Transient energy excitation in shortcuts to adiabaticity for the time-dependent harmonic oscillator

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(Received 28 September 2010; published 4 November 2010)

We study for the time-dependent harmonic oscillator the transient energy excitation in speed-up processes ("shortcuts to adiabaticity") designed to reproduce the initial populations at some predetermined final frequency and time. We provide lower bounds and examples. Implications for the limits imposed to the process times and for the principle of unattainability of the absolute zero, in a single expansion or in quantum refrigerator cycles, are drawn.

DOI: 10.1103/PhysRevA.82.053403 PACS number(s): 37.10.De, 03.65.−w, 42.50.−p

I. INTRODUCTION

Adiabatic processes in quantum systems are frequently useful to drive or prepare states in a robust and controllable manner, and have also been proposed to solve complicated computational problems, but they are, by definition, slow. (The definition of "adiabatic process" here is the usual one in quantum mechanics, namely, a slow change of Hamiltonian parameters keeping the populations of the instantaneous eigenstates constant all along.) Thus a natural objective is to cut down the time to arrive at the same final state, possibly up to phase factors, in other words, to find "shortcuts to adiabaticity," by designing optimal adiabatic pathways, or by admitting that the populations may not be preserved at intermediate times. Several works have recently proposed different ways to achieve this goal for general or specific cases [1–10]. One of the early applications considered has been particle transport without vibrational heating [1,2,5,9,10]. Another important case is frictionless harmonic trap compressions or expansions for state preparation [4–8,11], which were first addressed with "bang-bang" (piecewise constant frequency) methods [11]. A different route is to design by inverse engineering techniques a time-dependent frequency for which the expanding modes associated with Lewis-Riesenfeld invariants [12] take the state from the initial to the final potential configuration without transitions [6,7]. This has been implemented experimentally to decompress ^{87}Rb cold atoms in a harmonic magnetic trap [13]. The extension to Bose-Einstein condensates may be carried out with a variational ansatz [4] and realized experimentally as well [14]. Invariant-based inverse engineering has been also proposed to cool mechanical resonators [15].

In the same vein, Berry has provided an algorithm to construct a Hamiltonian $\widetilde{H}(t)$ for which the adiabatic approximation for the state evolution under a time-dependent reference Hamiltonian $H(t)$ becomes the exact dynamics with $\widetilde{H}(t)$. This algorithm has been applied to spins in magnetic fields [3], harmonic oscillators [6], or to speed up adiabatic state-preparation methods such as rapid adiabatic passage (RAP), stimulated rapid adiabatic passage (STIRAP), and its variants [8]. Also, Masuda and Nakamura have adapted for adiabatic processes [5] a ("fast-forward") scaling technique to speed up the state dynamics [16], with application examples to particle transport or time-dependent harmonic potentials [5] and spins or charged particles in electromagnetic fields [10]. Finally, optimal control theory has been used for nonadiabatic cooling under imposed costs [11,17]; and Lyapunov control methods have been proposed to speed up quantum adiabatic computing without information of the Hamiltonian eigenstates [18].

In this paper we shall examine the energy "cost" of such processes; more precisely, their transient excitation energies. Our central study case is the expansion (or compression) of a harmonic oscillator, a basic model for many operations in any cold atoms laboratory [7]. Intuitively, one expects the transient system energy and the time of the process to be "conjugate," that is, an increase of the former when decreasing the later, but the details of this relation, and the role played by other parameters defining the process (such as initial and final frequencies) have to be clarified both for fundamental reasons and for the applications. In particular, the energy excitation will set limits to the possible speed-up. In a trap which is harmonic near the ground state, but not for higher energies, large transient energies will imply perturbing effects of anharmonicities and thus undesired excitations of the final state, or even atom loss. The transient excitation energy also has implications for quantifying the principle of unattainability of zero temperature, first enunciated by Nernst [19]. Fowler and Guggenheim [20] formulated it as follows: "It is impossible by any procedure no matter how idealized to reduce the temperature of any system to the absolute zero in a finite number of operations." They identify it with the third law of thermodynamics although this is sometimes disputed. More recently, Kosloff and co-workers [11,21,22] have restated the unattainability principle as the vanishing of the cooling rate in quantum refrigerators when the temperature of the cold bath approaches zero, and quantify it by the scaling law relating the cooling rate and cold bath temperature. We shall examine the consequences of the transient energy excitation on the unattainability principle at two levels, namely, for a single, isolated expansion, and considering the expansion as one of the branches of a quantum refrigerator cycle.

When describing these cycles and indeed in many intersection areas between quantum mechanics and thermodynamics one finds the need to use the word "adiabatic" in two different ways: the thermodynamical one (meaning that there is no heat transfer between the system and environment) and the quantum one. Many authors have pointed out this duality

as an unfortunate source of confusion. It may prove useful to distinguish them and avoid ambiguities and the hassle of detailed explanations with a shorthand notation. Following the example of Dirac's q number versus c number distinction, we propose to refer to a process as "t adiabatic" if it is thermodynamically adiabatic and as "q adiabatic" if it is a quantum-mechanically adiabatic (i.e., slow) process.

II. BANG-BANG METHODS

The Hamiltonian for a particle with mass m in a time-dependent harmonic oscillator is given by

$$H = \hat{p}^2/2m + m\omega^2(t)\hat{q}^2/2. \tag{1}$$

Let us assume an expansion (compressions are treated similarly) with initial angular frequency $\omega_0 \equiv \omega(0)$ at time $t = 0$ and final frequency $\omega_f \equiv \omega(t_f) < \omega(0)$ at time t_f.

In the "bang-bang" approach the frequency is shaped as a stepwise constant function of time, choosing the step values and durations so as to preserve the initial-state populations in the final configuration. For *real* trap intermediate frequencies this requires a minimal total expansion time [11]

$$t_f > \frac{\sqrt{1-\omega_f/\omega_0}}{\sqrt{\omega_f\omega_0}}. \tag{2}$$

The limit can be realized by only three jumps, that is, two real intermediate frequencies (it cannot be improved by using more intermediate frequencies), but one of the intermediate frequencies should be infinite.

Up to a constant factor the main dependence in the bound (2) already appears in a simpler process that reproduces for (ω_f,t_f) the initial populations with just one intermediate frequency, the geometric average $\omega_1 = (\omega_0\omega_f)^{1/2}$, and a total time

$$t_f = \frac{\pi}{2\sqrt{\omega_0\omega_f}}, \tag{3}$$

which is a quarter of the corresponding period [23]. For an initial nth state of the oscillator the instantaneous mean energy during the transient period becomes the arithmetic mean of the initial and final energies

$$\langle H\rangle = \hbar\left(n+\frac{1}{2}\right)\frac{\omega_0+\omega_f}{2}. \tag{4}$$

Equation (3) and the bound (2) are relevant because the t-adiabatic expansion is actually the speed bottleneck in quantum Otto refrigerator cycles which use particles in a harmonic oscillator as the "working medium" [11,22]. The cooling rate R as the cold bath temperature T_c approaches zero is dominated by the expansion time and scales as $R \propto \omega_f/t_f$. Since $\omega_f \propto T_c$ as $T_c \to 0$, the dependence of t_f on ω_f quantifies the unattainability principle. In particular, q-adiabatic expansions lead to $R \propto T_c^3$ scalings, in contrast with the $R \propto T_c^{3/2}$ scaling achieved with the times in Eqs. (2) or (3) [11,22].

In [7] it was demonstrated, however, that the minimal time in Eq. (2) can be beaten with bang-bang methods and inverse engineering methods (see also the next section) by allowing for imaginary intermediate frequencies, i.e., transients in which the harmonic oscillator becomes a parabolic repeller. It was pointed out [7] that this new freedom leads to the absence, at least in principle, of a lower bound for the expansion time, which could obviously affect the optimal scaling of cooling rates. We shall analyze the impact of these ultrafast expansions on the third law in the following sections.

III. ENERGY BOUNDS FOR INVERSE-ENGINEERED TIME-DEPENDENT HARMONIC OSCILLATORS

A. Bound for time-averaged energy

In this section, we will set a lower bound for the time-averaged energy in the transitionless expansions and compressions of the time-dependent harmonic oscillator. A shortcut to adiabaticity taking the nth state of the initial trap to the final nth state of the final trap up to phase factors is achieved [7] by designing the frequency from the Ermakov equation

$$\ddot{b} + \omega^2(t)b = \frac{\omega_0^2}{b^3}, \tag{5}$$

where b is an engineered scaling function which satisfies the following boundary conditions at $t = 0$ and t_f

$$\begin{aligned} b(0) = 1, \quad \dot{b}(0) = 0, \quad \ddot{b}(0) = 0, \\ b(t_f) = \gamma, \quad \dot{b}(t_f) = 0, \quad \ddot{b}(t_f) = 0. \end{aligned} \tag{6}$$

Here $\gamma = \sqrt{\omega_0/\omega_f}$ and the single and double dots denote first and second derivatives with respect to time. [The simplest choice for interpolating $b(t)$ between 0 and t_f is a polynomial form, $b(t) = 6(\gamma-1)s^5 - 15(\gamma-1)s^4 + 10(\gamma-1)s^3 + 1$, where $s = t/t_f$.] In this manner the nth stationary state of the initial oscillator will evolve according to the "expanding mode"

$$\begin{aligned} \Psi_n(t,x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \frac{e^{-i(n+1/2)\int_0^t dt' \frac{\omega_0}{b(t')^2}}}{(2^n n!b)^{1/2}} \\ \times e^{i\frac{m}{2\hbar}(\frac{\dot{b}}{b(t)}+\frac{i\omega_0}{b^2})x^2} \mathcal{H}_n\left[\left(\frac{m\omega_0}{\hbar}\right)^{1/2}\frac{x}{b}\right], \end{aligned} \tag{7}$$

where $\mathcal{H}_n$ is a Hermite polynomial, and will become eventually, up to a phase, the nth eigenstate of the final trap at t_f. At intermediate times $|\Psi_n\rangle$ does not coincide, in general, with the instantaneous eigenvectors $|n\rangle$ of $H(t)$, $H(t)|n(t)\rangle = \epsilon_n(t)|n(t)\rangle$.

For the nth expanding mode, the instantaneous average energy $E_n(t) \equiv \langle\Psi_n|H(t)|\Psi_n\rangle$ is

$$E_n(t) = \frac{(2n+1)\hbar}{4\omega_0}\left[\dot{b}^2 + \omega^2(t)b^2 + \frac{\omega_0^2}{b^2}\right], \tag{8}$$

which is, in general, different, except at the initial and final times, from ϵ_n. The time average of E_n is defined by

$$\overline{E_n} \equiv \frac{1}{t_f}\int_0^{t_f} E_n(t)\,dt. \tag{9}$$

To find a lower bound for $\overline{E_n}$ we substitute Eq. (8) into Eq. (9) and integrate by parts making use of the boundary conditions (6)

$$\overline{E_n} = \frac{(2n+1)\hbar}{2\omega_0 t_f}\int_0^{t_f}\left(\dot{b}^2 + \frac{\omega_0^2}{b^2}\right)dt. \tag{10}$$

The integrand has the form of the Lagrangian of a particle in an attractive inverse square potential, but the minimization problem [i.e., finding an optimal function $b(t)$ subjected to the boundary conditions (6)] cannot be solved with a Euler-Lagrange equation (see, e.g., [24]) since there are too many boundary conditions which affect not only b, but also $\dot{b}$ and $\ddot{b}$ at the edges of the time interval. We can nevertheless find easily, using the Euler-Lagrange equation, the quasi-optimal "trajectory" $b(t)$ that minimizes the integral subjected only to the boundary values of b, that is, $b(0) = 1$ and $b(t_f) = \gamma$. Since these two conditions define a broader set of functions than the ones satisfying (6), the quasi-optimal b provides at least a lower bound for the time-averaged energy. For the function

$$f(t,b,\dot{b}) = \dot{b}^2 + \omega_0^2/b^2, \tag{11}$$

the Euler-Lagrange differential equation $f_b - \frac{d}{dt} f_{\dot{b}} = 0$ is

$$b^3\ddot{b} = -\omega_0^2. \tag{12}$$

The solution satisfying $b(0) = 1$ and $b(t_f) = \gamma$ is

$$b(t) = \sqrt{\left(B^2 - \omega_0^2 t_f^2\right)s^2 + 2Bs + 1}, \tag{13}$$

where $B = -1 + \sqrt{\gamma^2 + \omega_0^2 t_f^2}$ and the positive root should be taken. Substituting Eq. (13) into the integral (10), we finally obtain a lower bound for the time-averaged energy

$$\mathcal{B}_n = \frac{(2n+1)\hbar}{2\omega_0 t_f^2}\left\{\left(B^2 - \omega_0^2 t_f^2\right) - 2\omega_0 t_f \times\left[\operatorname{arctanh}\left(\frac{B^2 + B - \omega_0^2 t_f^2}{\omega_0 t_f}\right) - \operatorname{arctanh}\left(\frac{B}{\omega_0 t_f}\right)\right]\right\}, \tag{14}$$

such that $\overline{E_n} \geqslant \mathcal{B}_n$. When the final frequency ω_f is small enough to satisfy $t_f \ll 1/\sqrt{\omega_0\omega_f}$ and $\gamma \gg 1$, the lower bound has the following simple asymptotic form

$$\mathcal{B}_n \approx \frac{(2n+1)\hbar}{2\omega_f t_f^2}. \tag{15}$$

Incidentally, $\mathcal{B}_n$ sets also a lower bound for the maximum of the instantaneous energy $E_n(t)$.

Figure 1 is a contour plot of $\mathcal{B}_0$ as a function of t_f and ω_f for $\omega_0 = 2\pi \times 250$ Hz, which will also be the initial frequency in the following examples. The important point is that the transient energy increases not only with decreasing final time t_f, but also with decreasing final frequency ω_f. Figure 2 shows the exponents of the scaling for the bound (14), its asymptotic form (15), and the time-averaged energy for a polynomial b. In all cases $\overline{E_0}$, or $\mathcal{B}_0 \propto 1/(\omega_f t_f^2)$ asymptotically, which confirms the relevance (tightness) of the bound.

A consequence of Eq. (15) is

$$t_f \geqslant \sqrt{\frac{(2n+1)\hbar}{2\omega_f \overline{E_n}}}. \tag{16}$$

The interest of Eq. (16) compared to Eq. (2) is that, in principle, for fixed ω_0, it is possible to beat the bang-bang minimal time, but the price is an increase in the transient energy. In practice, this energy cannot be arbitrarily large, if only because there are no perfect harmonic oscillators. In particular, if we consider in Eq. (16) that $\overline{E_n}$ is limited by some maximal value because of anharmonicities or a finite trap depth, the obtained scaling is fundamentally the same as

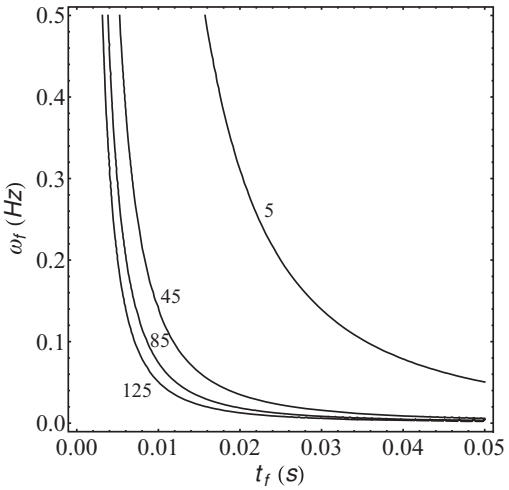

FIG. 1. Contour plot of the lower bound for time-averaged energy [in units of $E_0(0) = \hbar\omega_0/2$] as a function of t_f and ω_f. $\omega_0 = 2\pi \times 250$ Hz.

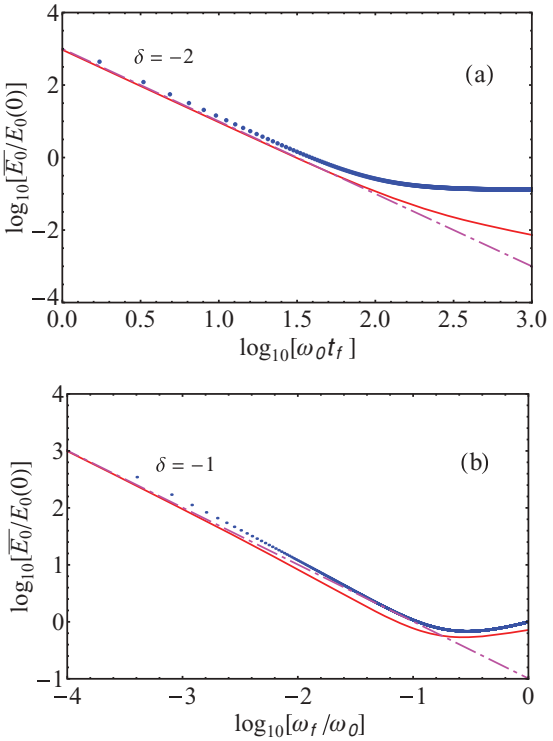

FIG. 2. (Color online) Dependence of the time-averaged energy for the ground state $\overline{E_0}/E_0(0)$ on the (a) short time t_f ($\omega_f = 2\pi \times 0.25$ Hz) and (b) final frequency ω_f ($t_f = 2$ ms). In both cases $\omega_0 = 2\pi \times 250$ Hz. Bound given by Eq. (14) (solid red line); asymptotic expression Eq. (15) (dot-dashed magenta line), and time-averaged energy for polynomial b (dotted blue line). The δ's are the asymptotic exponents of t_f and ω_f, respectively, as they go to zero.

for bang-bang methods, and leads to a cooling rate $R \propto T_c^{3/2}$ in an inverse quantum Otto cycle, although an opportunity is offered to improve the proportionality factor by increasing the allowed $\overline{E_n}$.

Independently of the participation of the harmonic trap expansion as a branch in a refrigerator cycle, we may see rather directly the effect of the previous analysis on a single expansion by assuming that the initial and final states are described by canonical density operators characterized by temperatures T_0 and T_f, related by $T_f = (\omega_f/\omega_0)T_0$ for a population-preserving process. Within the idealized but specific context of a pure parabolic potential expansion, the unattainability of a zero temperature can be reformulated microscopically as follows: The transient excitation energy becomes infinite for any population-preserving and finite-time process if the final temperature is zero (which requires $\omega_f = 0$). This excitation energy has to be provided by an external device, so the absence of a lower process time limit should not lead us astray since there remains a fundamental obstruction to reach $T_f = 0$ in a finite time, in the form of the need for a source of infinite power.

B. Minimization of time-averaged energy

To minimize the time-averaged energy and approach the lower bound, we can use the quasi-optimal $b(t)$, Eq. (13), in a central time segment $[\tau, 1-\tau]$ and match it at the extremes with two "cap polynomials," each of them satisfying three of the boundary conditions at Eq. (6) (at $t = 0$ or t_f), plus three boundary conditions for b, $\dot{b}$, and $\ddot{b}$ at the matching point. The idea is illustrated in Fig. 3.

The resulting hybrid b takes the form

$$b = \begin{cases} \sum_{j=0}^{5} c_j s^j & (0 \leqslant s \leqslant \tau) \\ \sqrt{(B^2 - \omega_0^2 t_f^2)s^2 + 2Bs + 1} & (\tau \leqslant s \leqslant 1-\tau), \\ \sum_{j=0}^{5} d_j s^j & (1-\tau \leqslant s \leqslant 1) \end{cases} \tag{17}$$

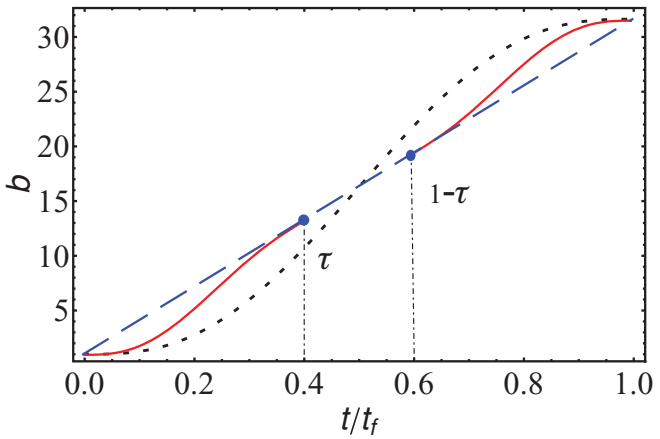

FIG. 3. (Color online) Example of hybrid b combining the quasi-optimal trajectory in the central segment $[\tau, 1-\tau]$ and "cap polynomials" with the right boundary conditions, where $\omega_0 = 2\pi \times 250$ Hz, $\omega_f = 2\pi \times 0.25$ Hz, $t_f = 2$ ms, and $\tau = 0.4$ (solid red line); quasi-optimal square-root b of Eq. (13) (dotted black line); and polynomial trajectory (blue dashed line).

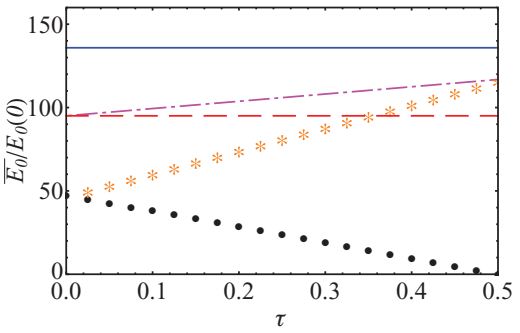

FIG. 4. (Color online) Time-averaged energy versus τ for polynomial b (solid blue line), bound in Eq. (14) (red dashed line), and hybrid b (dot-dashed magenta line). The contribution from the "cap polynomials" (orange line with "∗") and the central segment (black line with "•") are also depicted. Parameters are the same as in Fig. 3.

where the coefficients $\{c_j\}$ and $\{d_j\}$ have lengthy expressions, but are easily obtained from the matching conditions so we omit their explicit forms here.

Figure 4 demonstrates that this hybrid b can indeed minimize the time-averaged energy by making τ smaller and smaller, approaching the lower-energy bound as $\tau \to 0$. A detailed calculation shows that the contribution from the caps does not vanish as $\tau \to 0$ so the value of the time-averaged bound is reached at the price of a singular instantaneous energy, see Fig. 4.

IV. ENERGY VARIANCE AND ANANDAN-AHARONOV RELATION

We shall now discuss the impact of the shortcuts to adiabaticity on the standard deviation of the energy $\Delta H \equiv (\langle H^2\rangle - \langle H\rangle^2)^{1/2}$. This is important because a small averaged energy could, in principle, be spoiled by a large standard deviation. Anandan and Aharonov [25] found a relation between the time average of the standard deviation of the energy and the time of a process connecting two given states, irrespective of the Hamiltonian used to connect them. The so called "Anandan-Aharonov" (AA) relation provides a lower bound for the average uncertainty of the energy, which is extensively used to minimize the time t_f required for the evolution between the two orthogonal quantum states. Based on the Fubini-Study metric, the following distance may be defined

$$S = 2\int_0^{t_f} \frac{\Delta H(t)}{\hbar} dt \geqslant S_0, \tag{18}$$

where the minimal value $S_0 = 2\arccos(|\langle\Psi(t=0)|\Psi(t=t_f)\rangle|)$ corresponds to the "geodesic" [26]. For orthogonal states $S_0 = \pi$ and

$$\overline{\Delta H}\, t_f \geqslant \frac{h}{4}, \tag{19}$$

where

$$\overline{\Delta H} = \frac{\int_0^{t_f} \Delta H(t)\, dt}{t_f}, \tag{20}$$

but more generally, for arbitrary (possibly nonorthogonal) initial and final states, the AA relation is

$$\overline{\Delta H}\, t_f \geqslant \frac{hS_0}{4\pi}. \tag{21}$$

This may be applied to the harmonic oscillator and, in particular, to any process taking an nth initial eigenstate to an nth final eigenstate, up to phase factors. Defining $\Delta H_n(t) \equiv [\langle\Psi_n|H^2(t)|\Psi_n\rangle - E_n^2]^{1/2}$, we have, for the ground state, see Eq. (7),

$$\cos^2\left(\frac{S_0}{2}\right) \equiv |\langle\Psi_0(0)|\Psi_0(t_f)\rangle|^2 = \frac{2\sqrt{\omega_0\omega_f}}{\omega_0+\omega_f}, \tag{22}$$

and the relation

$$\overline{\Delta H_0}\, t_f \geqslant \hbar \arccos\left[\frac{\sqrt{2}(\omega_0\omega_f)^{1/4}}{(\omega_0+\omega_f)^{1/2}}\right]. \tag{23}$$

In the $\omega_f \to 0$ limit one finds again the time-energy uncertainty relation for two orthogonal states, that is, $\overline{\Delta H_0} \geqslant h/(4t_f)$, independent of ω_f. This bound, although certainly correct, is not tight and does not capture the actual dependences found for time-averaged standard deviations, which, in fact, scale on ω_f and t_f in the same way as the corresponding time-averaged energies of the previous section. Similarly to Fig. 2, Fig. 5 makes the exponents explicit. We have used the standard deviation for the nth expanding mode, which takes the form

$$\Delta H_n(t) = \frac{\sqrt{2}(n^2+n+1)^{1/2}\hbar}{4\omega_0} \times \left[\left(\dot{b}^2+\omega^2(t)b^2+\frac{\omega_0^2}{b^2}\right)^2+\frac{4\omega_0^2\dot{b}^2}{b^2}\right]^{1/2}, \tag{24}$$

and its time average

$$\overline{\Delta H_n} \equiv \frac{1}{t_f}\int_0^{t_f} \Delta H_n(t)\, dt. \tag{25}$$

The dependence of $\overline{\Delta H_0}$ on t_f and ω_f as they approach zero independently is summarized by the scaling exponents. In the limit of $t_f \to 0$, $\overline{\Delta H_0} \propto t_f^{\delta}$. Figure 5 shows that $\delta = -2$ for the calculated standard deviation, whereas AA provides $\delta = -1$. Similarly, as $\omega_f \to 0$, $\overline{\Delta H_0} \propto \omega_f^{\delta}$. We find $\delta = -1$ in the calculated standard deviation versus $\delta = 0$ from the AA relation. The looseness of the AA relation in this case is not necessarily a generalized feature of all time-dependent Hamiltonians. We shall discuss this point elsewhere in connection with the shortcut-to-diabaticity processes for internal atomic state preparation [8].

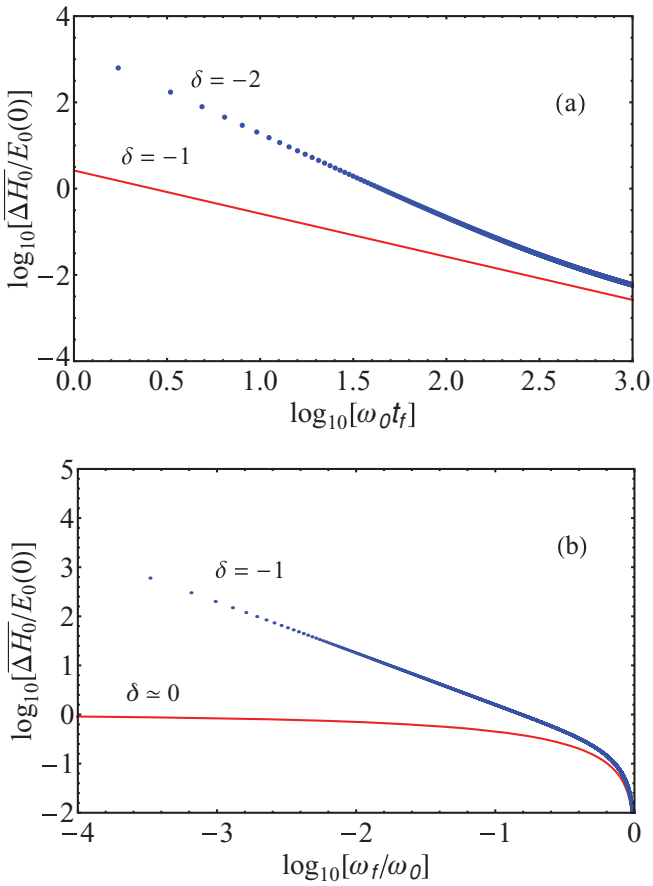

FIG. 5. (Color online) Dependence of the time-averaged uncertainty of energy $\overline{\Delta H_0}$ on the (a) short time t_f and (b) final frequency ω_f, where the parameters are the same as Fig. 2, polynomial trajectory (dotted blue line), AA relation (solid red line).

V. ADDING TERMS TO THE HAMILTONIAN

Motivated by recent experimental realizations [13,14], we have considered up to now simple processes in which the only external manipulation consists in shaping $\omega(t)$. Other possibilities exist in which the Hamiltonian is complemented with additional terms [3,6]. We shall analyze now the energy excitations for a Hamiltonian that results from the transitionless inverse engineering algorithm proposed by Berry [3] and applied to the time-dependent harmonic oscillator in [6]

$$\widetilde{H} = H + H_1, \tag{26}$$

$$H = \hbar\omega(t)(a_t^\dagger a_t + 1/2), \tag{27}$$

$$H_1 = i\hbar\frac{\dot{\omega}}{4\omega}\left(a_t^2 - a_t^{\dagger 2}\right). \tag{28}$$

We assume here that $\omega(t)$ remains positive. $\widetilde{H}$ would drive the system without transitions along the states of the instantaneous basis of the time-dependent harmonic oscillator H. In particular for the nth state,

$$|\phi_n(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t \epsilon_n(t')dt'}|n(t)\rangle, \tag{29}$$

is an exact solution of the time-dependent Schrödinger equation with $\widetilde{H}$.

The subscript t in the Schrödinger-picture creation and annihilation operators above denotes their fundamental time dependence because of the changing frequency and eigenstates, not to be confused with the time dependence of Heisenberg picture operators. H_1 is related to the squeezing operator [6], and may also be written as $H_1 = -\frac{\dot{\omega}}{4\omega}(\hat{q}\,\hat{p}+\hat{p}\hat{q})$, so that $\widetilde{H}$ is still a generalized harmonic oscillator quadratic in positions and momenta. The expectation values of $\widetilde{H}$ and $\widetilde{H}^2$

for the nth state are easily calculated

$$\langle\phi_n|\widetilde{H}|\phi_n\rangle = \epsilon_n = (n+1/2)\hbar\omega, \tag{30}$$

$$\langle\phi_n|\widetilde{H}^2|\phi_n\rangle = \frac{\hbar^2\dot{\omega}^2}{8\omega^2}(n^2+n+1)+\epsilon_n^2, \tag{31}$$

from which we deduce

$$\Delta\widetilde{H}_n \equiv \left(\langle\phi_n|\widetilde{H}^2|\phi_n\rangle - \epsilon_n^2\right)^{1/2} = \frac{\hbar}{4}\frac{|\dot{\omega}|}{\omega}[2(n^2+n+1)]^{1/2}, \tag{32}$$

and the time averages

$$\overline{\epsilon_n} = \frac{\hbar(n+1/2)}{t_f}\int_0^{t_f}\omega(t)\,dt, \tag{33}$$

$$\overline{\Delta\widetilde{H}_n} = \frac{\hbar}{4t_f}[2(n^2+n+1)]^{1/2}\int_0^{t_f}\frac{|\dot{\omega}|}{\omega}dt. \tag{34}$$

We shall first show with some specific examples that it is easy to find scalings which are, in principle, more favorable for implementing a fast transitionless process than the ones in the previous sections. For example, for a linear frequency ramp expansion $\omega = \omega_0 + (\omega_f - \omega_0)t/t_f$,

$$\overline{\epsilon_n} = \hbar(n+1/2)\frac{\omega_0+\omega_f}{2}, \tag{35}$$

which is independent of t_f, and

$$\overline{\Delta\widetilde{H}_n} = \frac{\hbar}{4t_f}[2(n^2+n+1)]^{1/2}\ln\left(\frac{\omega_0}{\omega_f}\right), \tag{36}$$

(in fact, a general result for $\dot{\omega}<0$) so, for a fixed $\overline{\Delta\widetilde{H}_n}$, t_f grows only logarithmically as $\omega_f \to 0$.

Note that some of the difficulties with high transient energies in the approaches which only control the time-dependent frequency, due to the particle exploration of regions far away from the trap center, disappear here since the system evolves at all times along the instantaneous eigenstates without transitions. Clearly $\langle\phi_n|H|\phi_n\rangle = \epsilon_n$ and $\langle\phi_n|H^2|\phi_n\rangle - \langle\phi_n|H|\phi_n\rangle^2 = 0$, so that the standard deviation (36) is entirely due to the complementary Hamiltonian H_1.

The main and so far important difficulty with this approach is that it is not clear how to implement $\widetilde{H}$ in practice [6]. H_1 involves a nonlocal interaction and the attempts to provide a quantum-optical realization have not yet succeeded.

VI. DISCUSSION AND CONCLUSION

We have studied the transient energy excitation in time-dependent quantum harmonic oscillators engineered so that the level populations at a final time are the same as the initial populations. We have considered first simple processes in which the only external manipulation consists in shaping $\omega(t)$. The populations of the instantaneous levels at intermediate times are, however, not preserved, so the transient excitation should be understood and possibly controlled. We have obtained bounds, shown examples, and determined the dominant dependences, which are different from the ones in the Anandan-Aharonov relation [25].

In a realistic application the oscillator will not be perfectly harmonic and it is natural to set some maximum value to the allowed excitation. Then the minimal time required for a fast expansion scales with the final frequency as $t_f \propto \omega_f^{-1/2}$. As the velocity determining step in quantum refrigerator Otto cycles this implies a dependence $R \propto T_c^{3/2}$ of the cooling rate, which had been previously conjectured to be a universal dependence characterizing the unattainability principle for any cooling cycle [22]. The present results provide strong support for the validity of this conjecture within the set of processes defined exclusively by time-dependent frequencies (without the added terms in the Hamiltonian), and call for further testing and study.

In Sec. V we have seen that, at least at a formal level, one could design even faster processes by adding terms to the harmonic oscillator Hamiltonian, but their physical implementation remains a challenge.

ACKNOWLEDGMENTS

We thank M. Berry, R. Kosloff, A. Ruschhaupt, D. Guéry-Odelin, and E. Torróntegui for discussions. Funding by the Basque Government (Grant No. IT472-10), the Ministerio de Ciencia e Innovación (FIS2009-12773-C02-01), the National Natural Science Foundation of China (Grant No. 60806041), the Shanghai Rising-Star Program (Grant No. 08QA14030), the Shanghai Leading Academic Discipline (Grant No. S30105), and Juan de la Cierva Programme is acknowledged.

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