i_2,\ldots,i_n$, i.e., $\sum_{i_1,i_2,\ldots,i_n} = \sum_{i_1=0}^{d_{j_1}-1}\sum_{i_2=0}^{d_{j_2}-1}\cdots\sum_{i_n=0}^{d_{j_n}-1}$, $d_{n+1}=1$.

We will distinguish between cases in which both indices i and j correspond to different, or the same, parts A and B in the bipartition with respect to $|\psi\rangle$. By calculation, one has

$$\begin{aligned}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} &= |\langle\phi_i|\rho|\phi_j\rangle| = \sqrt{\langle\phi_i|\rho|\phi_i\rangle\langle\phi_j|\rho|\phi_j\rangle}\\ &\leqslant \frac{\langle\phi_i|\rho|\phi_i\rangle + \langle\phi_j|\rho|\phi_j\rangle}{2}\\ &= \frac{\sqrt{\langle\Phi_{ii}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ii}\rangle} + \sqrt{\langle\Phi_{jj}|P_j^{+}\rho^{\otimes 2}P_j|\Phi_{jj}\rangle}}{2}\end{aligned} \tag{8}$$

in case of either $i,j \in A$ or $i,j \in B$, and

$$\begin{aligned}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} &= |\langle\phi_i|\rho|\phi_j\rangle| = \sqrt{\langle\phi_0|\rho|\phi_0\rangle\langle\phi_{ij}|\rho|\phi_{ij}\rangle}\\ &= \sqrt{\langle\Phi_{ij}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}\end{aligned} \tag{9}$$

in case of either $i \in A, j \in B$ or $i \in B, j \in A$. Here $|\phi_0\rangle = |xx\cdots x\rangle$ and $|\phi_{ij}\rangle = |x\cdots xyx\cdots xyx\cdots x\rangle$ such that all particles are in the state $|x\rangle$, except that the ith and jth particles are in the state $|y\rangle$. Combining (8) and (9) gives

$$\begin{aligned}\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} &= \sum_{\substack{i\in A, j\in B\\ \text{or } i\in B, j\in A}}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} + \sum_{\substack{i\neq j \text{ with}\\ i,j\in A\\ \text{or } i,j\in B}}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle}\\ &\leqslant \sum_{\substack{i\in A, j\in B\\ \text{or } i\in B, j\in A}}\sqrt{\langle\Phi_{ij}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ij}\rangle} + \sum_{\substack{i\neq j \text{ with}\\ i,j\in A\\ \text{or } i,j\in B}}\left(\frac{\sqrt{\langle\Phi_{ii}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ii}\rangle} + \sqrt{\langle\Phi_{jj}|P_j^{+}\rho^{\otimes 2}P_j|\Phi_{jj}\rangle}}{2}\right)\\ &\leqslant \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ij}\rangle} + (n-2)\sum_i\sqrt{\langle\Phi_{ii}|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ii}\rangle}.\end{aligned} \tag{10}$$

Hence, inequality (5) is satisfied by all biseparable n-partite pure states.

Next, we show that inequality (5) is also true for all biseparable n-partite mixed states. Indeed, the generalization of inequality (5) to mixed states is a direct consequence of the convexity of its left-hand side and the concavity of its right-hand side, which we can see in the following.

Suppose that

$$\rho = \sum_m p_m\rho_m = \sum_m p_m|\psi_m\rangle\langle\psi_m| \tag{11}$$

is biseparable n-partite mixed state, where $\rho_m = |\psi_m\rangle\langle\psi_m|$ is biseparable. Then, by the Cauchy-Schwarz inequality $(\sum_{k=1}^m x_k y_k)^2 \leqslant (\sum_{k=1}^m x_k^2)(\sum_{k=1}^m y_k^2)$, one has

$$\begin{aligned}\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} &\leqslant \sum_m p_m \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho_m^{\otimes 2}P|\Phi_{ij}\rangle}\\ &\leqslant \sum_m p_m\left(\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho_m^{\otimes 2}P_i|\Phi_{ij}\rangle} + (n-2)\sum_i\sqrt{\langle\Phi_{ii}|P_i^+\rho_m^{\otimes 2}P_i|\Phi_{ii}\rangle}\right)\\ &= \sum_{i\neq j}\sum_m\sqrt{\langle\phi_0|p_m\rho_m|\phi_0\rangle}\sqrt{\langle\phi_{ij}|p_m\rho_m|\phi_{ij}\rangle} + (n-2)\sum_i\sum_m p_m\langle\phi_i|\rho_m|\phi_i\rangle\\ &\leqslant \sum_{i\neq j}\sqrt{\sum_m\langle\phi_0|p_m\rho_m|\phi_0\rangle\sum_m\langle\phi_{ij}|p_m\rho_m|\phi_{ij}\rangle} + (n-2)\sum_i\langle\phi_i|\rho|\phi_i\rangle\\ &= \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle} + (n-2)\sum_i\sqrt{\langle\Phi_{ii}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ii}\rangle},\end{aligned} \tag{12}$$

which finishes the proof of inequality (5). ■

It is worth pointing out that inequality (III) of Ref. [27], which can be rewritten as

$$\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} \leqslant (n-2)\sum_{i,j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}, \tag{13}$$

is the corollary of this theorem. The reason is as follows: Note that the second summation in inequality (III) of Ref. [27], the right-hand side of inequality (13), can be reexpressed as

$$\begin{aligned}&(n-2)\sum_{i,j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}\\ &\quad= \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}\\ &\qquad+(n-2)\sum_i\sqrt{\langle\Phi_{ii}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ii}\rangle}\\ &\qquad+(n-3)\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}\end{aligned} \tag{14}$$

in the case of $n \geqslant 3$, and all terms in the third summation term of the right-hand side of Eq. (14) are expectation values of positive operators, which implies that

$$\begin{aligned}&\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle} + (n-2)\sum_i\sqrt{\langle\Phi_{ii}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ii}\rangle}\\ &\quad\leqslant (n-2)\sum_{i,j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}.\end{aligned} \tag{15}$$

Thus, inequality (13), i.e., inequality (III) of Ref. [27], follows from Theorem 1 and inequality (15).

Theorem 1 deserves comments. It is better than inequality (III) of Ref. [27] in the case of genuine multipartite-entanglement detection for n-partite quantum states. This criterion detects genuine n-partite entanglement [for n-qubit states such as W states mixed with white noise, and the mixture of the Greenberger-Horne-Zeilinger (GHZ) state and the W state, dampened by isotropic noise] that had not been identified so far.

Example 1. We consider the family of n-qubit states

$$\rho^{(G-W_n)} = \frac{1-\alpha-\beta}{2^n}\mathbb{I} + \alpha|\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n| + \beta|W_n\rangle\langle W_n|, \tag{16}$$

the mixture of the GHZ state and the W state, dampened by isotropic noise. Here

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|00\cdots0\rangle + |11\cdots1\rangle) \tag{17}$$

and

$$|W_n\rangle = \frac{1}{\sqrt{n}}(|00\cdots001\rangle + |00\cdots010\rangle + \cdots + |10\cdots00\rangle) \tag{18}$$

are the n-qubit GHZ state and the W state, respectively. For this family, our criteria can detect genuine n-partite ($n \geqslant 4$) entanglement that had not been identified so far. The detection parameter spaces of inequality (5) in Theorem 1, inequality (III) in Ref. [27], inequality in Ref. [23], and inequality (II) in Ref. [27] for $n = 10$ are illustrated in Fig. 1.

Example 2. Let us consider the n-qubit state, W states mixed with white noise,

$$\rho^{(W_n)}(p) = \frac{p}{2^n}\mathbb{I} + (1-p)|W_n\rangle\langle W_n|. \tag{19}$$

By our Theorem 1, and Theorem 3 of Ref. [29], we derive that if $0 \leqslant p < \frac{2^n}{n(2n-3)+2^n}$, then $\rho^{(W_n)}(p)$ is genuinely n-partite entangled, while from inequality (III) of Ref. [27], one can obtain that if $0 \leqslant p < \frac{2^n}{n^2(n-2)+2^n}$, then $\rho^{(W_n)}(p)$ is genuinely n-partite entangled. That is, our criteria detect W states mixed with white noise, $\rho^{(W_n)}(p)$, for $0 \leqslant p < \frac{2^n}{n(2n-3)+2^n}$ as genuinely n-partite entangled, whereas inequality (III) of

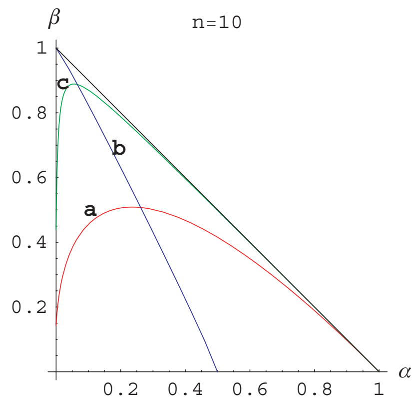

FIG. 1. (Color) Detection quality for the state $\rho^{(G-W_n)} = \frac{1-\alpha-\beta}{2^n}\mathbb{I} + \alpha|\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n| + \beta|W_n\rangle\langle W_n|$, $n = 10$. Here the (red) line a represents the threshold given by inequality (5) in Theorem 1 such that the region above it identifies genuine 10-partite entanglement. The regions above lines b (blue) and c (green) correspond to the genuine entanglement detected by inequalities (II) in Ref. [27] (also Ref. [23]) and (III) in Ref. [27], respectively. The area enclosed by the red curve a, the blue curve b, the green curve c, and the β axis contains the genuine 10-partite entanglement detected only by inequality (5) in Theorem 1.

Ref. [27] detects them only for $0 \leqslant p < \frac{2^n}{n^2(n-2)+2^n}$. For the special case $n = 3$ our criteria coincide. When $n = 3$, in Ref. [30] $\rho^{(W_n)}(p)$ was found to be genuinely multipartite entangled by means of the best known entanglement witness up to a threshold of $p < \frac{8}{19}$. This bound was then improved to $p < \frac{8}{17}$ [23,27], which is also our result. When $n = 4$, both Theorem 1 and the previous results [23,29] detect $\rho^{(W_n)}(p)$ for $p < \frac{4}{9} \approx 0.444$ as genuinely four-partite entangled, while inequality (III) in Ref. [27] detects it only for $p < \frac{1}{3} \approx 0.333$, the fidelity-based witness detects it only for $p < \frac{4}{15} \approx 0.267$, and the improved witness detects it only for $p < \frac{16}{45} \approx 0.356$ [30]. However, when $n = 5$, 6, 7, 8, and 9, Theorem 1 shows that $\rho^{(W_n)}(p)$ is genuinely multipartite entangled in the cases of $p < \frac{32}{67}$, $p < \frac{32}{59}$, $p < \frac{128}{205}$, $p < \frac{32}{45}$, and $p < \frac{512}{647}$, respectively, while inequality (III) in Ref. [27] shows that $\rho^{(W_n)}(p)$ is genuinely multipartite entangled in the cases of $p < \frac{32}{107}$, $p < \frac{4}{13}$, $p < \frac{128}{373}$, $p < \frac{2}{5}$, and $p < \frac{512}{1079}$, respectively. Therefore, our criterion is better than that in Ref. [27]. W states mixed with white noise, $\rho^{(W_n)}(p)$, for $\frac{2^n}{n^2(n-2)+2^n} \leqslant p < \frac{2^n}{n(2n-3)+2^n}$, as genuinely n-partite ($n = 5,6,7,8,9,\ldots$) entangled, are detected only by our criterion. We summarize our results in Table I.

TABLE I. Thresholds of the detection for genuine n-partite entanglement for W states mixed with white noise, $\rho^{(W_n)}(p) = \frac{p}{2^n}\mathbb{I} + (1-p)|W_n\rangle\langle W_n|$. The first row represents the number of qubits, while the second row and the last row are the thresholds identified by inequality (5) in Theorem 1 and inequality (III) in Ref. [27], respectively. $\rho^{(W_n)}(p)$, for $\frac{2^n}{n^2(n-2)+2^n} \leqslant p < \frac{2^n}{n(2n-3)+2^n}$, as genuinely n-partite ($n \geqslant 5$) entangled, are detected only by inequality (5) in Theorem 1.

3	4	5	6	7	8	9	n
$\frac{8}{17}$	$\frac{4}{9}$	$\frac{32}{67}$	$\frac{32}{59}$	$\frac{128}{205}$	$\frac{32}{45}$	$\frac{512}{647}$	$\frac{2^n}{n(2n-3)+2^n}$
$\frac{8}{17}$	$\frac{1}{3}$	$\frac{32}{107}$	$\frac{4}{13}$	$\frac{128}{373}$	$\frac{2}{5}$	$\frac{512}{1079}$	$\frac{2^n}{n^2(n-2)+2^n}$

IV. DETECTION OF NONSEPARABLE n-PARTITE QUANTUM STATES

Theorem 2. Every fully separable n-partite state ρ satisfies

$$\sqrt{\langle\Phi|\rho^{\otimes 2}P|\Phi\rangle} \leqslant \left(\prod_{A\in S}\langle\Phi|P_A^{+}\rho^{\otimes 2}P_A|\Phi\rangle\right)^{\frac{1}{2^{n+1}-4}} \tag{20}$$

for fully separable states $|\Phi\rangle$, where S is the set of all nonempty proper subsets of $\{1,2,\ldots,n\}$, the permutation operators P_A are the operators permuting the two copies of all subsystems contained in the set A, and P is the total permutation operator, permuting the two copies.

This inequality is equality for fully separable n-partite pure states.

Proof. We start by showing that inequality (20) holds for pure states. Let us suppose, then, that ρ is an n-partite fully separable pure state and $|\Phi\rangle = |\Phi_1\rangle|\Phi_2\rangle$ with fully separable n-partite states $|\Phi_1\rangle$ and $|\Phi_2\rangle$. The left-hand side of inequality (20) is the absolute value of the matrix element $\langle\Phi_1|\rho|\Phi_2\rangle$:

$$\sqrt{\langle\Phi|\rho^{\otimes 2}P|\Phi\rangle} = |\langle\Phi_1|\rho|\Phi_2\rangle|, \tag{21}$$

since P simply permutes $|\Phi_1\rangle$ and $|\Phi_2\rangle$, i.e., $P|\Phi_1\rangle\otimes|\Phi_2\rangle = |\Phi_2\rangle\otimes|\Phi_1\rangle$. Beacuse it is fully separable, $\rho^{\otimes 2}$ is invariant under permutation of each element A of S:

$$P_A^{+}\rho^{\otimes 2}P_A = \rho^{\otimes 2}. \tag{22}$$

Thus,

$$\begin{aligned}\sqrt{\langle\Phi|\rho^{\otimes 2}P|\Phi\rangle} &= |\langle\Phi_1|\rho|\Phi_2\rangle| \leqslant \sqrt{\langle\Phi_1|\rho|\Phi_1\rangle\langle\Phi_2|\rho|\Phi_2\rangle}\\ &= \sqrt{\langle\Phi|\rho^{\otimes 2}|\Phi\rangle} = \left(\prod_{A\in S}\sqrt{\langle\Phi|\rho^{\otimes 2}|\Phi\rangle}\right)^{\frac{1}{2^n-2}}\\ &= \left(\prod_{A\in S}\sqrt{\langle\Phi|P_A^{+}\rho^{\otimes 2}P_A|\Phi\rangle}\right)^{\frac{1}{2^n-2}},\end{aligned} \tag{23}$$

as claimed. Here we have used the positivity of the density matrix in the first inequality and the cardinality $|S|$ of S being 2^n-2 (S has exactly 2^n-2 elements) in the third equality. In fact, for any fully separable pure state ρ, a straightforward computation yields

$$|\langle\Phi_1|\rho|\Phi_2\rangle| = \sqrt{\langle\Phi_1|\rho|\Phi_1\rangle\langle\Phi_2|\rho|\Phi_2\rangle}. \tag{24}$$

Therefore, inequality (20) holds with equality if ρ is a fully separable pure state.

It remains to show that inequality (20) holds if ρ is a mixed state. Now we suppose that $\rho = \sum p_i\rho_i$ is a fully separable n-partite mixed state, where ρ_i is a fully separable pure state. As the absolute value is convex, i.e., $|a+b| \leqslant |a|+|b|$ for arbitrary complex numbers a and b, and inequality (20) is satisfied by the fully separable pure state ρ_i, one gets

$$\sqrt{\langle\Phi|\rho^{\otimes 2}P|\Phi\rangle} = |\langle\Phi_1|\rho|\Phi_2\rangle| \leqslant \sum_i p_i|\langle\Phi_1|\rho_i|\Phi_2\rangle|$$

$$= \sum_i p_i\sqrt{\langle\Phi|\rho_i^{\otimes 2}|\Phi\rangle} = \sum_i p_i\left(\prod_{A\in S}\langle\Phi|P_A^+\rho_i^{\otimes 2}P_A|\Phi\rangle\right)^{\frac{1}{2^{n+1}-4}}. \tag{25}$$

By continuously using the Hölder inequality

$$\sum_{k=1}^m |x_k y_k| \leqslant \left(\sum_{k=1}^m |x_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^m |y_k|^q\right)^{\frac{1}{q}}, \quad p,q>1,\quad \frac{1}{p}+\frac{1}{q}=1, \tag{26}$$

we obtain that

$$\sum_i p_i\left(\prod_{A\in S}\langle\Phi|P_A^+\rho_i^{\otimes 2}P_A|\Phi\rangle\right)^{\frac{1}{2^{n+1}-4}} \leqslant \left[\prod_{A\in S}\langle\Phi|P_A^+\left(\sum_i p_i^2\rho_i^{\otimes 2}\right)P_A|\Phi\rangle\right]^{\frac{1}{2^{n+1}-4}} \leqslant \left(\prod_{A\in S}\langle\Phi|P_A^+\rho^{\otimes 2}P_A|\Phi\rangle\right)^{\frac{1}{2^{n+1}-4}}, \tag{27}$$

where, in the second inequality, we have used $\rho^{\otimes 2} - \sum_i p_i^2\rho_i^{\otimes 2} = \sum_{i\neq j} p_i p_j \rho_i \otimes \rho_j \geqslant 0$, since density matrices ρ_i are positive semidefinite, i.e., $\rho_i \geqslant 0$. Combining inequalities (25) and (27) gives inequality (20), as required. This completes the proof. ∎

In particular, if ρ is a fully separable n-qubit state, then this theorem for $|\Phi\rangle = |00\cdots 0\rangle|11\cdots 1\rangle$ implies

$$|\rho_{1,2^n}| \leqslant (\rho_{2,2}\rho_{3,3}\rho_{4,4}\cdots\rho_{2^n-1,2^n-1})^{\frac{1}{2^n-2}}, \tag{28}$$

the first inequality of Theorem 4 in Ref. [29], which is a necessary and sufficient condition [29] for the GHZ state mixed with white noise, $\rho(p) = (1-p)|\text{GHZ}_n\rangle\langle\text{GHZ}_n| + \frac{p}{2^n}\mathbb{I}$ as fully separable, where $|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle)$.

For detection of nonseparable quantum states, Theorem 2 is as strong as the PPT criterion and criterion (∗) in Ref. [28]. We consider the most general maximally entangled state (general GHZ state) for n-qudit mixed with white noise,

$$\rho = p|\Psi\rangle\langle\Psi| + \frac{1-p}{d^n}\mathbb{I}_{d^n}, \tag{29}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle^{\otimes n}. \tag{30}$$

A direct calculation of inequality (20) yields that these states are nonseparable (not fully separable) if

$$p > \frac{1}{1+d^{n-1}}, \tag{31}$$

which is exactly the threshold detected by the PPT criterion and criterion (∗) in Ref. [28].

Theorem 3. Suppose that ρ is a fully separable n-partite state. Then the inequality

$$\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} \leqslant \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle} \tag{32}$$

holds with equality if ρ is a pure state.

Proof. Note that the left-hand side of inequality (32) minus the right-hand side of (32) is a convex function of the matrix ρ entries (since the left-hand side is the summation of absolute values of density-matrix elements and the right-hand side is the summation of the square root of a product of two diagonal density-matrix elements). Consequently, it suffices to prove the validity for fully separable pure states, and the validity for mixed states is guaranteed.

As in the proof of Theorem 1, we need only to prove that inequality (32) holds for fully separable pure states. Suppose that ρ is a pure state. Since ρ is a fully separable pure state, this gives

$$|\langle\phi_i|\rho|\phi_j\rangle| = \sqrt{\langle\phi_i|\rho|\phi_i\rangle\langle\phi_j|\rho|\phi_j\rangle} = \sqrt{\langle\phi_0|\rho|\phi_0\rangle\langle\phi_{ij}|\rho|\phi_{ij}\rangle}, \tag{33}$$

$$P_i^+\rho^{\otimes 2}P_i = \rho^{\otimes 2}, \tag{34}$$

where $|\phi_0\rangle$ and $|\phi_{ij}\rangle$ are the same as in Theorem 1. By applying these two equalities, we have

$$\sum_{i\neq j}\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} = \sum_{i\neq j}|\langle\phi_i|\rho|\phi_j\rangle| = \sum_{i\neq j}\sqrt{\langle\phi_i|\rho|\phi_i\rangle\langle\phi_j|\rho|\phi_j\rangle} = \sum_{i\neq j}\sqrt{\langle\phi_0|\rho|\phi_0\rangle\langle\phi_{ij}|\rho|\phi_{ij}\rangle} = \sum_{i\neq j}\sqrt{\langle\Phi_{ij}|P_i^+\rho^{\otimes 2}P_i|\Phi_{ij}\rangle}, \tag{35}$$

as desired. This completes the proof. ∎

For the n-qubit W state mixed with white noise, $\rho^{(W_n)}(p)$, Eq. (32) detects entanglement for

$$p < \frac{2^n}{2^n+n}; \tag{36}$$

that is, $\rho^{W_n}(p)$ is entangled (not fully separable) if $p < \frac{2^n}{2^n+n}$.

V. EXPERIMENTAL IMPLEMENTATION

Our criteria are experimentally accessible without quantum-state tomography. Each term in the left-hand side of our criteria can be determined by measuring two observables, while each term in the right-hand side can be determined by measuring one observable. For any fixed $|\Phi_{ij}\rangle$, inequalities (5) and (32) can be determined by n^2+1 and n^2-n+1 density-matrix elements, respectively. For any fixed $|\Phi\rangle$, inequality (20) can be determined by 2^n-1 density-matrix elements. Compared to the $(d_1^2-1)(d_2^2-1)\cdots(d_n^2-1)$ measurements needed for quantum-state tomography, which requires an exponential increase since $(d_1^2-1)(d_2^2-1)\cdots(d_n^2-1) = (d^2-1)^n$ in the case of all subsystems with same dimension d, the numbers of density-matrix elements in our criteria not only grow significantly slower with n, but have the great advantage of being independent of the dimension d_l of the subsystem $l, l = 1,2,\dots,n$.

The observables associated with each term (diagonal matrix elements) of the right-hand sides of inequalities (5) and (32) can be implemented by means of local observables, which can be seen from the expressions $|\phi_0\rangle\langle\phi_0| = T^{\otimes n}$, $|\phi_{ij}\rangle\langle\phi_{ij}| = T^{\otimes(i-1)} \otimes Q \otimes T^{\otimes(j-i-1)} \otimes Q \otimes T^{\otimes(n-j)}$, and $|\phi_i\rangle\langle\phi_i| = T^{\otimes(i-1)} \otimes Q \otimes T^{\otimes(n-i)}$, where $i < j$, $T = |x\rangle\langle x|$, and $Q = |y\rangle\langle y|$. Similarly, each term of the right-hand side of inequality (20) can also be determined by local measurements. Thus, determining one diagonal matrix element requires only a single local observable.

Next, from $\sqrt{\langle\Phi_{ij}|\rho^{\otimes 2}P|\Phi_{ij}\rangle} = |\langle\phi_i|\rho|\phi_j\rangle|$ and $\sqrt{\langle\Phi|\rho^{\otimes 2}P|\Phi\rangle} = |\langle\Phi_1|\rho|\Phi_2\rangle|$, we should determine the modulus of the off-diagonal elements $|\langle\phi_i|\rho|\phi_j\rangle|$ by measuring two observables O_{ij} and $\tilde{O}_{ij}$, and $|\langle\Phi_1|\rho|\Phi_2\rangle|$ by measuring O and $\tilde{O}$, since $\langle O_{ij}\rangle = 2\mathrm{Re}\langle\phi_i|\rho|\phi_j\rangle$, $\langle\tilde{O}_{ij}\rangle = -2\mathrm{Im}\langle\phi_i|\rho|\phi_j\rangle$, $\langle O\rangle = 2\mathrm{Re}\langle\Phi_1|\rho|\Phi_2\rangle$, and $\langle\tilde{O}\rangle = -2\mathrm{Im}\langle\Phi_1|\rho|\Phi_2\rangle$. Here $O_{ij} = |\phi_i\rangle\langle\phi_j| + |\phi_j\rangle\langle\phi_i|$, $\tilde{O}_{ij} = -\mathrm{i}|\phi_i\rangle\langle\phi_j| + \mathrm{i}|\phi_j\rangle\langle\phi_i|$, $O = |\Phi_1\rangle\langle\Phi_2| + |\Phi_2\rangle\langle\Phi_1|$, and $\tilde{O} = -\mathrm{i}|\Phi_1\rangle\langle\Phi_2| + \mathrm{i}|\Phi_2\rangle\langle\Phi_1|$.

Without loss of generality, let $i < j$. From

$$\begin{aligned} O_{ij} &= \tfrac{1}{2}T^{\otimes(i-1)} \otimes M \otimes T^{\otimes(j-i-1)} \otimes M \otimes T^{\otimes(n-j)} \\ &\quad + \tfrac{1}{2}T^{\otimes(i-1)} \otimes \tilde{M} \otimes T^{\otimes(j-i-1)} \otimes \tilde{M} \otimes T^{\otimes(n-j)}, \end{aligned} \tag{37}$$

$$\begin{aligned} \tilde{O}_{ij} &= \tfrac{1}{2}T^{\otimes(i-1)} \otimes M \otimes T^{\otimes(j-i-1)} \otimes \tilde{M} \otimes T^{\otimes(n-j)} \\ &\quad - \tfrac{1}{2}T^{\otimes(i-1)} \otimes \tilde{M} \otimes T^{\otimes(j-i-1)} \otimes M \otimes T^{\otimes(n-j)}, \end{aligned} \tag{38}$$

where $M = |y\rangle\langle x| + |x\rangle\langle y|$, $\tilde{M} = \mathrm{i}|y\rangle\langle x| - \mathrm{i}|x\rangle\langle y|$, one can determine the left-hand side of inequality (5) by $2(n^2 - n)$ local observables.

Suppose $|\Phi_1\rangle = |x_1x_2\cdots x_n\rangle$ and $|\Phi_2\rangle = |y_1y_2\cdots y_n\rangle$. Let $R_l = |y_l\rangle\langle x_l| + |x_l\rangle\langle y_l|$ and $\tilde{R}_l = \mathrm{i}|y_l\rangle\langle x_l| - \mathrm{i}|x_l\rangle\langle y_l|$, $l = 1,2,\dots,n$. Following the method of Refs. [31,32], the element $\sqrt{\langle\Phi|\rho^{\otimes 2}P|\Phi\rangle}$ can be obtained from two local measurement settings R_l and $\tilde{R}_l$, given by

$$\mathcal{M}_l = \left[\cos\left(\frac{l\pi}{n}\right)R_l + \sin\left(\frac{l\pi}{n}\right)\tilde{R}_l\right]^{\otimes n}, \quad l = 1,2,\dots,n, \tag{39}$$

$$\tilde{\mathcal{M}}_l = \left[\cos\left(\frac{l\pi + \pi/2}{n}\right)R_l + \sin\left(\frac{l\pi + \pi/2}{n}\right)\tilde{R}_l\right]^{\otimes n}, \quad l = 1,2,\dots,n. \tag{40}$$

These operators obey

$$\sum_{l=1}^{n}(-1)^l\mathcal{M}_l = nO, \tag{41}$$

$$\sum_{l=1}^{n}(-1)^l\tilde{\mathcal{M}}_l = n\tilde{O}, \tag{42}$$

which can be proved in the same way as in Refs. [31,32]. Therefore, in total, at most $\frac{5(n^2-n)}{2} + n + 1$, $\frac{5(n^2-n)}{2} + 1$, and $2^n + 2n - 2$ local observables are needed to test our separability criteria [inequalities (5), (20), and (32), respectively].

VI. CONCLUSION

In conclusion, we investigate n-partite quantum states from elements of density matrices and derive practical separability criteria to identify genuinely entangled and nonseparable n-partite mixed quantum states. We show cases in which our criteria are stronger than all known separability criteria. In fact, our criteria detect genuine n-partite entanglement that had not been identified so far. Our approach has the added appeal of enabling relatively easy computations and requiring far fewer measurements to implement experimentally, compared to full quantum tomography.

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[1] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[2] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. **68**, 557 (1992).
[5] H. K. Lo and H. F. Chau, Science **283**, 2050 (1999).
[6] F. L. Yan and X. Q. Zhang, Eur. Phys. J. B **41**, 75 (2004).
[7] T. Gao, F. L. Yan, and Z. X. Wang, J. Phys. A **38**, 5761 (2005).
[8] C. H. Bennett and D. P. DiVincenzo, Nature (London) **404**, 247 (2000).
[9] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[10] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[11] C. H. Bennett, Phys. Scr., T **76**, 210 (1998).
[12] J. S. Bell, Physics (NY) **1**, 195 (1964).
[13] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[14] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[15] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[16] N. J. Cerf and C. Adami, Phys. Rev. Lett. **79**, 5194 (1997).
[17] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[18] M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86**, 5184 (2001).
[19] O. Rudolph, Phys. Rev. A **67**, 032312 (2003).
[20] K. Chen and L. A. Wu, Quantum Inf. Comput. **3**, 193 (2003).
[21] K. Chen and L. A. Wu, Phys. Lett. A **306**, 14 (2002).
[22] S. Albeverio, K. Chen, and S. M. Fei, Phys. Rev. A **68**, 062313 (2003).
[23] O. Gühne and M. Seevinck, New J. Phys. **12**, 053002 (2010).

[24] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).
[25] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. Lett. **99**, 250405 (2007).
[26] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
[27] M. Huber, F. Mintert, A. Gabriel, and B. C. Hiesmayr, Phys. Rev. Lett. **104**, 210501 (2010).
[28] A. Gabriel, B. C. Hiesmayr, and M. Huber, Quantum Inf. Comput. **10**, 829 (2010).
[29] T. Gao and Y. Hong (to be published in Eur. Phys. J. D.), e-print arXiv:1006.4516.
[30] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).
[31] O. Gühne, C.-Y. Lu, W.-B. Gao, and J.-W. Pan, Phys. Rev. A **76**, 030305(R) (2007).
[32] M. Seevinck and J. Uffink, Phys. Rev. A **78**, 032101 (2008).