Advanced Jones calculus for the classification of periodic metamaterials

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By relying on an advanced Jones calculus, we analyze the polarization properties of light upon propagation through metamaterial slabs in a comprehensive manner. Based on symmetry considerations, we show that all periodic metamaterials may be divided into five different classes only. It is shown that each class differently affects the polarization of the transmitted light and sustains different eigenmodes. We show how to deduce these five classes from symmetry considerations and provide a simple algorithm that can be applied to decide to which class a given metamaterial belongs by measuring only the transmitted intensities.

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I. INTRODUCTION

Metamaterials (MM) provide a large variety of unprecedented optical properties. Whereas at first, properties such as a dispersive permeability were at the focus of interest [1–3], the range of properties to be intentionally affected by suitably chosen MMs has significantly increased. Increasingly more complex [4–8] structures and most notably chiral [9–15] and quasiplanar [16–18] chiral structures have attracted a great deal of attention due to their polarization selective optical response and their potential to implement functional devices with unprecedented applications such as, e.g., broadband polarizers for circular light [19]. Moreover, from a scientific point of view, such complex MMs permit us to observe unexpected and counterintuitive effects such as asymmetric transmission for circularly [20–23] or even for linearly polarized light [24].

Recent studies have shown that the assignment of effective material parameters is doubtful in many cases [25,26] and generally requires the assumption of complex constitutive relations [27–29]. Thus a more suitable target function to be tailored by an appropriate MM design is the optical response itself. This optical response is completely involved in the response functions, such as complex reflection and transmission coefficients, for a given input illumination. This paradigm change reflects that for an actual application a certain value of some effective material coefficient is of minor importance, as long as the sample exhibits the desired optical response.

The response, in particular in transmission, can then be easily described by transmittances and polarization ellipses, averaged polarization rotation, and polarization conversion [9,14,30]. These phenomenological quantities can be completely determined from the frequency-dependent Jones matrix [31] that relates the complex amplitudes of the incident to the transmitted field. We will call this Jones matrix throughout the manuscript the T matrix, since it fully describes how light is transmitted through a metamaterial slab. This 2×2 matrix comprises, in general, four different complex and dispersive quantities, reflecting the spectral properties of the MM. The associated Jones calculus can be applied to describe the transmission of an arbitrarily polarized incident plane wave through a MM slab if only the zeroth-order Bloch mode emerges. This holds for MMs composed of periodically arranged subwavelength unit cells, and we will assume this throughout the manuscript. For the sake of simplicity we also assume that the structures are symmetrically embedded. We assume that all materials are linear and reciprocal, i.e., excluding Faraday media. No further restrictions on the symmetry of the unit cells and the generally complex permittivity of the constituting materials are necessary.

With this work we intend to introduce a classification of periodic metamaterials based on their symmetry properties and to link them to their specific T matrix. We will show that all metamaterials can be divided into only five distinct classes, each having an individual form of the T matrix and specific eigenstates. Each of these five classes leads to very specific transmission characteristics directly linked to the symmetry of the structure. Therefore, this investigation provides a useful tool to analyze the optical response of complex MMs, and it may serve as a guide to identifying designs for a desired polarization response. Although for fabricated MM the geometry is usually known, the application of combinatorial approaches to explore new MM geometries in the near future requires such a tool to classify the properties of MMs.

The paper is structured as follows. In Sec. II we present the necessary fundamentals to handle the generally complex valued T matrices and derive general expressions for the eigenpolarizations. In Sec. III we derive the form of the T matrix for the most relevant symmetry classes. In Sec. IV we provide examples of metamaterials for these symmetry classes and discuss briefly their optical behavior. In Sec. V a comprehensive tabular overview is given to summarize the results, and we present a simple scheme to classify MM samples without having *a priori* knowledge in terms of the presented formalism by measured transmittances only.

II. BASIC THEORY

It is assumed that the MM slab is illuminated by a plane wave propagating in the positive z direction

$$\mathbf{E}_i(\mathbf{r},t) = \begin{pmatrix} i_x \\ i_y \end{pmatrix} e^{i(kz-\omega t)},$$

with ω being its frequency, $k = \omega/c\sqrt{\epsilon(\omega)}$ the wave vector, and the complex amplitudes i_x and i_y describing the state of polarization. The transmitted field is then given by

$$\mathbf{E}_t(\mathbf{r},t) = \begin{pmatrix} t_x \\ t_y \end{pmatrix} e^{i(kz-\omega t)},$$

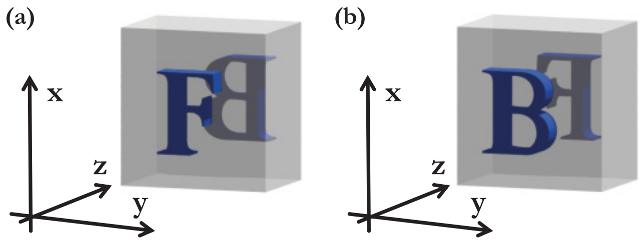

FIG. 1. (Color online) Schematic of the geometry. (a) and (b) show the sample from opposite sides with F and B indicating the front and back side, respectively.

where we have assumed that the medium is sandwiched between a medium characterized by the permittivity $\epsilon(\omega)$. A sketch of the geometry is depicted in Fig. 1. The unit cells are periodically arranged in the x and y directions without restriction to a particular lattice. We assume coherent, monochromatic plane waves in order to use a generalized Jones calculus instead of the Mueller calculus necessary for incoherent light [32,33]. The Jones calculus is said to be generalized since we allow for arbitrary complex Jones matrices which we will call T matrices (transmission matrices).

The T matrix connects the generally complex amplitudes of the incident and the transmitted field:

$$\begin{pmatrix} t_x \\ t_y \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \hat{T}^{\mathrm{f}} \begin{pmatrix} i_x \\ i_y \end{pmatrix}, \tag{1}$$

where for convenience we have replaced the entries T_{ij} by A,B,C,D, which form the actual T matrix. In the following few sections we will discuss some generic properties of this T matrix.

A. Directional-dependent properties

In the last term of Eq. (1), the T matrix superscript f designates propagation in the forward direction. Of course, the choice of forward (f) and backward (b) propagation is arbitrary. Thus $\hat{T}^{\mathrm{b}}$ describes the transmission matrix for light propagating through the structure rotated by 180° with respect to the x axis, where the choice of x or y is arbitrary.

Since only reciprocal media are considered, we have

$$\hat{T}^{\mathrm{b}} = \begin{pmatrix} A & -C \\ -B & D \end{pmatrix}, \tag{2}$$

where the minus sign in the off-diagonal elements accounts for the rotation of the system looking from the back side [34]. Therefore, the complex matrix $\hat{T}^{\mathrm{f}}$ already contains all information necessary to determine light transmission for arbitrarily polarized incident light from both main illumination directions. Its is important to stress that this relation between $\hat{T}^{\mathrm{f}}$ and $\hat{T}^{\mathrm{b}}$ is in general only valid for this particular base where the coordinate axis from the back side are given by replacing those of the front side by $x^{\mathrm{b}} = \pm x^{\mathrm{f}}$, $y^{\mathrm{b}} = \mp y^{\mathrm{f}}$. The actual sign depends on the definition of the rotation of the system.

B. Change of the base

For analytical as well as experimental concerns, it is useful to have at hand the transmission matrix in an arbitrary not necessarily orthogonal base. Let the vectors $\bar{\mathbf{i}}$ and $\bar{\mathbf{t}}$ denote the incident and transmitted light in a certain base. Then the incident and transmitted light in the Cartesian base is given by $\mathbf{i} = \hat{\Lambda}\,\bar{\mathbf{i}}$ and $\mathbf{t} = \hat{\Lambda}\bar{\mathbf{t}}$, respectively, with $\hat{\Lambda}$ being the change of the basis matrix. Hence, the T matrix for this new base is given by

$$\mathbf{t} = \hat{T}\mathbf{i} \rightarrow \bar{\mathbf{t}} = \hat{\Lambda}^{-1}\hat{T}\hat{\Lambda}\bar{\mathbf{i}} = \hat{T}_{\mathrm{new}}\bar{\mathbf{i}} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \bar{i}_1 \\ \bar{i}_2 \end{pmatrix}. \tag{3}$$

All representations of the system are completely equivalent, of course. A transformation of practical importance is the change from the Cartesian base to the circular base. Then the change of basis matrix reads as

$$\hat{\Lambda} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

where the columns of the $\hat{\Lambda}$ matrix are the new eigenstates. The T matrix for circular states is then given by

$$\begin{aligned} \hat{T}^{\mathrm{f}}_{\mathrm{circ}} &= \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} [A + D + i(B - C)] & [A - D - i(B + C)] \\ [A - D + i(B + C)] & [A + D - i(B - C)] \end{pmatrix}, \end{aligned} \tag{4}$$

connecting the amplitudes of circularly polarized incident and transmitted light:

$$\begin{pmatrix} t_+ \\ t_- \end{pmatrix} = T^{\mathrm{f}}_{\mathrm{circ}} \begin{pmatrix} i_+ \\ i_- \end{pmatrix},$$

By using Eqs. (2) and (4) it becomes obvious that the T matrix for backward propagation is given by

$$\hat{T}^{\mathrm{b}}_{\mathrm{circ}} = \begin{pmatrix} T_{++} & -T_{-+} \\ -T_{+-} & T_{--} \end{pmatrix}. \tag{5}$$

Note that the matrix $\hat{T}^{\mathrm{b}}$ in an arbitrary base is not simply given by Eq. (2), i.e., by interchanging the negative off-diagonal elements, but by applying the corresponding change of basis matrix $\hat{\Lambda}$ to $\hat{T}^{\mathrm{f}}$ and $\hat{T}^{\mathrm{b}}$ in the linear base individually.

C. Asymmetric transmission

Although not having discussed any symmetry property at all, we want to discuss at this point the special effect of asymmetric transmission, which has attracted considerable interest due to its counterintuitive occurrence, and discuss peculiarities related to a change of the base. The difference of the T matrices for opposite propagation directions is the key to that asymmetric transmission. By asymmetric transmission Δ, we understand the difference in the modulus of the total transmission between forward and backward propagation (see

Fig. 1) for a certain base vector, e.g., $\mathbf{i} = i_1\mathbf{e}_1$:

$$\Delta = \left|T^{\mathrm{f}}_{11}\right|^2 + \left|T^{\mathrm{f}}_{21}\right|^2 - \left|T^{\mathrm{b}}_{11}\right|^2 - \left|T^{\mathrm{b}}_{21}\right|^2.$$

This quantity obviously depends on the chosen base; e.g., for a linear state coinciding with the coordinate axis, we have ($\mathbf{i} = i_x\mathbf{e}_x$)

$$\Delta^{\mathrm{lin}} = |C|^2 - |B|^2,$$

whereas in the circular base we have ($\mathbf{i} = i_+\mathbf{e}_+$)

$$\Delta^{\mathrm{circ}} = |T_{-+}|^2 - |T_{+-}|^2 \neq \Delta^{\mathrm{lin}}$$

in general. This dependency on the base is exploited, e.g., in [20,21] where asymmetric transmission for circularly polarized light is observed without asymmetric transmission for linearly polarized light. Hence, the only proper choice is a linear base with base vectors parallel to the principal coordinate axes. Only in this base can we distinguish asymmetric transmission due to the structure from asymmetric transmission due to the chosen base.

D. Eigenpolarizations

To characterize the different structures, it is useful to determine the eigenstates of the polarization because they are uniquely related to the symmetry. Therefore, a simple eigenvalue problem has to be solved:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} i_x \\ i_y \end{pmatrix} = \kappa \begin{pmatrix} i_x \\ i_y \end{pmatrix}, \tag{6}$$

with the eigenvalue κ. By solving these equations we obtain

$$\kappa_{1,2} = \tfrac{1}{2}[(A + D) \pm \sqrt{(A - D)^2 + 4BC}], \tag{7}$$

where $\kappa_{1,2}$ gives the complex transmission for the eigenstates. The eigenpolarizations are then given by simply inserting $\kappa_{1,2}$ into Eq. (6) and solving for i_x and i_y. The eigenbasis in matrix form can be written as

$$\hat{\Lambda} = \begin{pmatrix} 1 & 1 \\ \frac{\kappa_1 - A}{B} & \frac{\kappa_2 - A}{B} \end{pmatrix}, \tag{8}$$

with

$$\mathbf{i}_1 = \begin{pmatrix} 1 \\ \frac{\kappa_1 - A}{B} \end{pmatrix}, \quad \mathbf{i}_2 = \begin{pmatrix} 1 \\ \frac{\kappa_2 - A}{B} \end{pmatrix}, \tag{9}$$

where the eigenvectors are not yet normalized. It is important to note that the eigenbasis depends in general on the frequency due to the dispersive behavior of the transmission. Only for highly symmetric structures is the eigenbasis frequency independent, as will be shown later. With the use of the characteristic polynomial of Eq. (6) the matrix $\hat{\Lambda}$ can be rewritten as

$$\hat{\Lambda} = \begin{pmatrix} 1 & \frac{\kappa_2 - D}{C} \\ \frac{\kappa_1 - A}{B} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{X}{2C} \\ \frac{X}{2B} & 1 \end{pmatrix}, \tag{10}$$

with $X = -(A - D) + \sqrt{(A - D)^2 + 4BC}$. Note that the matrices $\hat{\Lambda}$ in Eqs. (8) and (10) are different, but both are denoted simply by $\hat{\Lambda}$ not to confuse the reader with additional indices. They are only a concatenation of eigenvectors that are determined up to an arbitrary complex factor. The matrix $\hat{\Lambda}$ becomes unique as soon as the eigenvectors are normalized. The fractions in Eq. (10) are complex numbers, hence we can express the eigenbasis as

$$\hat{\Lambda} = \begin{pmatrix} 1 & 1 \\ R_1 e^{i\varphi_1} & \frac{1}{R_2} e^{-i\varphi_2} \end{pmatrix}, \tag{11}$$

with

$$R_1 e^{i\varphi_1} = \frac{X}{2B}, \quad R_2 e^{i\varphi_2} = -\frac{X}{2C}. \tag{12}$$

The eigenvectors are obviously orthogonal only if

$$R_1 = R_2 \ \text{ and } \ \varphi_1 + \varphi_2 = (2n + 1)\pi, \quad \text{with} \quad n \in \mathbb{Z}.$$

This is only the case for linear, circular, and a special class of elliptical polarization. In all other cases the eigenstates are nonorthogonal [35–37]. Note that systems with orthogonal eigenstates are sometimes termed homogeneous systems, whereas systems with nonorthogonal states are termed inhomogeneous ones [38].

Once the eigenstates are derived, the transmission matrix can be determined within this eigenbase by applying the transformation (3). The corresponding T matrix is then diagonal. Nevertheless, using the T matrix in the eigenbase is only appropriate and convenient if the eigenstates are orthogonal and frequency independent.

The five different classes of periodic metamaterials that can be distinguished are closely related to their eigenstates. These five possible sets of eigenstates are linear, circular, and elliptic ones, and the elliptic ones can be further separated into corotating, counterrotating, and general elliptic states with no fixed relation between ϕ_1 and ϕ_2. Later we will show how the symmetry class determines the respective eigenstate.

III. SYMMETRY CONSIDERATIONS

By the symmetry considerations in the next sections, we will show how the symmetry properties of the structure affect the symmetry of the T matrix. The arising T matrices can be reduced to five principal forms where in general a larger number of distinct matrices is possible by rotating the structure by an arbitrary angle with respect to the z axis. On the other hand, such rotations can be used to remove redundant information. Rotation by an angle φ is accomplished by applying the following matrix operation:

$$D_\varphi = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \rightarrow \hat{T}_{\mathrm{new}} = D_\varphi^{-1}\hat{T}D_\varphi, \tag{13}$$

resulting in the new T matrix $\hat{T}_{\mathrm{new}}$ of the rotated sample. Note that the eigenvalues of the rotated system are invariant to this operation and are uniquely related to the principal symmetry. The actual form of the matrices and the derived, redundant matrices will be given later in Sec. IV to keep this part consistent.

In general, all complex components of the T matrix are different if the metamaterial does not exhibit any reflection or rotational symmetry. If such type of symmetry exists, the components of the T matrix must reflect that. We will therefore briefly discuss various symmetries and their corresponding impacts on the T matrices.

If the metamaterial is mirror-symmetric with respect to the xz plane, the T matrix for the structure reflected at that plane is identical to the original one. Therefore we have

$$M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \quad M_x^{-1}\hat{T}^{\mathrm{f}}M_x = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} = \hat{T}^{\mathrm{f}}$$
$$\rightarrow \hat{T}^{\mathrm{f}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{14}$$

with M_x being the reflection matrix with respect to the x axis. So any structure that obeys that symmetry may be obviously described by a diagonal T matrix.

If the metamaterial is mirror-symmetric with respect to the yz plane, we have

$$M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}: \quad M_y^{-1}\hat{T}^{\mathrm{f}}M_y = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} = \hat{T}^{\mathrm{f}}$$
$$\rightarrow \hat{T}^{\mathrm{f}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \tag{15}$$

Hence, if there exists any mirror plane parallel to the z axis, the T matrix is diagonal provided that the mirror plane coincides with the x or y axes, respectively. In such a system the eigenstates of the polarization are obviously linear states.

If the structure is C_2-symmetric with respect to the z axis, we have

$$D_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}: \quad D_\pi^{-1}\hat{T}^{\mathrm{f}}D_\pi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \hat{T}^{\mathrm{f}}. \tag{16}$$

Hence rotating any structure by 180° with respect to the z axis does not change the response at all. Even if the structure does not have any further symmetry it fulfills that relation.

If the structure is C_3-symmetric with respect to the z axis, we have

$$\rightarrow \hat{T}^{\mathrm{f}} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \tag{17}$$

However, that symmetry is almost never met without additional metamaterial mirror symmetries, but it is given here for completeness.

If the structure is C_4-symmetric with respect to the z axis, we have

$$D_{\frac{\pi}{2}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: \quad D_{\frac{\pi}{2}}^{-1}\hat{T}^{\mathrm{f}}D_{\frac{\pi}{2}} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \hat{T}^{\mathrm{f}}$$
$$\rightarrow \hat{T}^{\mathrm{f}} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \tag{18}$$

Hence the structure is insensitive to linearly polarized light of any state. If there is an additional mirror symmetry with respect to a plane parallel or perpendicular to the z axis, the off-diagonal elements will vanish, resulting in a completely polarization-independent structure. Otherwise the eigenstates will be circularly polarized, as will be shown later in detail.

Further important conclusions can be drawn by investigating the possible mirror symmetries with respect to a plane perpendicular to the z axis. If the structure possesses this type of symmetry, the reflected structure is the same as seen from the back side:

$$M_x^{-1}\hat{T}^{\mathrm{f}}M_x = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} = \begin{pmatrix} A & -C \\ -B & D \end{pmatrix} = \hat{T}^{\mathrm{b}}$$
$$\rightarrow \hat{T}^{\mathrm{f}} = \begin{pmatrix} A & B \\ B & D \end{pmatrix}, \tag{19}$$

i.e., the off-diagonal elements are identical. If the system possesses a center of inversion, the matrix has also this form, because inversion is equivalent to applying a reflection and a subsequent rotation by π, where the latter does not change the response, as shown in Eq. (16).

By comparison with Eq. (18) it is obvious that the T matrix will have the form $\hat{T}^{\mathrm{f}} = \mathrm{diag}\{A,A\}$ if the structure is additionally C_4-symmetric with respect to the z axis.

That important relation [Eq. (19)] is valid for all truly two-dimensional (planar) structures and all structures that possess any mirror plane perpendicular to a coordinate axis, i.e., achiral structures. In general, any substrate will break this symmetry [16,17,39], but usually the substrate effect is negligible compared to the effect of anisotropy [23].

Most important for our investigations are structures that cannot be mapped onto their mirror image by proper rotations. Those structures are called chiral. In general, the components of the T matrix for those structures are all different. In the context of the basic geometry analyzed here, there exist only two exceptions. The first one has already been discussed within the context of Eq. (18). The second one is a C_2 symmetry with respect to the x or y axis. For this type of symmetry, the structure is identical from both sides, hence

$$\hat{T}^{\mathrm{f}} = \hat{T}^{\mathrm{b}} = \begin{pmatrix} A & B \\ -B & D \end{pmatrix}. \tag{20}$$

IV. EXAMPLES AND CLASSIFICATION

To understand the usefulness of the approach presented, we will discuss the different symmetry classes for simple examples. The metaatoms exemplarily shown in the following are assumed to be periodically arranged in x and y directions. Importantly, the symmetry constraints applied to the unit cell have to be consistent with the symmetry of the lattice. That is crucial since, e.g., even an achiral metaatom can result in a chiral structure by proper arrangement on a periodic lattice [40].

A. Simple anisotropic media

The most significant symmetry is that of reflection symmetry with respect to the xz or yz plane or both. As already explained within the context of Eqs. (14) and (15), the T matrix is then diagonal. The eigenvalues are simply $\kappa_1 = A$ and $\kappa_2 = D$. The eigenstates are linear states parallel and orthogonal to the mirror plane, respectively. Only a dichroitic behavior will be obtained, and no polarization rotation occurs for light being parallel or orthogonal to the mirror planes. If the coordinate system is not aligned parallel to the mirror plane, the T matrix for that system will have off-diagonal elements,

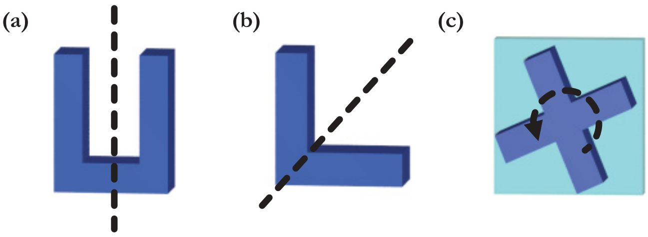

FIG. 2. (Color online) Examples for simple anisotropic (a) and (b) and simple chiral (c) metaatoms. The structures are located in the xy plane with light impinging normally to the structure in the z direction. The black dashed lines indicate the mirror planes and the rotation axis, respectively. (a) Split-ring resonator with mirror plane parallel to the y axis. (b) L-shaped particle with identical arms with mirror plane 45° inclined. (c) Cross on substrate with C_4 rotational symmetry with respect to the z axis. The square-shaped substrate indicates the arrangement on a square lattice, necessary for the C_4 of the entire system. Such an arrangement gives rise to so-called structural chirality, although the particle itself is achiral.

which disappear after a proper rotation. The most general form of the T matrix for systems with linear eigenstates is

$$\hat{T}^{\mathrm{f}} = \begin{pmatrix} A & B \\ B & D \end{pmatrix},$$

but in this case the components A, B, and D are not independent but connected by trigonometric functions, as is clear by explicitly evaluating Eq. (13) for a diagonal matrix.

An example for such a metamaterial is shown in Fig. 2(a). Other examples are the fishnet [41] and its variations [42], cut wire pairs [43], and similar structures. In Fig. 2(b) we show a special example of a structure with a symmetry plane which is 45° inclined with respect to both the x and y axes. In this case the T matrix has the form

$$\hat{T}^{\mathrm{f}} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The eigenstates are linearly orthogonal polarized, hence a rotation by an angle $\varphi = 45°$ leads to a diagonal form:

$$\hat{T}^{\mathrm{f}}_{\mathrm{new}} = D^{-1}_{\frac{\pi}{4}} \hat{T}^{\mathrm{f}} D_{\frac{\pi}{4}} = \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}.$$

A similar structure obeying the same relations is that published in [6]. There, the unit cell consisting of four split-ring resonators has no rotational symmetry. But reflecting the structure at a plane diagonal to the given unit cell leads to a structure that is shifted by half a period in the x or y direction. Due to the invariance of the optical response for periodic systems to any translation, this mirror plane leads in fact to linearly polarized eigenstates. Therefore, rotating the structure by 45° results in a diagonal T matrix.

B. Simple chiral media

The second important group are those structures exhibiting C_4 symmetry but without any additional reflection symmetry. The T matrix is then given by Eq. (18). Since these matrices are invariant to an arbitrary rotation D_φ, Eq. (18) is already the most general form of the T matrix in a linear orthogonal base for such systems. The T matrix in the circular base is then diagonal:

$$T^{\mathrm{f}}_{\mathrm{circ}} = \begin{pmatrix} T_{++} & 0 \\ 0 & T_{--} \end{pmatrix} = \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}.$$

Obviously the eigenpolarizations are circular states, since the T matrix is diagonal and the eigenvalues are simply $\kappa_1 = A + iB$ and $\kappa_2 = A - iB$. The y components of the eigenvectors [Eq. (9)] are $i_{y,1,2} = \pm i$, i.e., frequency independent. At such systems all effects related to circular dichroism are observable, whereas emphasis is put on the fact that circular dichroism is in general accompanied by a difference in the phase advance for right-(rcp) and left-circularly polarized (lcp) light due to causality; i.e., the real and imaginary parts of the wave number for rcp and lcp differ in general [44].

The difference $T_{++} - T_{--} = 2iB$ is given by the off-diagonal elements in the linear polarization representation and specifies the optical rotation power. Systems obeying that symmetry are prototypical optically active materials. Examples are gammadions (also called swastikas) [similar to the cross in Fig. 2(c)] and C_4 spirals. Note that the influence of the substrate is important for planar structures [17,39] where the chirality and hence the optical rotatory power is a result of the substrate only.

C. Generalized anisotropic media

The third group consists of those systems that have a mirror symmetry perpendicular to the z axis or a center of inversion and at most a C_2 symmetry with respect to the z axis. From the latter one, we know that it has no influence on the transmission matrix [Eq. (16)]. Examples are given in Fig. 3. The only necessary symmetry is the reflection symmetry perpendicular to the z axis without any further restrictions. Hence, there is no preferable alignment in the xy plane, and the basic form of the T matrix is unaffected by any rotation with respect to the z axis.

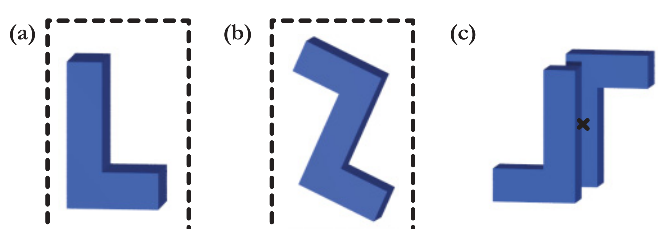

FIG. 3. (Color online) Examples for generalized anisotropic metaatoms. The metaatoms are located in the xy plane with light impinging normally to the structure in the z direction. The black dashed lines indicate the mirror planes in (a) and (b), and the black cross in (c) is a center of inversion symmetry. (a) A planar L-shaped metaatom with different arms. (b) A planar S-shaped metaatom with a C_2 symmetry with respect to the z axis. (c) A three-dimensional metaatom made of L-shaped particles with a center of inversion.

For those systems the T matrices in the linear and circular representation are given by

$$T^{\mathrm{f}} = \begin{pmatrix} A & B \\ B & D \end{pmatrix},$$
$$T^{\mathrm{f}}_{\mathrm{circ}} = \frac{1}{2}\begin{pmatrix} A+D & A-D+2iB \\ A-D-2iB & A+D \end{pmatrix},$$

hence the eigenstates are neither linearly nor circularly polarized states.

Since we have $T_{++} = T_{--}$, there is no polarization rotation due to chirality. In fact, it can be shown that the averaged polarization rotation accounting for chirality vanishes in such systems [45]. The off-diagonal elements in the circular basis are different, hence the polarization conversion from left- to right-hand polarized light and vice versa is different.

The difference in conversion is again given by the off-diagonal elements in the linear basis $T_{+-} - T_{-+} = 2iB$. This difference is also the source of the asymmetric transmission for circularly polarized light. Assuming (+)-polarized incident light, the total transmission τ in the forward direction is $\tau^{\mathrm{f}} = |T_{++}|^2 + |T_{-+}|^2$, whereas for the backward direction we have $\tau^{\mathrm{b}} = |T_{++}|^2 + |T_{+-}|^2$ due to Eq. (19). Therefore, the difference in the total transmission is determined by B. For (−)-polarized incident light, the results are identical. Note that there is no asymmetric transmission for linearly polarized light, because $\hat{T}^{\mathrm{f}}$ is symmetric.

It is important to note that the moduli of the off-diagonal elements are in general in the order of those of the diagonal elements (10^{-1}). Hence the asymmetric transmission can become quite large. As already indicated, any substrate will break the mirror symmetry in the z direction, resulting in $B \neq C$, $|B - C| \ll |B|$, and $B \approx C$. As this difference due to the small effect of the substrate is very weak (typically 10^{-3}), it is often neglected and hardly measurable compared to the asymmetric transmission effect.

The eigenstates for such an achiral system are elliptical, corotating states, as discussed e.g., in [46]. The effects of light propagating through such structures can be understood in terms of the concept of elliptical dichroism [47]. By using Eqs. (11) and (12) and $B = C$, they can be expressed in normalized form as

$$\mathbf{i}_1 = \frac{1}{\sqrt{1+R^2}}\begin{pmatrix} 1 \\ Re^{i\varphi} \end{pmatrix}, \quad \mathbf{i}_2 = \frac{R}{\sqrt{1+R^2}}\begin{pmatrix} 1 \\ -\frac{1}{R}e^{-i\varphi} \end{pmatrix}.$$

They are only orthogonal for $\varphi = n\pi$ with $n \in \mathbb{N}$ leading to linear eigenstates.

Note that planar structures with that symmetry can be described by an effective permittivity tensor independent of the wave vector, i.e., without magnetoelectric coupling [48]. That is why we call this group generalized anisotropic structures.

The most general form is again obtained by applying a rotation by an arbitrary angle φ, leading to

$$\hat{T}^{\mathrm{f}}_{\mathrm{new}} = D_{\varphi}^{-1}\hat{T}^{\mathrm{f}}D_{\varphi} = \begin{pmatrix} A' & B' \\ B' & D' \end{pmatrix},$$

hence the general form is invariant since no preferred alignment exists.

D. Generalized chiral media

The fourth group consists of chiral structures that have an additional C_2 symmetry with respect to the x or y axis. The T matrix obeys the form

$$T^{\mathrm{f}} = \begin{pmatrix} A & B \\ -B & D \end{pmatrix},$$
$$T^{\mathrm{f}}_{\mathrm{circ}} = \frac{1}{2}\begin{pmatrix} A+D+2iB & A-D \\ A-D & A+D-2iB \end{pmatrix}, \qquad (21)$$

hence there is no difference in the polarization conversion and hence no asymmetric transmission for neither linearly nor circularly polarized light. Furthermore there is obviously no asymmetric transmission in any base, since the structure is identical from both sides when the axis of rotation coincides with the x or y axis.

But there is a difference in the quantity $T_{++} - T_{--} = 2iB$ determining the optical rotation power typical for chiral structures. In contrast to the second group, there is an additional anisotropy ($A \neq D$), hence the eigenstates are not circular but elliptically counterrotating. Again, by using Eqs. (11) and (12) and $C = -B$, they can be expressed in normalized form as

$$\mathbf{i}_1 = \frac{1}{\sqrt{1+R^2}}\begin{pmatrix} 1 \\ Re^{i\varphi} \end{pmatrix}, \ \mathbf{i}_2 = \frac{R}{\sqrt{1+R^2}}\begin{pmatrix} 1 \\ \frac{1}{R}e^{-i\varphi} \end{pmatrix}.$$

They are only orthogonal if $\varphi = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{N}$ leading to circular counterpropagating eigenstates typical for chiral structures. That is why we term this group generalized chiral structures.

Typical examples are shown in Figs. 4(a) and 4(b). Another important example are three-dimensional spirals [49–53] with $\frac{N}{2}$ whorls aligned along the z axis. Spirals with integer whorls are clearly identical for both propagation directions, whereas spirals with half-integer whorls are identical after rotation by π around the z axis, keeping the response unaffected.

Note that for an arbitrary rotation D_{φ}, all matrix elements are different, hence the symmetry axis must be aligned with a principal coordinate axis to achieve the form of Eq. (21). In

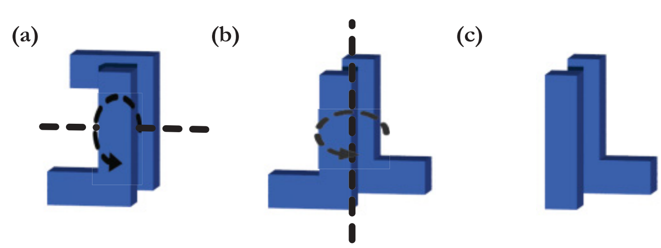

FIG. 4. (Color online) Examples for generalized chiral metaatoms (a) and (b) and a no-symmetry metaatom (c). The structures are located in the xy plane with light impinging normally to the structure in the z direction. The black dashed lines indicate the axes of rotational symmetry in (a) and (b), which show three-dimensional structures made of two L-shaped particles with C_2 symmetry with respect to the x or y axis, respectively. They are identical for forward and backward propagation. (c) A three-dimensional structure made of an L-shaped particle and an I-shaped particle with no symmetry at all.

TABLE I. (Color online) Overview of possible symmetries, typical metaatoms, the corresponding T matrices, and their eigenstates of the polarization. For every symmetry group, only a single example is shown. Other possible symmetries resulting in the same type of T matrices are given in brackets. Here M_{ij} designates mirror symmetry with respect to the ij plane, and C_{n},i means n-fold rotational symmetry with respect to the i axis.

Symmetry	Examples	T matrix	Eigenstates
M_{xz} (M_{yz})		$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$	Linear
$C_{4,z}$ ($C_{3,z}$)		$T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$	Circular
M_{xy} ($C_{2,z}$, inversion symmetry)		$T = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$	Elliptic, corotating
$C_{2,y}$ ($C_{2,x}$)		$T = \begin{pmatrix} A & B \\ -B & D \end{pmatrix}$	Elliptic, counterrotating
No symmetry ($C_{2,z}$)		$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$	Elliptic

particular if the system is rotated by 45° the T matrix has the form

$$T^{\mathrm{f}} = \begin{pmatrix} A' & B' \\ C' & A' \end{pmatrix}.$$

Nevertheless, if the eigenvectors of the arbitrarily oriented system are elliptically counterrotating, the convenient form of Eq. (21) can be achieved by a proper alignment of the system.

E. Arbitrary complex media

The fifth and last group are chiral structures without any symmetry. A simple example is shown in Fig. 4(c). Here all elements of the T matrices in the linear as well as in the circular base are different:

$$T^{\mathrm{f}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{22}$$

It is impossible to achieve $|B| = |C|$ by a proper rotation. Therefore, independent of the base, asymmetric transmission occurs always and in particular also for linearly polarized light. All effects of generalized anisotropy as well as generalized chirality can be observed. The normalized eigenvectors can be expressed as

$$\mathbf{i}_1 = \frac{1}{\sqrt{1+R_1^2}}\begin{pmatrix} 1 \\ R_1 e^{i\varphi_1} \end{pmatrix}, \quad \mathbf{i}_2 = \frac{R_2}{\sqrt{1+R_2^2}}\begin{pmatrix} 1 \\ \frac{1}{R_2} e^{-i\varphi_2} \end{pmatrix},$$

whereas $R_1(\omega) \neq R_2(\omega)$ and $\varphi_1(\omega) \neq \varphi_2(\omega)$. The eigenstates are strongly dependent on the actual value of the components of $\hat{T}$ and are simply elliptical, whereas no principal rotation direction is assignable. Linear as well as elliptical counter- and corotating states and combinations of them with no fixed phase relation can be found in general.

An example of such a structure is investigated in detail both numerically and experimentally in [24].

V. SUMMARY

A summarizing overview of possible structures and the corresponding basic forms of the T matrices are shown in Table I. Once the general form of the T matrix is known, all effects regarding the observable polarization phenomena can be fully deduced. Based on our investigations it is easy to provide an algorithm to determine the general form of the T matrix for an unknown sample by measuring transmitted intensities with the help of linear polarizers only.

A possible approach can be as follows:

(1) Use linearly polarized light and measure the orthogonally polarized output while rotating the sample. If the output vanishes for every rotation angle, the medium is polarization independent, i.e., a simple isotropic medium. If the output vanishes for some rotation angles and this angle is independent of the wavelength, the structure is simple anisotropic. If no such rotation angle can be found there is obviously no mirror plane parallel to the z axis.

(2) If the transmitted intensity is independent of the rotation of the sample for both co- and cross-polarized light, the eigenstates are circularly polarized and the structure is simple chiral.

(3) If both aforementioned procedures do not provide a positive results, the structure is more complex and the measurements become more difficult, too. To distinguish between the remaining possible forms, it is necessary to measure the off-diagonal entries of the T matrix simultaneously. If these off-diagonal elements are identical for a fixed wavelength and a fixed rotation angle independent of their particular choice, the structure is generalized anisotropic. If the off-diagonal elements are identical only for a fixed rotation angle but for every wavelength, the structure is generalized chiral. In all other cases, we have $A \neq B \neq C \neq D$.

By using circularly polarized light, a similar scheme can be obtained; however, it would require circular analyzers as well.

VI. CONCLUSION

Taking advantage of symmetry considerations we have analyzed the potential of various MMs to affect the polarization state of light upon transmission. By focusing the attention on any optical response that is directly accessible in an experiment, the properties of MMs may become so involved that the establishment of valid constitutive relation may be beyond what is possible for structures with an ever increasing complexity. We have explicitly shown that all MMs belong to one of five different classes; each being characterized by certain relations that connect the entries of the T matrix and each class is able to support specific polarization phenomena. The sub-wavelength nature of MMs is the only requirement for these considerations. Moreover, the symmetry operations applied to the metaatoms have to be consistent with the symmetry of the lattice and it is required that the MM is sandwiched between identical media. Nonetheless, we have explicitly listed all relevant structures where a violation of this assumption causes deviations. To foster practical application of this classification we have finally provided a protocol useful to reveal the underlying symmetry of an unknown MM and its T matrix from far-field measurements of the transmitted intensities only. Once it is identified, all the achievable optical properties that affect the state of polarization are fully disclosed.

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