# Geometric phase of mixed states for three-level open systems

Yanyan Jiang,[1,2] Y. H. Ji,[1,3] Hualan Xu,[1] Li-yun Hu,[1] Z. S. Wang,[1,3,*] Z. Q. Chen,[4] and L. P. Guo[4]

[1]*College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang 330022, People's Republic of China*
[2]*Department of Physics, Anqing Teachers College, Anqing 246011, People's Republic of China*
[3]*Key Laboratory of Optoelectronic and Telecommunication of Jiangxi, Nanchang, Jiangxi 330022, People's Republic of China*
[4]*Key Laboratory of Artificial Micro- and Nano-Structures of Ministry of Education and School of Physics and Technology, Wuhan University, Wuhan 430072, People's Republic of China*

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Geometric phase of mixed state for three-level open system is defined by establishing in connecting density matrix with nonunit vector ray in a three-dimensional complex Hilbert space. Because the geometric phase depends only on the smooth curve on this space, it is formulated entirely in terms of geometric structures. Under the limiting of pure state, our approach is in agreement with the Berry phase, Pantcharatnam phase, and Aharonov and Anandan phase. We find that, furthermore, the Berry phase of mixed state correlated to population inversions of three-level open system.

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## I. INTRODUCTION

A pure quantum state retains a memory of its evolution in terms of a geometric phase [1–3] when it undergoes a closed evolution in parameter space, where the geometric phase essentially arises as an effect of parallel transport in the Poincaré representation of the manifold. More precisely, the amplitudes of wave functions are mapped onto given points on the Poincaré sphere for a pure state [4–6]. Geometric phase has been observed in spin-$1/2$ systems through nuclear magnetic resonance (NMR) experiments [7] and with polarized photons using interferometers (PPI) [8].

The quantum geometric phase has been extensively explored for physical systems with two-level systems (qubits) [9,10]. Common examples of qubits are an atom [11,12] with spin $S_z = \pm\frac{1}{2}$ or photons [13] with two polarization states. Indeed, quantum interference with a photon beam, generated from a spontaneous parametric down-conversion (SPDC), has been employed in a large number of experiments [14–16] to verify and probe the intriguing properties of quantum geometric phase and quantum entanglement. Such a system has also been shown to be a versatile tool for testing many concepts and approaches used in quantum information theory.

Application of geometric phases in quantum computation [17–19] has motivated their studies under more realistic situations [20–35]. In a real system, it is unavoidable interaction of a quantum system with its surrounding environment. The interaction may lead to an irreversible loss of information on the system so the process limits the ability to maintain pure quantum states in quantum information. Therefore, it is necessary to include the effect of decoherence. Up to now, however, the definition of the geometric phase for the open system is still a controversial issue. It is therefore extremely important to understand all aspects of the geometric phase in open system [36,37].

It is known that a $d$-dimensional qudit possesses a much more complex but richer structure than the ordinary qubit. In quantum information science, thus, information processing tasks can not only be implemented using two-dimensional qubits but also are sometimes more efficiently performed using the qudits as carriers of information [38–41]. It has been found that the qudits are better adapted for certain purposes, such as quantum cryptography [42].

Following the pioneering experiment on SPDC [43], various protocols have been demonstrated using orbital angular momentum states of photons [44], where the photons are a promising carrier of quantum information. In optical implementations, therefore, the degrees of freedom of photons may be used to define the qudits in which a physical realization of the qutrits and four-dimensional quantum states (ququarts) may be obtained by biphoton states arising in the processes of SPDC. In order to construct the qutrits, for example, one has sufficiently to use the collinear degenerate SPDC processes in which state vectors of two photons in a SPDC pair are strictly parallel to each other and frequencies are also given and equal to each other. For the ququarts it is to use either the noncollinear frequency-degenerate or collinear but frequency-nondegenerate SPDC processes, where either directions of state vectors or frequencies of two photons in SPDC pairs differ from each other. When photon polarization is a two-dimensional degree of freedom, furthermore, the spatial (transverse momentum or angular momentum) and spectral (time-frequency) degrees of freedom are intrinsically continuous and can be used to define arbitrarily dimensional qudits by appropriate discretization. Therefore, it is interesting to extend the geometric phase to a higher-dimensional open system [45].

It is known that there have been many proposals tackling the geometric phase of two-level open systems from different generalizations of the parallel transport condition [20,26]. However, it may be difficult to expand it to the three-level open system. The generalizations especially are not unique so as to give out different results. In addition, a general belief is that the Berry phases are geometric in their nature, i.e., proportional to the area spanned in parameter space. Therefore, we will

*zishengwang@yahoo.com

 

express the geometric phase [31–33] for the three-level open system in terms of geometric structures on a three-dimensional complex Hilbert space.

## II. GENERALIZED BLOCH SPHERE OF THREE-LEVEL SYSTEM

Because a higher-dimensional Hilbert space may associate with quantum cryptography and entanglement, an increasing interest is to study the geometric phase of the three-level open system.

Differing from the pure state, the mixed state for the three-level open system is always written in many different ways as a probabilistic mixture of distinct but not necessarily orthogonal pure states. Thus, the density matrix was introduced as a way of describing the quantum open system and the state of open system is not completely known.

It is known that the unit matrices $1_{3\times 3}$ and eight Gell-mann matrices $\lambda$ construct a complete basis of any $3\times 3$ matrices. Any $3\times 3$ density matrix $\rho$ for the three-level open system, therefore, may be expanded as

$$\rho = c + \mathbf{d}\cdot\lambda, \tag{1}$$

this expansion is unique, i.e.,

$$c = \tfrac{1}{3}\mathrm{Tr}\,\rho, \tag{2}$$

$$d_i = \tfrac{1}{2}\mathrm{Tr}\,(\rho\lambda_i), \quad i = 1,2,\ldots,8. \tag{3}$$

Thus the density matrix $\rho$ of a three-level system may be expressed as

$$\rho = \tfrac{1}{3}(1+\sqrt{3}\mathbf{n}\cdot\lambda), \tag{4}$$

where $\mathbf{n}$ is a Bloch vector,

$$n_i = \frac{\sqrt{3}}{2}\mathrm{Tr}\,(\rho\lambda_i), \tag{5}$$

with $\mathbf{n}\cdot\mathbf{n} = r^2 \leqslant 1$ as a radius of an eight-dimensional Bloch sphere, where

$$\begin{aligned} r^2 = {}& \tfrac{3}{4}[(\rho_{12}+\rho_{21})^2 - (\rho_{12}-\rho_{21})^2 + (\rho_{11}-\rho_{22})^2 \\ &+ (\rho_{13}+\rho_{31})^2 - (\rho_{13}-\rho_{31})^2 + (\rho_{23}+\rho_{32})^2 \\ &- (\rho_{23}-\rho_{32})^2] + \tfrac{1}{4}(\rho_{11}+\rho_{22}-2\rho_{33})^2 \end{aligned} \tag{6}$$

describes a mixed degree of a three-level open system and eight components of Bloch vectors may be defined as $u_{mn} = \rho_{mn}+\rho_{nm}, v_{mn} = i(\rho_{mn}-\rho_{nm})\,(m<n=1,2,3), w_2 = \rho_{11}-\rho_{22}$ and $w_3 = \frac{1}{2}(\rho_{11}+\rho_{22}-2\rho_{33})$, which have obviously physical meanings, i.e., $u_{mn}$ and $v_{mn}(m<n=1,2,3)$ measure overlaps between energy levels $m$ and $n$, respectively. While $w_2$ and $w_3$ are physical quantities to describe population inversions.

Moreover, the set

$$\mathcal{S} = \{\mathbf{n}\in\mathcal{R}^{\mathbf{8}}|\mathbf{n}\cdot\mathbf{n} = \mathbf{r^2}, \mathbf{n}^* = \mathbf{n}\}, \tag{7}$$

is an analog of the generalized Bloch sphere with eight dimensions for the three-level system.

Similarly to the two-level system [4–6,31–33], we may introduce eight parameters, i.e., seven angles $\theta, \phi, \alpha, \beta, \gamma, \chi, \xi$, and radius $r$, to parametrize the eight-dimensional Bloch sphere, where the Bloch vector $\mathbf{n}$, its eight components, and $r$ are real.

According to Eqs. (5) and (6), the Bloch parameters in the three-level open system may be defined by

$$\cos\theta = \frac{1}{\sqrt{3}}\left[1-\frac{\sqrt{3}}{2r}(\rho_{11}+\rho_{22}-2\rho_{33})\right]^{\frac{1}{2}}, \tag{8}$$

$$\cos\phi = \frac{1}{\sqrt{2}}\left[1+\frac{\rho_{11}-\rho_{22}}{r\sin^2\theta}\right]^{\frac{1}{2}}, \tag{9}$$

$$\tan(\beta-\chi-\alpha+\gamma) = \frac{n_2}{n_1} = i\frac{\rho_{12}-\rho_{21}}{\rho_{12}+\rho_{21}}, \tag{10}$$

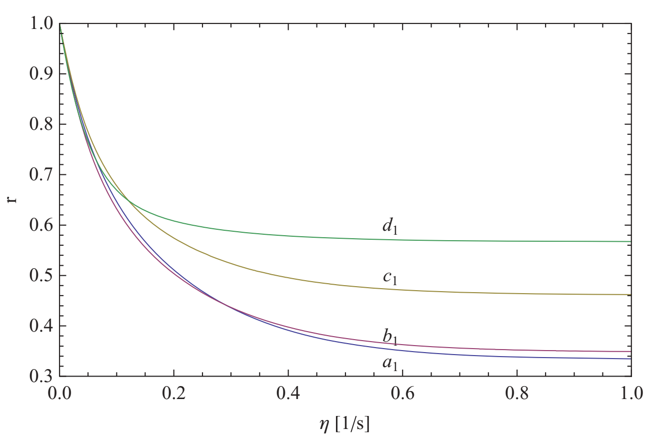


FIG. 1. (Color online) Bloch radius as a function of decay coefficient $\eta$ with unit of $\Omega = 1[1/s]$ for different initial conditions $(a_1)$ $\delta_1 = \sqrt{5}/3$, $\delta_2 = \sqrt{2}/3$, and $\delta_3 = \sqrt{2}/3$; $(b_1)$ $\delta_1 = 3/4$, $\delta_2 = 2/4$, and $\delta_3 = \sqrt{3}/4$; $(c_1)$ $\delta_1 = 4/5$, $\delta_2 = \sqrt{5}/5$, and $\delta_3 = 2/5$; and $(d_1)$ $\delta_1 = 5/6$, $\delta_2 = 3/6$, and $\delta_3 = \sqrt{2}/6$ at the quasicyclicity $T = 2\pi/\Omega$, where the corresponding population inversions of system are $w_1(t)\rangle 0$ and $w_2(t)\rangle 0$.

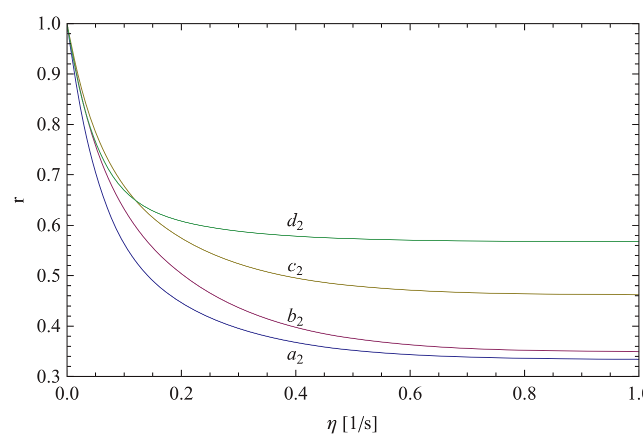


FIG. 2. (Color online) Same as described in the caption to Fig. 1 for the different initial conditions $(a_2)$ $\delta_1 = 2/3$, $\delta_2 = 2/3$, and $\delta_3 = 1/3$; $(b_2)$ $\delta_1 = 2/4$, $\delta_2 = 3/4$, and $\delta_3 = \sqrt{3}/4$; $(c_2)$ $\delta_1 = \sqrt{5}/5$, $\delta_2 = 4/5$, and $\delta_3 = 2/5$; and $(d_2)$ $\delta_1 = 3/6$, $\delta_2 = 5/6$, and $\delta_3 = \sqrt{2}/6$ at the quasicyclicity $T = 2\pi/\Omega$, where the corresponding population inversions of system are $w_1(t)\langle 0$ and $w_2(t)\langle 0$.

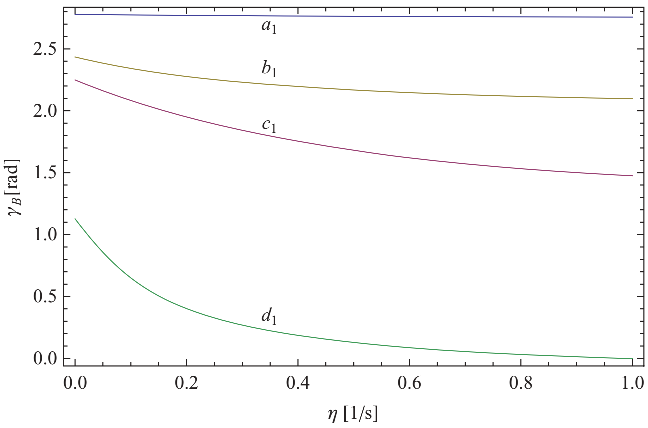


FIG. 3. (Color online) Same as described in the caption to Fig. 1 for the Berry phase of the mixed state in the three-level open system.

$$\tan(\alpha - \gamma - \xi) = \frac{n_5}{n_4} = i\frac{\rho_{13} - \rho_{31}}{\rho_{13} + \rho_{31}}, \quad (11)$$

$$\tan(\beta - \chi + \xi) = \frac{n_7}{n_6} = i\frac{\rho_{23} - \rho_{32}}{\rho_{23} + \rho_{32}}. \quad (12)$$

Inserting Eqs. (8)–(12) into Eqs. (5) and (6), the Bloch vector **n** is parameterized as

$$\begin{aligned}\mathbf{n} = r\sqrt{3}\Big( & \sin^2\theta \sin\phi \cos\phi \cos(\beta - \chi - \alpha + \gamma),\\ & \sin^2\theta \sin\phi \cos\phi \sin(\beta - \chi - \alpha + \gamma),\\ & \frac{1}{2}\sin^2\theta(\cos^2\phi - \sin^2\phi),\\ & \sin\theta\cos\theta\cos\phi\cos(\alpha - \gamma - \xi),\\ & -\sin\theta\cos\theta\cos\phi\sin(\alpha - \gamma - \xi),\\ & \sin\theta\cos\theta\sin\phi\cos(\beta - \chi + \xi),\\ & -\sin\theta\cos\theta\sin\phi\sin(\beta - \chi + \xi),\\ & \frac{1}{2\sqrt{3}}(1 - 3\cos^2\theta)\Big),\end{aligned} \quad (13)$$

where the physical properties of eight components of Bolch vectors are effectively used, which differ from the standard parametrization of an $n$-dimensional sphere.

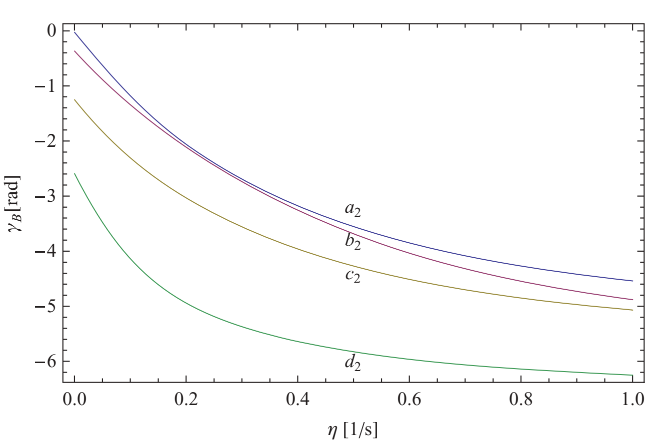


FIG. 4. (Color online). Same as described in the caption to Fig. 2 for the Berry phase of the mixed state in the three-level open system.

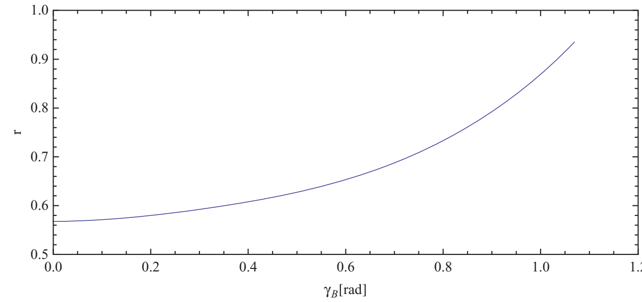


FIG. 5. (Color online) Bloch radius at the quasicyclicity $T = 2\pi/\Omega$ as a function of the Berry phase with the initial conditions $c_1 = 5/6$, $c_2 = 3/6$, and $c_3 = \sqrt{2}/6$.

It is known that, under the case $r = 1$, the eight-dimensional Bloch sphere becomes unit Poincaré sphere and the three-level system is in pure state. Thus the Bolch vector **n** is on the unit Poincaré sphere. In the case of $r < 0$, the Bloch vector **n** is in interior of this unit Poincaré sphere. Therefore, the mixed states in the three-level system may be identified with the interior points of this generalized unit Poincaré sphere.

## III. A NONUNIT STATE VECTOR FOR MIXED STATE OF THREE-LEVEL SYSTEM

It is known that the pure state is a special case of mixed state. It is very easy to unify them by using representation of Poincaré sphere, where the pure and mixed states correspond to the given points on the sphere and interior points in the sphere, respectively.

In the two-level open system, a mapping of the nonunit vector states in the two-dimensional complex Hilbert space $\mathcal{H}^2$ onto the interior points in the three-dimensional Poincaré sphere is used [31–33]. The mapping plays an important role in connecting the open system with nonunit vector ray. By using the nonunit vector ray in $\mathcal{H}^2$, the geometric phase may be defined for the two-level open system and it is proved that it is an agreement with the definition of nonunitary evolution [31,46].

In the three-level open system, the state vector are nonunit vector ray in the three-dimensional complex Hilbert space $\mathcal{H}^3$ and the generalized Poincaré sphere is eight dimensional. In order to define the geometric phase, therefore, a reasonable approach is

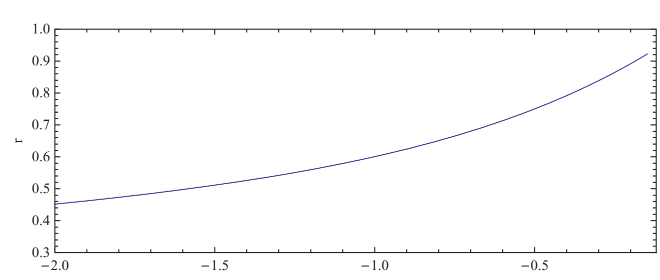


FIG. 6. (Color online) Bloch radius at the quasicyclicity $T = 2\pi/\Omega$ as a function of the Berry phase with the initial conditions $c_1 = 2/3$, $c_2 = 2/3$, and $c_3 = 1/3$.

to seek for a mapping between the eight-dimensional Poincaré sphere and the three-dimensional Hilbert space.

According to Eq. (5), we find that a mapping of $\mathbf{n} \in \mathcal{S}$ in the eight-dimensional Bloch sphere for the three-state system onto $|\psi(t)\rangle = \Pi^{-1}\rho(t) \in \mathcal{H}^3$ may be expressed as

$$n_i = \frac{\sqrt{3}}{2}\langle\psi|\lambda_i|\psi\rangle,\ i = 1,2,\ldots,8, \tag{14}$$

where $\rho = |\psi\rangle\langle\psi|$ is used in Eq. (5). It is obvious that the state vector $|\psi\rangle$, differing from the pure state, is a nonunit vector ray in $\mathcal{H}^3$ to describe the mixed state. The solution of Eq. (14) is direct, one has

$$|\psi\rangle = \sqrt{r}\begin{pmatrix} e^{i(\alpha-\gamma)}\sin\theta\cos\phi \\ e^{i(\beta-\chi)}\sin\theta\sin\phi \\ e^{i\xi}\cos\theta \end{pmatrix}, \tag{15}$$

which is a dressed three-level state vector including the effect of interaction between the physical system and environment.

In terms of Eqs. (14) and (15), the action of SU(3) on the vectors in $\mathcal{H}^3$ leads to an adjoint action on $\mathcal{S}$. Under the SU(3) transformation, the state vector becomes

$$|\psi'\rangle = u|\psi\rangle (u \in \mathrm{SU(3)}), \tag{16}$$

the corresponding Bloch vectors are transformed by SO(8) group, such as

$$n_i' = R_{ik}(u)n_k, \quad (R(u) \in \mathrm{SO(8)}). \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (14), one finds

$$R_{ik}(u) = \tfrac{1}{2}\mathrm{Tr}\,(\lambda_i u \lambda_k u^+), \tag{18}$$

which means that $\mathcal{S}^2$ may be a coset space SU(3)/U(1) × U(1).

## IV. GEOMETRIC PHASE OF THREE-LEVEL OPEN SYSTEM

We now begin with the smooth (open or closed) curve $\mathcal{C} = \{|\psi(t)\rangle = \Pi^{-1}\rho(t)\}$ and subdivide it into $N$ parts. The points of subdivision are at $t_0,t_1,\ldots,t_N$ and $|\psi_i\rangle = |\psi(t_i)\rangle = \Pi^{-1}\rho(t_i)$ are values at these points. Each trajectory, then, is represented by a discrete sequence of quantum states $\{|\psi_0\rangle,|\psi_1\rangle,\ldots,|\psi_N\rangle\}$. Thus the geometric phase for the three-level open system, expressed by the state vector in $\mathcal{H}^3$, is given by the Pantcharatnam formula

$$\begin{aligned}\gamma_g &= -\mathcal{L}t_{N\to\infty}\arg\{\langle\psi_0|\psi_1\rangle\langle\psi_1|\psi_2\rangle\cdots\langle\psi_{N-1}|\psi_N\rangle\langle\psi_N|\psi_0\rangle\} \\ &= \arg\langle\psi(t_0)|\psi(t)\rangle - \mathrm{Im}\left(\int_{t_0}^{t} d\tau\,\frac{\langle\psi(\tau)|\frac{d}{d\tau}|\psi(\tau)\rangle}{\langle\psi(\tau)|\psi(\tau)\rangle}\right),\end{aligned} \tag{19}$$

where the total phase is given by

$$\arg\langle\psi(t_0)|\psi(t)\rangle = \tan^{-1}\frac{\sin(\alpha_t-\gamma_t-\alpha_0-\gamma_0)A + \sin(\beta_t-\gamma_t-\beta_0-\gamma_0)B + \sin(\xi_t-\xi_0)C}{\cos(\alpha_t-\gamma_t-\alpha_0-\gamma_0)A + \cos(\beta_t-\gamma_t-\beta_0-\gamma_0)B + \cos(\xi_t-\xi_0)C}, \tag{20}$$

where $A = \sin\theta(t)\cos\phi(t)\sin\theta(t_0)\cos\phi(t_0)$, $B = \sin\theta(t)\sin\phi(t)\sin\theta(t_0)\sin\phi(t_0)$, $C = \cos\theta(t)\cos\theta(t_0)$, $\alpha_t = \alpha(t)$ $\gamma_t = \gamma(t)$, $\beta_t = \beta(t)$, $\xi_t = \xi(t)$.

Equation (19) is a gauge and reparametrized invariance. Therefore, $\gamma_g$ is a geometric phase associated with an evolution of a quantum three-level open system. It may be proved that, similarly to the two-level open system [31–33], Eq. (19) was in agreement with the result directly from nounitary evolution, where the geometric phase may be expressed by the density matrix. It is interesting to note that, furthermore, under the U(1) gauge transformation, $|\psi'(t)\rangle = \exp[-i\arg\langle\psi(t_0)|\psi(t)\rangle]|\psi(t)\rangle$, the geometric phase (19) may be rewritten as

$$\gamma_g^A = -\mathrm{Im}\int_{t_0}^{t}\frac{\langle\psi'(t)|d|\psi'(t)\rangle}{\langle\psi'(t)|\psi'(t)\rangle}, \tag{21}$$

which is a generalization of the Aharonov and Anandan phase for the pure state with the condition of $\langle\psi'(t)|\psi'(t)\rangle = \langle\psi(t)|\psi(t)\rangle = 1$. Therefore, $\gamma_g^A$ is called as the Aharonov and Anandan phase for the mixed state.

Considering the interaction between the physical system and environment, the system no longer undergoes a cyclic evolution, where the exponent decay factors are included in the density matrices, Bloch vectors, and nonunit state vectors. When it is isolated from the environment, however, the system may be regarded as a quasicyclic process so the total phase in Eq. (20) is equal to $2\pi$, which is not important and may

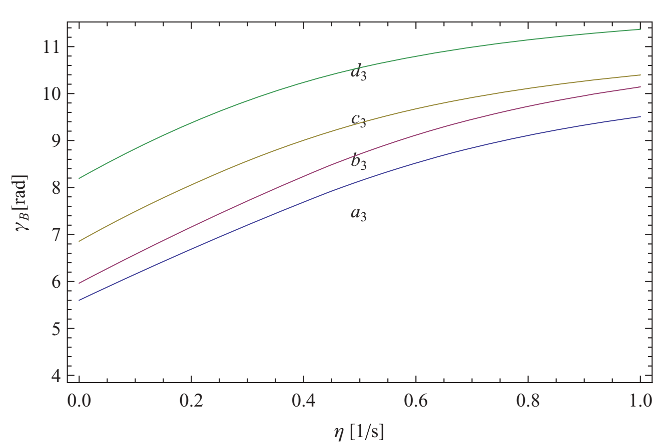


FIG. 7. (Color online) Berry phase of the mixed state for the different initial conditions $(a_3)\delta_1 = 2/3$, $\delta_2 = 1/3$, and $\delta_3 = 2/3$; $(b_3)\delta_1 = 2/4$, $\delta_2 = \sqrt{3}/4$, and $\delta_3 = 3/4$; $(c_3)\delta_1 = \sqrt{5}/5$, $\delta_2 = 2/5$, and $\delta_3 = 4/5$; and $(d_3)\delta_1 = 3/6$, $\delta_2 = \sqrt{2}/6$, and $\delta_3 = 5/6$, where the corresponding population inversions of system are $w_1(t)\langle 0$ and $w_2(t)\rangle 0$.

be dropped off in quantum computation. Thus, the geometric phase under the quasicyclic process may be expressed as

$$\gamma_g(\mathcal{C}) = -\Im \oint_{\mathcal{C}} \frac{\langle \psi|d|\psi\rangle}{\langle \psi|\psi\rangle} = -\oint_{\mathcal{C}} \{\sin^2\theta[\cos^2\phi d(\alpha-\gamma) + \sin^2\phi d(\beta-\chi)] + \cos^2\theta d\xi\}, \tag{22}$$

which is called as Berry phase of mixed state, while $\mathcal{C}$ is a closed circle arc on the generalized eight-dimensional sphere for the three-level system.

Let us consider a projective map described by $\Pi(|\psi\rangle) = \{|\overline{\psi}(t)\rangle : |\overline{\psi}(t)\rangle = e^{i\alpha(t)}|\psi(t)\rangle\}$. Then $|\psi(t)\rangle$ defines a curve, $\mathcal{C}$, in Hilbert space: $[0,\tau] \to \mathcal{H}^3$ with $\hat{C} = \Pi(\mathcal{C})$ being a closed curve in $\mathcal{H}^3$ for the quasicyclic evolution. It is obvious that the same $|\overline{\psi}(t)\rangle$ can be chosen for every curve $\mathcal{C}$ for which $\Pi^{-1}(\mathcal{C}) = \hat{C}$ by an appropriate choice of $\alpha(t)$.

It is noted that the Berry phase of mixed state, described by Eq. (22), is an integral of the Mead-Berry connection one-form $K = \Im \frac{\langle\overline{\psi}(t)|d|\overline{\psi}(t)\rangle}{\langle\overline{\psi}(t)|\overline{\psi}(t)\rangle}$ around the closed curve $\hat{C}$ in $\mathcal{H}^3$. If there is a different choice for $\alpha(t)$, such as $|\overline{\psi}^*(t)\rangle = e^{i\beta(t)}|\overline{\psi}\rangle$, then $K \to K^* = K - d\beta$ because $\langle\overline{\psi}(t)|\overline{\psi}(t)\rangle = r$ is real so that $dK^* = dK$, which means that $\gamma_g$ is invariant. Thus, the Berry phase of mixed state, described by Eq. (22), is independent of $\alpha(t)$ for a given closed curve $\hat{C}$ under the quasicyclic case.

## V. MASTER EQUATION FOR DISSIPATIVE SYSTEM

As an example, let us consider a three-level system interacting with environment. When a relevant dynamical time scale of the open quantum system is long compared to the time for the environment to the forgetting quantum information, the evolution of system is effectively local in time (the Markovian approximation) and may be described by the Lindblad's master equation [47],

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[\hat{\mathcal{H}},\rho] + \sum_i \left(\Gamma_i^+\rho\Gamma_i - \frac{1}{2}\{\Gamma_i^+\Gamma_i,\rho\}\right). \tag{23}$$

The first term on the right of Eq. (23) is a usual Schrödinger term that generates a unitary evolution. The others describe all possible transitions that the open system may undergo due to interacting with the reservoir. The operators $\Gamma_i (i = 1,2,\ldots,8)$ are called Lindblad operators or quantum jump operators. We can readily check, using Eq. (23), that $\dot{\rho}$ is Hermitian and $\mathrm{Tr}\,\dot{\rho} = 0$, which implies that the Lindblad's master equation in Eq. (23) preserves positivity of the density operator $\rho(t)$ for the open system.

It is noted that the Lindblad operators $\Gamma_i = \sqrt{\eta_i(t)}\lambda_i$ represent the couplings to the environment. The decoherent time is approximately given by $1/\Gamma_i(t)$. The noise can be controlled by switching on and off $\eta_i(t)$. Now suppose the dephasing noise $\Gamma = \sqrt{\eta}\lambda_3$ is applied to our three-level system with the Hamiltonian $\mathcal{H} = \frac{1}{2}\hbar\Omega\lambda_3$. Thus the solution of master equation can be obtained as

$$\rho_{11}(t) = \rho_{11}(0), \tag{24}$$

$$\rho_{12}(t) = \rho_{12}(0)\exp\{(-2i\Omega - 2\eta)t\}, \tag{25}$$

$$\rho_{13}(t) = \rho_{13}(0)\exp\{(-i\Omega - \eta/2)t\}, \tag{26}$$

$$\rho_{21}(t) = \rho_{21}(0)\exp\{(2i\Omega - 2\eta)t\}, \tag{27}$$

$$\rho_{22}(t) = \rho_{22}(0), \tag{28}$$

$$\rho_{23}(t) = \rho_{23}(0)\exp\{(i\Omega - \eta/2)t\}, \tag{29}$$

$$\rho_{31}(t) = \rho_{31}(0)\exp\{(i\Omega - \eta/2)t\}, \tag{30}$$

$$\rho_{32}(t) = \rho_{32}(0)\exp\{(-i\Omega - \eta/2)t\}, \tag{31}$$

$$\rho_{33}(t) = \rho_{33}(0). \tag{32}$$

Under the case of initial pure state $|\psi(0)\rangle = \delta_1|1\rangle + \delta_2|2\rangle + \delta_3|3\rangle$, $\rho_{11}(0) = |\delta_1|^2$, $\rho_{12}(0) = \delta_1\delta_2^*$, $\rho_{13}(0) = \delta_1\delta_3^*$, $\rho_{21}(0) = \delta_2\delta_1^*$, $\rho_{22}(0) = |\delta_2|^2$, $\rho_{23}(0) = \delta_2\delta_3^*$, $\rho_{31}(0) = \delta_3\delta_1^*$, $\rho_{32}(0) = \delta_3\delta_2^*$, and $\rho_{33}(0) = |\delta_3|^2$, where $\delta_i (i = 1,2,3)$ are independent of the evolving time and may be controlled by the external conditions.

In Eqs. (24)–(32) the dephasing factors $\exp(-2\eta t)$ and $\exp(-\eta t/2)$ parametrize the amount of decoherence. The effects of dephasing are to decrease the size of the nondiagonal elements of density matrix in a basis determined by the dephasing interaction with the environment so that a single-qutrit is corrupted by the dephasing.

Inserting Eqs. (24)–(32) into Eqs. (6) and (8)–(12), one may obtain the eight Bloch parameters. Then the Berry phase of mixed state (22) can be calculated by the Bloch parameters.

## VI. DISCUSSIONS AND CONCLUSIONS

The Bloch radius as a function of decay rate $\eta$ is shown in Figs. 1 and 2 in units of $\Omega = 1[1/s]$ for different initial conditions at the quasicyclicity $T = 2\pi/\Omega$, respectively. From Figs. 1–2, we see that, with increasing of the decay rate, the Bloch radius decays, which means that the physical state of three-level open system is from the pure state to mixed state. The mixed degree is obviously dependent on the decay rate.

We find that, furthermore, the corresponding Berry phase of the mixed state may be separated into two groups with positive (see Fig. 3) and negative (see Fig. 4) values for the different initial conditions, where Figs. 3 and 4 have the same parameters as Figs. 1 and 2, respectively. It is obvious that the sign of the phase depends on population inversions $w_1(t) = \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t)$ and $w_2(t) = \rho_{11}(t) - \rho_{22}(t)$. The Berry phases of the mixed state at Fig. 3 correspond to $w_1(t)\rangle 0$ and $w_2(t)\rangle 0$; the others at Fig. 4 correspond to $w_1(t) \leqslant 0$ and $w_2(t) \leqslant 0$.

Figures 3 and 4 show that, as the decay rate increases,+ the Berry phases decrease. However, the absolute values of the phases at Fig. 4 increase, which may be understand by the relations between Bloch radius and Berry phase. From Figs. 5 and 6, we see that the Bloch radius almost exponentially increase with the Berry phase. At Fig. 5, the maximum value for positive Berry phase corresponds to the pure state with $r = 1$. At Fig. 6, the minimum value for negative one corresponds to the pure state.

It is interesting that, when the population inversions have differing signs, i.e., $w_1(t)\langle 0$ and $w_2(t)\rangle 0$, the Berry phase is an increasing function of the decay rate, differing from the previous situations (see Fig. 7). The similar cases are observed for the other signs among $w_1(t)$, $w_2(t)$ and $w_3(t)$. It is known that the population inversions are physical quantities describing the evolution of the system. It is shown that, therefore, the three-level open system retains a memory of its evolution in terms of the Berry phase of the mixed state.

Thus, our definition of the geometric phase for the three-level mixed state may have a hidden rich physics.

In conclusion, a way is expanded to calculate geometric phase for the three-level open system. By mapping the interior points of the eight-dimensional Poincaré sphere onto field amplitudes, we establish in connecting density matrices with nonunit vector rays in $\mathcal{H}^3$. Geometric (Berry) phases, defined according to the vector rays, depend only on the geometric structure on this space, where the simple structure is exploited so any state can be described as a (generally non-normalized) vector in a Block sphere. Under the limiting of pure state, moreover, our approach is in agreement with the Berry phase, Pantcharatnam phase, and Aharonov and Anandan phases.

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