Interpreting quantum states of electromagnetic field in time-dependent linear media

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Recently, Pedrosa and Rosas [Phys. Rev. Lett. **103**, 010402 (2009)] investigated the quantum states of an electromagnetic field in time-dependent linear media using a Hermitian linear invariant. The wave function obtained by them is represented in terms of an arbitrary weight function $g(\lambda_l)$. Since the type of wave function varies depending on the choice of $g(\lambda_l)$ in their problem, it may be a difficult task to construct a coherent state that resembles the classical state from their theory. We suggest, on the basis of a non-Hermitian linear invariant, another quantum state that is a kind of coherent state. The expectation value of canonical variables in this alternate state follows an exact classical trajectory. For a simple case in which the time dependence of the parameters $\epsilon(t)$, $\mu(t)$, and $\sigma(t)$ disappears, we showed that the quantum energy expectation value in the alternate quantum state recovers exactly to the classical energy in the limit $\hbar \to 0$. This alternate state leads to the correspondence between the quantum and the classical behaviors of physical observables in a high-energy limit.

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I. INTRODUCTION

Planck's pioneering work [1] in the field of quantum physics highlighted the importance of the physical nature of the radiation field in a cavity. Quantum electrodynamics in vacuum and in dielectric media have long been studied, and various theories of fundamental quantum effects in this area have been developed thus far. It appears natural to extend the initial quantization scheme for the cavity field into a more general problem, such as that of fields in conducting media, nonlinear media, and time-dependent linear media. Choi [2] quantized the electromagnetic field in conducting linear media using the Lewis-Riesenfeld dynamical invariant method [3,4]. This led to renewed interest in the study of the quantum properties of light that undergoes dissipation principally due to the conductivity of media. The quantization of electromagnetic fields in time-dependent linear media may also be an interesting problem in quantum optics. Media are classified as time-dependent media if their parameters, such as electric permittivity, magnetic permeability, and conductivity, vary with time. Among various fields in electrodynamics, examples of the possible application of the electromagnetic wave propagation in time-dependent media include the modulation of microwave power [5], wave propagation in ionized plasmas [6], and magnetoelastic delay lines [7]. Cirone *et al.* investigated the problem of photon creation produced by a time-dependent dielectric [8,9]. Choi and Yeon [10] later quantized electromagnetic fields in time-dependent linear media.

Pedrosa and Rosas [11] recently used a linear Hermitian invariant to obtain the quantum state of an electromagnetic field with a choice of the Coulomb gauge. (Hereafter, we call Ref. [11] paper I for convenience.) They demonstrated that the time dependence of the electric permittivity leads to the attenuation of the electromagnetic fields. However, it is hard to discuss quantum and classical correspondence on the basis of their quantum solution since the wave function they obtained varies depending on the choice of an arbitrary weight function. In this paper, we introduce a different class of the quantum state for an electromagnetic field in time-dependent linear media on the basis of another type of linear invariant. This linear invariant is non-Hermitian, whereas that used in paper I is Hermitian. This is the main difference between our procedure and that described in paper I with regard to the development of the relevant quantum theory. We compare the results of our quantization scheme with those obtained in paper I and discuss the usefulness of quantizing the electromagnetic field on the basis of the non-Hermitian linear invariant. We show that the quantum behavior of the system analyzed using our quantum solution follows a classical trajectory and that the quantum energy expectation value in this state exactly coincides with the classical energy in the limit $\hbar \to 0$, where $\hbar$ is Planck's constant.

II. ELECTROMAGNETIC FIELDS IN TIME-DEPENDENT LINEAR MEDIA

The relations between the fields and the current in time-dependent linear media are given as $\mathbf{D} = \epsilon(t)\mathbf{E}$, $\mathbf{B} = \mu(t)\mathbf{H}$, and $\mathbf{J} = \sigma(t)\mathbf{E}$, where $\epsilon(t)$ is the electric permittivity, $\mu(t)$ the magnetic permeability, and $\sigma(t)$ the conductivity. In this case, the speed of light is explicitly time-dependent and is given as $c(t) = [\epsilon(t)\mu(t)]^{-1/2}$. For the sake of simplicity, we assume that the media have no net charge density. In the Coulomb gauge, we then need to expand the electric and the magnetic fields in terms of only the vector potential because the scalar potential disappears in this situation. From Maxwell's fundamental equations, one can verify that the vector potential satisfies the equation [11,12]

$$\nabla^2\mathbf{A} - [\sigma(t) + \dot{\epsilon}(t)]\mu(t)\frac{\partial \mathbf{A}}{\partial t} - \mu(t)\epsilon(t)\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{1}$$

To decouple the position and the time variables in the vector potential, we substitute $\mathbf{A}(\mathbf{r},t) = \sum_l \mathbf{u}_l(\mathbf{r})q_l(t)$ into Eq. (1) to obtain

$$\nabla^2\mathbf{u}_l(\mathbf{r}) + k_l^2\mathbf{u}_l(\mathbf{r}) = 0, \tag{2}$$

$$\ddot{q}_l(t) + \frac{\sigma(t) + \dot{\epsilon}(t)}{\epsilon(t)}\dot{q}_l(t) + \omega_l^2(t)q_l(t) = 0. \tag{3}$$

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Here, $\omega_l(t)$ is the natural frequency of the fields, and $k_l(=|\mathbf{k}_l|)=\omega_l(0)/c(0)$, the wave number. If we consider that k_l is constant, the frequency is given by $\omega_l(t)=[c(t)/c(0)]\omega_l(0)$. Generally, different polarization modes of $\mathbf{u}_l(\mathbf{r})$, distinguished by the subscript l, are determined by the geometry and boundary conditions. For example, the mode function for the electromagnetic field propagating under a periodic boundary condition is given by $\mathbf{u}_{l\nu}(\mathbf{r})=(1/\sqrt{V})\hat{\epsilon}_{l\nu}e^{\pm i\mathbf{k}_l\cdot\mathbf{r}}$, where V is the volume of a sector and $\hat{\epsilon}_{l\nu}$ the unit vector that indicates the polarization direction of the field.

The corresponding time evolution of the fields is described by a time-dependent Hamiltonian of the form

$$\hat{H}_l(\hat{q}_l,\hat{p}_l,t)=e^{-\Gamma(t)}\frac{\hat{p}_l^2}{2\epsilon_0}+\frac{1}{2}e^{\Gamma(t)}\epsilon_0\omega_l^2(t)\hat{q}_l^2, \tag{4}$$

where $\hat{p}_l=-i\hbar\partial/\partial q_l$, $\epsilon_0=\epsilon(0)$ and

$$\Gamma(t)=\int_0^t dt'[\sigma(t')+\dot{\epsilon}(t')]/\epsilon(t'). \tag{5}$$

Using Hamilton's equations, it is not difficult to verify that this Hamiltonian gives the exact classical equation of motion given in Eq. (3).

If the Hamiltonian is explicitly time-dependent as in this case, the energy operator is somewhat different from the Hamiltonian [13,14]. From the fundamentals of electrodynamics, we see that the classical energy is given by $U_l=\frac{1}{2}\epsilon\dot{q}_l^2+\frac{1}{2}\epsilon\omega_l^2q_l^2$. Considering the relation $\dot{q}_l=e^{-\Gamma}p_l/\epsilon_0$, the energy operator is obtained in the form

$$\hat{U}_l(\hat{q}_l,\hat{p}_l,t)=e^{-2\Gamma(t)}\epsilon(t)\frac{\hat{p}_l^2}{2\epsilon_0^2}+\frac{1}{2}\epsilon(t)\omega_l^2(t)\hat{q}_l^2. \tag{6}$$

To quantize the electromagnetic field, we need to solve the Schrödinger equation $i\hbar\partial\psi_{\lambda_l}/\partial t=\hat{H}_l\psi_{\lambda_l}$, where the subscript λ_l denotes an eigenvalue that is described later.

Because the Hamiltonian given in Eq. (4) is time-dependent, the invariant formulation of the quantum electrodynamics may be useful for the further development of the theory. If a system admits a dynamical invariant $\hat{I}(\hat{q},\hat{p},t)$, the Schrödinger solution is represented in terms of its eigenstate $\phi_\lambda(q,t)$ and a time-dependent phase $\theta_\lambda(t)$. Thus, the quantum state can be written in the form

$$\psi_{\lambda_l}(q_l,t)=e^{i\theta_{\lambda_l}(t)}\phi_{\lambda_l}(q_l,t). \tag{7}$$

We can determine $\phi_{\lambda_l}(q_l,t)$ by solving the eigenvalue equation of the invariant:

$$\hat{I}_l(\hat{q}_l,\hat{p}_l,t)\phi_{\lambda_l}(q_l,t)=\lambda_l\phi_{\lambda_l}(q_l,t), \tag{8}$$

whereas $\theta_{\lambda_l}(t)$ is obtained from

$$\hbar\frac{d\theta_{\lambda_l}(t)}{dt}=\langle\phi_{\lambda_l}|\left(i\hbar\frac{\partial}{\partial t}-\hat{H}_l\right)|\phi_{\lambda_l}\rangle. \tag{9}$$

Now let us cast a linear invariant of the system in the form

$$\hat{I}_l=b_1(t)\hat{p}_l+b_2(t)\hat{q}_l, \tag{10}$$

where $b_1(t)$ and $b_2(t)$ are time functions that are determined later. Applying basic algebraic techniques after substituting Eq. (10) into the Liouville–von Neumann equation,

$$d\hat{I}_l/dt=\partial\hat{I}_l/\partial t+[\hat{I}_l,\hat{H}_l]/(i\hbar)=0 \tag{11}$$

yields

$$\dot{b}_1(t)=-\frac{e^{-\Gamma(t)}}{\epsilon_0}b_2(t),\quad \dot{b}_2(t)=e^{\Gamma(t)}\epsilon_0\omega_l^2(t)b_1(t). \tag{12}$$

By combining these two equations, we see that $b_1(t)$ is obtained from the differential equation

$$\ddot{b}_1(t)+\dot{\Gamma}(t)\dot{b}_1(t)+\omega_l^2(t)b_1(t)=0. \tag{13}$$

Once $b_1(t)$ is determined by solving this equation, one can also obtain $b_2(t)$ from the first term in Eq. (12). Using these procedures, it is possible to construct the complete form of the linear invariant for the system.

In the next section, we investigate the quantum states derived in paper I using a linear invariant. We then suggest an alternate linear invariant that gives a different class of quantum states that appear to be useful in investigating quantum and classical correspondence in time-dependent linear media.

III. QUANTIZATION OF ELECTROMAGNETIC FIELDS

A. Overview of previous method

In paper I, $b_1(t)=\beta_l(t)$ is used, where $\beta_l(t)$ is a time function that is real and obeys the classical equation of motion

$$\ddot{\beta}_l(t)+\frac{\sigma+\dot{\epsilon}}{\epsilon}\dot{\beta}_l(t)+\omega_l^2(t)\beta_l(t)=0. \tag{14}$$

Because this is a second-order differential equation, there generally exist two independent solutions, $\beta_{l,1}(t)$ and $\beta_{l,2}(t)$; i.e., we can write

$$\beta_l(t)=c_1\beta_{l,1}(t)+c_2\beta_{l,2}(t). \tag{15}$$

Then, the corresponding linear invariant is given by

$$\hat{I}_l=\beta_l(t)\hat{p}_l-\epsilon_0e^{\Gamma(t)}\dot{\beta}_l(t)\hat{q}_l. \tag{16}$$

By evaluating Eqs. (8) and (9) using this invariant, we derive

$$\phi_{\lambda_l}(q_l,t)=\sqrt{\frac{1}{2\pi\hbar\beta_l(t)}}\exp\left[\frac{i}{\hbar\beta_l(t)}\left(\frac{\epsilon_0\dot{\beta}_l(t)e^{\Gamma(t)}}{2}q_l^2+\lambda_lq_l\right)\right], \tag{17}$$

$$\theta_{\lambda_l}(t)=\frac{\lambda_l^2}{2\hbar\epsilon_0}\int_0^t\frac{e^{-\Gamma(t')}dt'}{\beta_l^2(t')}. \tag{18}$$

Equation (7) along with the above two equations is the corresponding quantum state obtained by Pedrosa and Rosas. One can easily verify that the expectation value of $\hat{q}_l$ in this state diverges as follows:

$$\begin{aligned}\langle\hat{q}_l\rangle&=\int_{-\infty}^{\infty}\psi_{\lambda_l}^*(q_l,t)\hat{q}_l\psi_{\lambda_l}(q_l,t)dq_l\\&=\frac{1}{2\pi\hbar\beta_l(t)}\int_{-\infty}^{\infty}q_ldq_l\longrightarrow\infty.\end{aligned} \tag{19}$$

The reason for this divergence is that the solution Eq. (17) is given in terms of plane wave. Further, the authors of paper I suggested a more general state by superposing $\psi_{\lambda_l}(q_l,t)$ such that

$$\psi(q_l,t)=\int_{-\infty}^{\infty}g(\lambda_l)\psi_{\lambda_l}(q_l,t)d\lambda_l, \tag{20}$$

where $g(\lambda_l)$ is a weight function. In principle, a coherent state whose behavior resembles a classical one might be constructed as a wave packet with a suitable weight function in Eq. (20). However, it may be not an easy task. In the next subsection, we suggest another type of quantum state that does not give the divergence of $\langle\hat{q}_l\rangle$ even when we do not prescribe the superposition rule considering $g(\lambda_l)$.

B. Alternate method

Remember that β_l (or $\beta_{l,1}$ and $\beta_{l,2}$) introduced in the previous subsection is real. Instead of this, we use a complex function of time, expressed in terms of $\beta_{l,1}$ and $\beta_{l,2}$, as a solution of Eq. (13) in order to obtain a more plausible quantum state. That is,

$$b_{1,\mathrm{new}}(t) = \frac{\beta_{l,0}(t)}{\sqrt{2\hbar W_l}}e^{i\eta_l(t)}, \tag{21}$$

where W_l is a Wronskian that has the form $W_l = \epsilon_0 e^{\Gamma}(\beta_{l,1}\dot{\beta}_{l,2} - \dot{\beta}_{l,1}\beta_{l,2})$, $\eta_l(t)$ is a phase given by

$$\eta_l(t) = (W_l/\epsilon_0)\int_0^t dt'\left[\beta_{l,0}^2(t')e^{\Gamma(t')}\right]^{-1}, \tag{22}$$

and $\beta_{l,0}(t)$ is the real classical solution of a modified type of the Milne equation [15] that is given by

$$\ddot{\beta}_{l,0}(t) + \frac{\sigma+\dot{\epsilon}}{\epsilon}\dot{\beta}_{l,0}(t) + \omega_l^2(t)\beta_{l,0}(t) - \frac{W_l^2}{(\epsilon_0 e^{\Gamma(t)})^2\beta_{l,0}^3} = 0. \tag{23}$$

One can find that $dW_l/dt = 0$ from the direct differentiation of W_l. This implies that W_l is a time-constant. In fact, the relation between $\beta_{l,0}(t)$ and the two time functions $\beta_{l,1}(t)$ and $\beta_{l,2}(t)$ introduced in the previous subsection is [16] $\beta_{l,0}(t) = \sqrt{\beta_{l,1}^2(t) + \beta_{l,2}^2(t)}$. Another time function $b_{2,\mathrm{new}}(t)$ is now easily derived by substituting the time derivative of Eq. (21) into the first term of Eq. (12). Thus, we eventually obtain an alternate linear invariant such that

$$\hat{\mathcal{I}}_l(\hat{q}_l,\hat{p}_l,t) = \hat{a}_l e^{i\eta_l(t)}, \tag{24}$$

where

$$\begin{aligned}\hat{a}_l &= \frac{1}{\sqrt{2\hbar W_l}}[\beta_{l,0}(t)\hat{p}_l - iZ_l(t)\hat{q}_l],\\ Z_l(t) &= \frac{W_l}{\beta_{l,0}(t)} - i\epsilon_0 e^{\Gamma(t)}\dot{\beta}_{l,0}(t).\end{aligned} \tag{25}$$

The Hermitian adjoint of Eq. (24), $\hat{\mathcal{I}}_l^\dagger(\hat{q}_l,\hat{p}_l,t)$, which is represented in terms of $\hat{a}_l^\dagger$, is also an invariant. It may be easy to find that $\hat{a}_l$ and $\hat{a}_l^\dagger$ satisfy the boson commutation relation $[\hat{a}_l,\hat{a}_l^\dagger] = 1$. Hence, we can regard $\hat{a}_l$ and $\hat{a}_l^\dagger$ as the annihilation and the creation operators, respectively.

Because the direct time derivative of $\hat{a}_l$ in Eq. (25) results in $d\hat{a}_l/dt = -iW_l\hat{a}_l/(\beta_{l,0}^2\epsilon_0 e^{\Gamma})$, the time evolution of $\hat{a}_l$ is given by

$$\hat{a}_l(t) = \hat{a}_l(0)e^{-i\eta_l(t)}. \tag{26}$$

From the inverse representation of $\hat{a}_l$ and $\hat{a}_l^\dagger$ together, we have

$$\hat{q}_l = i\sqrt{\frac{\hbar\beta_{l,0}^2}{2W_l}}(\hat{a}_l - \hat{a}_l^\dagger), \tag{27}$$

$$\hat{p}_l = \sqrt{\frac{\hbar}{2W_l}}[Z_l^*(t)\hat{a}_l + Z_l(t)\hat{a}_l^\dagger]. \tag{28}$$

To avoid confusion with the previous case, let us express the Schrödinger solution in the form $\Psi_{\Lambda_l}(q_l,t) = e^{i\Theta_{\Lambda_l}(t)}\Phi_{\Lambda_l}(q_l,t)$ instead of Eq. (7), where Λ_l is the eigenvalue of the alternate invariant given in Eq. (24). By solving the eigenvalue equation of $\hat{\mathcal{I}}_l$ straightforwardly using Eq. (24), we obtain the eigenvalue as $\Lambda_l = \alpha_l e^{i\eta_l(t)}$, where α_l is the eigenvalue of $\hat{a}_l$, and the eigenstate is given such that

$$\begin{aligned}\Phi_{\Lambda_l}(q_l,t) = \left(\frac{W_l}{\beta_{l,0}^2\hbar\pi}\right)^{1/4}\exp\Bigg[\frac{1}{\beta_{l,0}\hbar}\left(\sqrt{2\hbar W_l}\alpha_l q_l - \frac{Z_l(t)}{2}q_l^2\right)\\ - \frac{1}{2}|\alpha_l|^2 - \frac{1}{2}\alpha_l^2\Bigg].\end{aligned} \tag{29}$$

Further, from Eq. (9), we have $\Theta_l(t) = -\eta_l(t)/2$. Because Eq. (29) is also an eigenstate of $\hat{a}_l$,[1] the quantum state derived here is a class of a coherent state whose wave packet is Gaussian. The expectation values of canonical variables in this state are finite and are in good agreement with their corresponding classical results.

To see how well the quantum behavior of the system agrees with that of the classical state, at this stage it may be instructive to apply our theory to a simple case in which all three parameters are constant; i.e., $\epsilon(t) = \epsilon_0$, $\mu(t) = \mu_0$, and $\sigma(t) = \sigma_0$. Although the time dependence of the parameters disappears in this limit, we can confirm from Eq. (4) that the Hamiltonian remains time-dependent because of the existence of nonzero conductivity. Thus, the problem associated with this system still preserves nontrivialness from the viewpoint of quantum mechanics while we know its classical behaviors completely. In this case, ω_l also becomes constant, and $\eta_l(t) = \tilde{\omega}_l t$, where $\tilde{\omega}_l$ is a modified frequency of the form $\tilde{\omega}_l = \sqrt{\omega_l^2 - \sigma_0^2/(4\epsilon_0^2)}$. Then, the solution of Eq. (23) is given by

$$\beta_{l,0}(t) = \sqrt{W_l/(\epsilon_0\tilde{\omega}_l)}e^{-\sigma_0 t/(2\epsilon_0)}. \tag{30}$$

Considering Eq. (26) and the fact that α_l is the eigenvalue of $\hat{a}_l$ ($\hat{a}_l|\Phi_{\Lambda_l}\rangle = \alpha_l|\Phi_{\Lambda_l}\rangle$), we see that

$$\alpha_l(t) = \alpha_l(0)e^{-i\tilde{\omega}_l t}. \tag{31}$$

The initial value $\alpha_l(0)$ can also be derived from an elementary method (e.g., see Ref. [17]). By representing the classical solution of the canonical variable q_l in the form

$$q_{l,\mathrm{cl}} = q_{l,0}e^{-\sigma_0 t/(2\epsilon_0)}\cos(\tilde{\omega}_l t + \varphi_l), \tag{32}$$

where $q_{l,0}$ and φ_l are arbitrary real constants, we are able to obtain

$$\alpha_l(0) = -i\sqrt{\epsilon_0\tilde{\omega}_l/(2\hbar)}q_{l,0}e^{-i\varphi_l}. \tag{33}$$

[1] Φ_{Λ_l} is a common eigenstate of $\hat{\mathcal{I}}_l$ and $\hat{a}_l$.

Then, it is easy to verify that the expectation values of the quantum canonical variables, Eqs. (27) and (28), in the alternate state are given by[2] $\langle\Phi_{\Lambda_l}|\hat{q}_l|\Phi_{\Lambda_l}\rangle = q_{l,\text{cl}}$ and $\langle\Phi_{\Lambda_l}|\hat{p}_l|\Phi_{\Lambda_l}\rangle = \epsilon_0 e^{\Gamma(t)}\dot{q}_{l,\text{cl}}$, respectively; these are exactly identical to their classical trajectories. Moreover, using Eqs. (6), (27), (28), and (31), the expectation value of the energy operator in the alternate quantum state is evaluated to be

$$\langle\Phi_{\Lambda_l}|\hat{U}_l|\Phi_{\Lambda_l}\rangle = e^{-\sigma_0 t/\epsilon_0}\left(A_l\left\{1+\frac{\sigma_0}{2\omega_l\epsilon_0}\cos[2(\tilde{\omega}_l t+\varphi_l)-\delta_l]\right\}+\frac{\omega_l^2}{2\tilde{\omega}_l}\hbar\right), \tag{34}$$

where $A_l = \frac{1}{2}\epsilon_0\omega_l^2 q_{l,0}^2$ and $\delta_l = \tan^{-1}(2\epsilon_0\tilde{\omega}_l/\sigma_0)$. It is noticeable that only the last term is represented in terms of the constant $\hbar$, which is indispensable in the development of quantum theory. This implies that the quantum effect is given by the last term, whereas the first two terms appear equally in the classical energy. From the exponential factor $e^{-\sigma_0 t/\epsilon_0}$ of this equation, we can confirm that the quantum energy dissipates with time in a manner similar to the classical energy. Thus, the electromagnetic field in a medium with nonzero conductivity generally undergoes dissipation with time regardless of the quantum or classical viewpoints. The first two terms in the above equation involve the classical initial amplitude $q_{l,0}$, whereas the last term does not. We can hence neglect the last term when the energy of the system is sufficiently high, yielding the correspondence between the quantum and the classical aspects. Indeed, in the limit $\hbar \to 0$, the quantum energy $\langle\Phi_{\Lambda_l}|\hat{U}_l|\Phi_{\Lambda_l}\rangle$ reduces exactly to the corresponding classical energy, which is well-known from literature on classical mechanics (e.g., see Ref. [18]). When the conductivity of the medium disappears, the last term in Eq. (34) reduces to $\frac{1}{2}\hbar\omega_l$, the zero-point energy of an electromagnetic field in free space.

IV. CONCLUSION

An alternate quantum state is proposed for the electromagnetic field in time-dependent linear media, and we compared its characteristics with those of a state previously reported in paper I. The linear invariant we used to derive the new quantum state is non-Hermitian, whereas that used in paper I is Hermitian.

The expectation value of $\hat{q}_l$ in the quantum state $\psi_{\lambda_l}(q_l,t)$ in paper I does not follow the classical trajectory, but it diverges. A more general quantum state $\psi(q_l,t)$ is introduced in paper I, which is a superposition of the states $\psi_{\lambda_l}(q_l,t)$ with the consideration of the weight function $g(\lambda_l)$ [see Eq. (20)]. It may be possible to construct a coherent state from $\psi(q_l,t)$ with a suitable choice of the function $g(\lambda_l)$. However, it is expected that this might be not an easy task. For this reason, we proposed an alternate wave function $\Psi_{\Lambda_l}(q_l,t)$ that is derived from another type of linear invariant. The quantum expectation values of canonical variables $\hat{q}_l$ and $\hat{p}_l$ in the alternate quantum state exactly follow classical trajectories. Moreover, we showed that the quantum energy obtained from a rigorous evaluation in the state $\Psi_{\Lambda_l}(q_l,t)$ reduces to the exact classical energy when $\hbar \to 0$. The expectation values of other quantum observables in this state may also correspond to their classical counterparts in the classical limit.

Actually, the quantization of the electromagnetic field in connection with the alternate quantum state leads to a correspondence between the quantum and the classical aspects in a high-energy limit ($|\alpha_l| \gg 1$), which is an important concept in theoretical physics. It should be noted that when we interpret these quantum properties of the system on the basis of the alternate invariant, it is not necessary to employ a superposition rule for $\Psi_{\Lambda_l}(q_l,t)$ with consideration of the weight function $g(\Lambda_l)$, unlike what is described in paper I.

Besides, the Gawssian wave packets that have been obtained in similar problems, presented, for example, in Ref. [19], are also essential to analyze several quantum behaviors and do not reproduce the classical trajectory. Quantum states other than those that reproduce the classical trajectory are important and widely studied in quantum physics.

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[2]The expectation value with respect to $|\Psi_{\Lambda_l}\rangle$ is the same as that with respect to $|\Phi_{\Lambda_l}\rangle$.

[1] M. Planck, *Vorlesungen über die Theorie der Wärmestrahlung*, 5. Aufl., (J. A. Barth, Leipzig, 1923), p. 136.
[2] J. R. Choi, Chinese J. Phys. **41**, 257 (2003).
[3] H. R. Lewis, Jr., Phys. Rev. Lett. **18**, 510 (1967).
[4] H. R. Lewis Jr. and W. B. R. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[5] F. R. Morgenthaler, IRE Transactions on Microwave Theory and Techniques **6**, 167 (1958).
[6] S. Kozaki, Electron. Lett. **14**, 826 (1978).
[7] S. M. Rezende and F. R. Morgenthaler, J. Appl. Phys. **40**, 524 (1969).
[8] M. Cirone, K. Rzążewski, and J. Mostowski, Phys. Rev. A **55**, 62 (1997).
[9] M. A. Cirone and K. Rzążewski, Phys. Rev. A **60**, 886 (1999).
[10] J. R. Choi and K. H. Yeon, Int. J. Mod. Phys. B **19**, 2213 (2005).
[11] I. A. Pedrosa and A. Rosas, Phys. Rev. Lett. **103**, 010402 (2009).
[12] J. R. Choi and M. Maamache, PIERS Online **6**, 113 (2010).
[13] M. A. Marchiolli and S. S. Mizrahi, J. Phys. A: Math. Gen. **30**, 2619 (1997).
[14] K. H. Yeon, C. I. Um, and T. F. George, Phys. Rev. A **36**, 5287 (1987).
[15] W. E. Milne, Phys. Rev. **35**, 863 (1930).
[16] K.-E. Thylwe and H. J. Korsch, J. Phys. A: Math. Gen. **34**, 3497 (2001).
[17] J. R. Choi and I. H. Nahm, Int. J. Theor. Phys. **46**, 1 (2007).
[18] J. G. Goh, *Mechanics* [in Korean] (Cheong Moon Gak, Seoul, 1988), Chap. 3.
[19] R. Heather and H. Metiu, Chem. Phys. Lett. **118**, 558 (1985); G. V. Dunne and V. Khemani, J. Phys. A: Math. Gen. **38**, 7757 (2005).