Local unitary invariants for N-qubit pure states

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The concept of negativity font, a basic unit of multipartite entanglement, is introduced. Transformation properties of determinants of negativity fonts under local unitary (LU) transformations are exploited to obtain relevant N-qubit polynomial invariants and construct entanglement monotones from first principles. It is shown that entanglement monotones that detect the entanglement of specific parts of the composite system may be constructed to distinguish between states with distinct types of entanglement. The structural difference between entanglement monotones for an odd and even number of qubits is brought out.

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I. INTRODUCTION

In 1935, Schrödinger [1] coined the term "entanglement" to describe quantum correlations that make it possible to alter the properties of a distant system instantaneously by acting on a local system. A spin singlet is an example of an entangled state of two spin-half particles. A qubit is any two-level quantum system with basis states represented by $|i\rangle$, $i = 0$ and 1. The spin singlet is an entangled state of two qubits. For a bipartite quantum system consisting of two distinguishable parts A and B, each of arbitrary dimension, negativity [2] of a partially transposed state operator [3] is known to be an entanglement monotone [4]. How properties of one part of a multipartite quantum system are altered by local operations on other parts at distinct remote locations is a complex question. In this article, we present an approach to construct meaningful local unitary (LU) invariants for multiqubit systems from first principles, that is, by examining the effect of local unitaries on different parts of the composite system. Our method, illustrated for the four-qubit case in Ref. [5], introduces the basic units of entanglement, referred to as negativity fonts. A negativity font is defined as a 2×2 matrix of probability amplitudes that determines the negative eigenvalues of a specific 4×4 submatrix of a partially transposed state operator. It was shown earlier [6] that a partial transpose can be written as a sum of K-way ($2 \leqslant K \leqslant N$) partial transposes. A K-way partial transpose contains information about K-body correlations of a multipartite system. Contributions of partial transposes to global negativity, referred to as partial K-way negativities are not unitary invariants, but when calculated for canonical states for three qubits [7,8] and four qubits [9] coincide with entanglement monotones. This article complements our earlier work by outlining a direct method to obtain multiqubit invariants relevant to the construction of entanglement monotones without reaching the canonical state or calculating partial K-way negativities. Multiqubit unitary invariants are obtained by examining the transformation properties of negativity fonts present in global partial transpose [3] and K-way ($2 \leqslant K \leqslant N$) partially transposed matrices [9] constructed from the N-qubit state operator. The mathematical form of the resulting multiqubit invariants for a given state reveals the entanglement microstructure of the state.

Multiqubit invariants, written in terms of determinants of negativity fonts, are essentially relations between intrinsic negative eigenvalues of selected 4×4 submatrices of K-way partially transposed matrices. A canonical form for the pure states of a general multipartite system was found in Ref. [10] by generalizing the concept of the Schmidt decomposition to N-party systems. In the case of four qubits, the standard approach from invariant theory has lead to the construction of a complete set of Stochastic Local operations (SL) invariants [11,12] and the algorithm for constructing N-qubit invariants by computation of the Hilbert series of the LU and special unitary invariants from the knowledge of the polynomial covariants of the group of invertible local filtering operations is given in Ref. [13]. The results for five qubits have also been reported [12]. The N-qubit invariants for an even number of qubits were reported earlier in Ref. [14] and for an even and odd number of qubits in Ref. [15]. The focus is on the geometric aspects of such invariants in Refs. [16–18]. Independent of these approaches, a method based on the expectation values of antilinear operators, with an emphasis on the permutation invariance of the global entanglement measure [19,20], has been suggested. The number of polynomial invariants is known to increase very quickly with the number of qubits. However, in general, a small number of invariants is needed to qualify and quantify the entanglement. The advantage of our approach is that it is easily applied to obtain the relevant invariants for any state at hand, not necessarily the general state or canonical state. Our results bring out the structural difference between LU invariants for N-odd and N-even qubits through the nature of K-way negativity fonts present in the respective invariants. For the multipartite case, one needs in-equivalent entanglement measures [14,21,22]. To show that the method can be used to construct entanglement monotones that detect the entanglement of specific parts of the composite system, four-qubit invariants to detect the

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entanglement of a pair of qubits due to four-way correlations are obtained.

The entanglement of qubits A_1 and A_2 in pure state $\widehat{\rho}^{A_1A_2} = |\Psi^{A_1A_2}\rangle\langle\Psi^{A_1A_2}|$ where

$$|\Psi^{A_1A_2}\rangle = a_{00}|00\rangle_{A_1A_2} + a_{10}|10\rangle_{A_1A_2} + a_{01}|01\rangle_{A_1A_2} + a_{11}|11\rangle_{A_1A_2}, \quad (1)$$

is measured by the negativity of a 4×4 matrix $(\rho^{A_1A_2})_G^{T_1}$ obtained by partially transposing the state of qubit A_1 in $\widehat{\rho}^{A_1A_2}$. We refer to the 2×2 matrix $\nu^{00} = [\begin{smallmatrix} a_{00} & a_{01} \\ a_{10} & a_{11}\end{smallmatrix}]$ as a negativity font of $(\rho^{A_1A_2})_G^{T_1}$. The squared negativity of $(\rho^{A_1A_2})_G^{T_1}$ is given by $(N_G^{A_1})^2 = 4|\det \nu^{00}|^2$. If $\det \nu^{00} = 0$, the state is separable. A general N-qubit pure state reads as

$$|\Psi^{A_1,A_2,\ldots,A_N}\rangle = \sum_{i_1 i_2,\ldots,i_N} a_{i_1 i_2,\ldots,i_N}|i_1 i_2,\ldots,i_N\rangle, \quad (2)$$

where $|i_1 i_2,\ldots,i_N\rangle$ are the basis vectors spanning the 2^N-dimensional Hilbert space, and A_p is the location of qubit p. The coefficients $a_{i_1 i_2,\ldots,i_N}$ are complex numbers. The basis states of a single qubit are labeled by $i_m = 0$ and 1, where $m = 1,\ldots,N$. The global partial transpose of the N-qubit state $\widehat{\rho} = |\Psi^{A_1A_2,\ldots,A_N}\rangle\langle\Psi^{A_1A_2,\ldots,A_N}|$ with respect to qubit p is constructed from the matrix elements of $\widehat{\rho}$ through

$$\begin{aligned}\langle i_1 i_2,\ldots,i_N|\widehat{\rho}_G^{T_p}|j_1 j_2,\ldots,j_N\rangle &= \langle i_1 i_2,\ldots,i_{p-1} j_p i_{p+1},\ldots,i_N| \\ &\quad\times \widehat{\rho}\,|j_1 j_2,\ldots,j_{p-1} i_p j_{p+1},\ldots,j_N\rangle. \quad (3)\end{aligned}$$

If $\widehat{\rho}$ is a pure state, then the negative eigenvalue of the 4×4 submatrix of $\widehat{\rho}_G^{T_p}$ in the space spanned by distinct basis vectors $|i_1 i_2,\ldots,i_p,\ldots,i_N\rangle$, $|j_1 j_2,\ldots,j_p,\ldots,j_N\rangle$, $|i_1 i_2,\ldots,j_p,\ldots,i_N\rangle$, and $|j_1 j_2,\ldots,i_p,\ldots,j_N\rangle$ is $\lambda^- = -|\det(\nu_K^{i_1 i_2,\ldots,i_p,\ldots,i_N})|$ with $\nu_K^{i_1 i_2,\ldots,i_p,\ldots,i_N}$ defined as

$$\nu_K^{i_1 i_2,\ldots,i_p,\ldots,i_N} = \begin{bmatrix} a_{i_1 i_2,\ldots,i_p,\ldots,i_N} & a_{j_1 j_2,\ldots,i_p,\ldots,j_N} \\ a_{i_1 i_2,\ldots,j_p,\ldots,i_N} & a_{j_1 j_2,\ldots,j_p,\ldots,j_N}\end{bmatrix}, \quad (4)$$

where $K = \sum_{m=1}^{N}(1-\delta_{i_m,j_m})$ $(2 \leqslant K \leqslant N)$, $\delta_{i_m,j_m} = 1$ for $i_m = j_m$, and $\delta_{i_m,j_m} = 0$ for $i_m \neq j_m$. In analogy with ν^{00}, the 2×2 matrix $\nu_K^{i_1 i_2,\ldots,i_p,\ldots,i_N}$ is defined as a K-way negativity font. The subscript K is used to group together the negativity fonts arising due to the K-way coherences of the composite system, that is, the correlations responsible for the Greenberger-Horne-Zeilinger (GHZ)-state-like entanglement of a K-partite system. For a given value of K, the negativity of the K-way partial transpose $\widehat{\rho}_K^{T_p}$ with respect to subsystem p, as defined in Ref. [9], arises solely from K-way negativity fonts. The determinants of negativity fonts are, in a sense, intrinsic negative eigenvalues of a global or a K-way partial transpose of the state operator. The global partial transpose of an N-qubit state is a combination of K-way partially transposed operators $(2 \leqslant K \leqslant N)$ [9] and can be expanded as $\widehat{\rho}_G^{T_p} = \sum_{K=2}^{N}\widehat{\rho}_K^{T_p} - (N-2)\widehat{\rho}$. The negativity of $\widehat{\rho}_G^{T_p}$, defined as $N_G^A = (\|\rho_G^{T_A}\|_1 - 1)$, where $\|\widehat{\rho}\|_1$ is the trace norm of $\widehat{\rho}$, arises due to all possible negativity fonts present in $\widehat{\rho}_G^{T_p}$. Since K qubits may be chosen in $(\frac{N!}{(N-K)!K!})$ ways the form of a K-way font must specify the set of K qubits it refers to. To distinguish between different K-way negativity fonts we shall replace subscript K in Eq. (4) by a list of qubit states for which $\delta_{i_m,j_m} = 1$. In other words, a K-way font involving qubits A_1 to A_K, that is, $\sum_{m=1}^{N}(1-\delta_{i_m,j_m}) = \sum_{m=1}^{K}(1-\delta_{i_m,j_m}) = K$, reads as

$$\nu_{(A_{K+1})_{i_{K+1}}(A_{K+2})_{i_{K+2}},\ldots,(A_N)_{i_N}}^{i_1 i_2,\ldots,i_p,\ldots,i_N} = \begin{bmatrix} a_{i_1 i_2,\ldots,i_p,\ldots,i_K i_{K+1} i_{K+2},\ldots,i_N} & a_{i_1+1 i_2+1,\ldots,i_p,\ldots,i_K+1 i_{K+1} i_{K+2},\ldots,i_N} \\ a_{i_1 i_2,\ldots,i_p+1,\ldots,i_K i_{K+1} i_{K+2},\ldots,i_N} & a_{i_1+1 i_2+1,\ldots,i_p+1,\ldots,i_K+1 i_{K+1},i_{K+2},\ldots,i_N}\end{bmatrix}, \quad (5)$$

and its determinant is represented by

$$D_{(A_{K+1})_{i_{K+1}},(A_{K+2})_{i_{K+2}},\ldots,(A_N)_{i_N}}^{i_1 i_2,\ldots,i_p,\ldots,i_K} = \det\left(\nu_{(A_{K+1})_{i_{K+1}},(A_{K+2})_{i_{K+2}},\ldots,(A_N)_{i_N}}^{i_1 i_2,\ldots,i_p,\ldots,i_N}\right). \quad (6)$$

Here $i_m+1 = 0$ for $i_m = 1$ and $i_m + 1 = 1$ for $i_m = 0$. In this notation no subscript is needed for an N-way negativity font that is $\nu_N^{i_1 i_2,\ldots,i_p,\ldots,i_N} = \nu^{i_1 i_2,\ldots,i_p,\ldots,i_N}$.

II. TRANSFORMATION OF N-WAY NEGATIVITY FONTS UNDER LOCAL UNITARY ON A SINGLE QUBIT

Determinant of an N-way negativity font

$$D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N} = \det\begin{bmatrix} a_{i_1 i_2,\ldots,i_p=0,\ldots,i_N} & a_{i_1+1,i_2+1,\ldots,i_p=0,\ldots,i_N+1} \\ a_{i_1 i_2,\ldots,i_p=1,\ldots,i_N} & a_{i_1+1,i_2+1,\ldots,i_p=1,\ldots,i_N+1}\end{bmatrix}, \quad (7)$$

is an invariant of LU U^{A_p} acting on qubit A_p. After applying unitary transformation $U^{A_q} = \frac{1}{\sqrt{1+|x|^2}}[\begin{smallmatrix} 1 & -x^* \\ x & 1\end{smallmatrix}]$ on qubit A_q with $q \neq p$ we obtain

$$U^{A_q}|\Psi^{A_1,A_2,\ldots,A_N}\rangle = \sum_{i_1 i_2,\ldots,i_N} b_{i_1 i_2,\ldots,i_N}|i_1 i_2,\ldots,i_N\rangle. \quad (8)$$

Using primed symbols for determinants of negativity fonts calculated from coefficients $b_{i_1 i_2,\ldots,i_N}$, we can write four transformation equations

$$(D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N})' = \frac{1}{1+|x|^2}\left[D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} - |x|^2 D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N} + x D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_0} - x^* D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_1}\right], \tag{9}$$

$$(D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N})' = \frac{1}{1+|x|^2}\left[D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N} - |x|^2 D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} + x D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_0} - x^* D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_1}\right], \tag{10}$$

$$\left[D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_0}\right]' = \frac{1}{1+|x|^2}\left[-x^*(D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} + D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N}) + D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_0} + (x^*)^2 D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_1}\right], \tag{11}$$

$$\left[D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_1}\right]' = \frac{1}{1+|x|^2}\left[x(D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} + D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N}) + D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_1} + x^2 D^{i_1 i_2,\ldots,i_p=0,\ldots,i_{q-1},i_{q+1},\ldots,i_N}_{(A_q)_0}\right] \tag{12}$$

relating N-way and $(N-1)$-way negativity fonts. Eliminating the variable x, the invariants of $U^{A_p}U^{A_q}$ are found to be

$$(D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N})' - (D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N})' = D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} - D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N}, \tag{13}$$

$$\begin{aligned}&[(D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N})' + (D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N})']^2 - 4\left(D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_0}\right)'\left(D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_1}\right)'\\ &= (D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} + D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N})^2 - 4D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_0} D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_1},\end{aligned} \tag{14}$$

$$\begin{aligned}&(D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N})'(D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N})' - \left(D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_0}\right)'\left(D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_1}\right)'\\ &= D^{i_1 i_2,\ldots,i_p=0,i_q=0,\ldots,i_N} D^{i_1 i_2,\ldots,i_p=0,i_q=1,\ldots,i_N} - D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_0} D^{i_1 i_2,\ldots,i_p=0,\ldots,i_N}_{(A_q)_1}.\end{aligned} \tag{15}$$

Relevant multiqubit invariants for a given value of N can be written down from these general results. The invariants of $U^{A_p}U^{A_r}$ for K-way fonts ($2 \leqslant K \leqslant N$) with qubits p and r in the superscript and $N-K$ subscripts are analogous to those for the N-way fonts.

III. N-EVEN N-WAY INVARIANT

The invariant of $U^{A_1}U^{A_2}U^{A_3}$ is obtained by taking a combination of N-way invariants of $U^{A_1}U^{A_2}$ such that Eq. (13) is satisfied for the third qubit, for example,

$$I(U^{A_1}U^{A_2}U^{A_3}) = D^{0000,\ldots,0} - D^{0100,\ldots,0} - D^{0010,\ldots,0} + D^{0110,\ldots,0}. \tag{16}$$

Using the same reasoning, the four-qubit N-way invariant looks like

$$\begin{aligned}I(U^{A_1}U^{A_2}U^{A_3}U^{A_4}) &= D^{0000,\ldots,0} - D^{0100,\ldots,0} - D^{0010,\ldots,0} + D^{0110,\ldots,0}\\ &\quad - D^{0001,\ldots,0} + D^{0101,\ldots,0} + D^{0011,\ldots,0} - D^{0111,\ldots,0},\end{aligned} \tag{17}$$

and the N-way invariant for N qubits reads as

$$I_N = \sum_{i_2,\ldots,i_N} (-1)^{i_1+i_2,\ldots,+i_p,\ldots,+i_N} D^{0i_2,\ldots,i_N}. \tag{18}$$

Noting that $D^{00i_3,\ldots,i_N} = -D^{01i_3+1,\ldots,i_N+1}$, we have

$$\begin{aligned}&[D^{00i_3,\ldots,i_N} + (-1)^{N-1} D^{01i_3+1,\ldots,i_N+1}]\\ &= D^{i_1 i_2,\ldots,i_p=0 i_q=0,\ldots,i_N}[1+(-1)^N],\end{aligned} \tag{19}$$

giving $I_{N-\text{odd}} = 0$, while for N-even

$$I_{N-\text{even}} = \sum_{i_3,\ldots,i_N} (-1)^{i_3+i_4+,\ldots,+i_N} D^{00i_3,\ldots,i_N}. \tag{20}$$

The invariant for N even has permutation symmetry, as such it may be used to define N tangle as

$$\tau_{N-\text{even}} = 4\left|\sum_{i_1,i_2,\ldots,i_{p-1},i_{p+1},\ldots,i_{q-1},i_{q+1},\ldots,i_N} (-1)^{i_1+i_2,\ldots,+i_p,\ldots,+i_N} D^{i_1 i_2,\ldots,i_p=0 i_q=0,\ldots,i_N}\right|^2. \tag{21}$$

Degree-four invariants for N qubits are obtained by starting with the $N-2$-qubit N-way invariants and using Eq. (14) to obtain an N-qubit invariant.

A four-qubit four-way invariant with negativity fonts lying solely in a four-way partial transpose is written from Eq. (20) as $I_4 = D^{0000} + D^{0011} - D^{0010} - D^{0001}$. We identify I_4 with invariant H of degree two as given in Ref. [11]. A four-qubit state with four-qubit entanglement arising due to quantum correlations of the type present in a four-qubit GHZ state is distinguished from other entangled states by a nonzero I_4. This entanglement is lost without leaving any residue on the loss of a single qubit. The entanglement monotone based on I_4 is

$$\tau_4 = 4|[D^{0000} + D^{0011} - (D^{0010} + D^{0001})]^2|,$$

called four tangle in analogy with three tangle [21]. Four tangle τ_4 vanishes on the W-like state of four qubits, however, it fails to vanish on the product of two-qubit entangled states.

We now apply the method to construct entanglement monotones that detect the entanglement of specific parts of the composite system, an entangled qubit pair in this case. To obtain degree-four invariants that detect products of two-qubit states, consider the combination of four-way fonts $J = D^{0000} - D^{0100} + D^{0010} - D^{0110}$, which is an invariant of $U^{A_1}U^{A_2}$. Using Eq. (14), applied to four-way and three-way fonts, the four-qubit invariant is found to be

$$\begin{aligned} J^{A_1A_2} &= (D^{0000} - D^{0100} + D^{0010} - D^{0110})^2 \\ &\quad + 8D^{00}_{(A_3)_0(A_4)_0}D^{00}_{(A_3)_1(A_4)_1} + 8D^{00}_{(A_3)_1(A_4)_0}D^{00}_{(A_3)_0(A_4)_1} \\ &\quad - 4\big[D^{000}_{(A_3)_0} - D^{010}_{(A_3)_0}\big]\big[D^{000}_{(A_3)_1} - D^{010}_{(A_3)_1}\big] \\ &\quad - 4\big[D^{000}_{(A_4)_0} - D^{010}_{(A_4)_0}\big]\big[D^{000}_{(A_4)_1} - D^{010}_{(A_4)_1}\big]. \end{aligned} \tag{22}$$

Similarly, an invariant obtained by starting with the four-way $U^{A_1}U^{A_3}$ invariant form is

$$\begin{aligned} J^{A_1A_3} &= (D^{0000} - D^{0010} + D^{0001} - D^{0011})^2 \\ &\quad + 8D^{00}_{(A_2)_0(A_4)_0}D^{00}_{(A_2)_1(A_4)_1} + 8D^{00}_{(A_2)_1(A_4)_0}D^{00}_{(A_2)_0(A_4)_1} \\ &\quad - 4\big[D^{000}_{(A_2)_0} - D^{010}_{(A_2)_0}\big]\big[D^{000}_{(A_2)_1} - D^{010}_{(A_2)_1}\big] \\ &\quad - 4\big[D^{000}_{(A_4)_0} - D^{001}_{(A_4)_0}\big]\big[D^{000}_{(A_4)_1} - D^{001}_{(A_4)_1}\big], \end{aligned} \tag{23}$$

and starting with the $U^{A_1}U^{A_4}$ invariant we get

$$\begin{aligned} J^{A_1A_4} &= (D^{0000} - D^{0001} + D^{0010} - D^{0011})^2 \\ &\quad + 8D^{00}_{(A_2)_0(A_3)_0}D^{00}_{(A_2)_1(A_3)_1} + 8D^{00}_{(A_2)_1(A_3)_0}D^{00}_{(A_2)_0(A_3)_1} \\ &\quad - 4\big[D^{000}_{(A_2)_0} - D^{001}_{(A_2)_0}\big]\big[D^{000}_{(A_2)_1} - D^{001}_{(A_2)_1}\big] \\ &\quad - 4\big[D^{000}_{(A_3)_0} - D^{001}_{(A_3)_0}\big]\big[D^{000}_{(A_3)_1} - D^{001}_{(A_3)_1}\big], \end{aligned} \tag{24}$$

with the corresponding entanglement monotones defined as $\beta^{A_1A_i} = \frac{4}{3}|J^{A_1A_i}|$, $i = 2$–4. By construction, $|J^{A_1A_i}|$ detects entanglement between qubits A_1A_i, provided the pair A_1A_i is entangled to its complement in the four-qubit state. For qubit A_1, the invariants $J^{A_1A_2}$, $J^{A_1A_3}$, and $J^{A_1A_4}$ satisfy the relation $(I_4)^2 = \frac{1}{3}(J^{A_1A_2} + J^{A_1A_3} + J^{A_1A_4})$. An interesting four-qubit state reported in Ref. [23] is

$$\begin{aligned} |\chi\rangle &= \frac{1}{\sqrt{8}}(|0000\rangle + |1111\rangle - |0011\rangle + |1100\rangle + |1010\rangle \\ &\quad - |0101\rangle + |0110\rangle + |1001\rangle), \end{aligned} \tag{25}$$

which is known to have maximal entanglement of the pair A_1A_2 with the pair of qubits A_2A_4. The state can be rewritten as an entangled state of A_1A_4 and A_2A_3 Bell pairs

$$\begin{aligned} |\chi\rangle &= \frac{1}{\sqrt{8}}(|00\rangle_{A_1A_4} + |11\rangle_{A_1A_4})(|00\rangle_{A_2A_3} + |11\rangle_{A_2A_3}) \\ &\quad + \frac{1}{\sqrt{8}}(|10\rangle_{A_1A_4} - |01\rangle_{A_1A_4})(|10\rangle_{A_2A_3} + |01\rangle_{A_2A_3}), \end{aligned}$$

however, is not reducible to a pair of Bell states. We verify that for this state $I_4 = 0$, $J^{A_1A_2} = J^{A_1A_3} = J^{A_2A_4} = J^{A_3A_4} = -\frac{1}{4}$, and $J^{A_1A_4} = J^{A_2A_3} = \frac{1}{2}$. Therefore the state is characterized by $\tau_4 = 0$, $\beta^{A_1A_2} = \beta^{A_1A_3} = \beta^{A_2A_4} = \beta^{A_3A_4} = \frac{1}{3}$, while $\beta^{A_1A_4} = \beta^{A_2A_3} = \frac{2}{3}$, indicating that the entanglement of state $|\chi\rangle$ is distinct from that of the GHZ state of four qubits having $\tau_4 = 1$, $\beta^{A_1A_2} = \beta^{A_1A_3} = \beta^{A_1A_4} = \frac{1}{3}$, as well as $\beta^{A_2A_3} = \beta^{A_2A_4} = \beta^{A_3A_4} = \frac{1}{3}$.

The degree-four invariants for four qubits, denoted as L, M, and N in Ref. [12], are combinations of $J^{A_1A_2}$, $J^{A_1A_3}$, $J^{A_1A_4}$, and $(I_4)^2$. Additional invariants are easily constructed to detect all possible types of four-qubit entanglement. One can verify that different types of four-qubit entanglement, detected by the antilinear operators of Ref. [19], are quantified by entanglement monotones constructed from four-qubit invariants.

IV. N-ODD N-WAY INVARIANT

Since $I_{N-\text{odd}} = 0$, there is no degree-two invariant of N-way fonts for a general state of N-odd qubits. But we can single out a qubit by writing $N-1$ qubit invariants and then use Eq. (14) to obtain the N-qubit invariant. If we single out the Nth qubit and look at negativity fonts of $\rho^{T_{A_1}}$, then two $(N-1)$ qubit N-way invariants are

$$I^{A_1(A_N)_0}_{N-\text{way}} = \sum_{i_3,\ldots,i_{N-1}} (-1)^{i_3+,\ldots,+i_{N-1}} D^{00i_3,\ldots,i_{N-1}i_N=0}, \tag{26}$$

$$I^{A_1(A_N)_1}_{N-\text{way}} = \sum_{i_3,\ldots,i_{N-1}} (-1)^{i_3+,\ldots,+i_{N-1}} D^{00i_3,\ldots,i_{N-1}i_N=1}. \tag{27}$$

Transformation equations for $I^{A_1(A_N)_0}_{N-\text{way}}$ and $I^{A_1(A_N)_1}_{N-\text{way}}$, under unitary U^{A_N} are written by using Eqs. (9) to (12) and yield an N-qubit invariant

$$I^{A_1A_N}_{N-\text{odd}} = \big[I^{A_1(A_N)_0}_{N-\text{way}} + I^{A_1(A_N)_1}_{N-\text{way}}\big]^2 - 4I^{A_1(A_N)_0}_{(N-1)-\text{way}}I^{A_1(A_N)_1}_{(N-1)-\text{way}}, \tag{28}$$

with negativity fonts in $\rho^{T_{A_1}}$, where

$$I^{A_1(A_N)_0}_{(N-1)-\text{way}} = \sum_{i_3,\ldots,i_{N-1}} (-1)^{i_3+,\ldots,+i_{N-1}} D^{00i_3,\ldots,i_{N-1}}_{(A_N)_0}, \tag{29}$$

and

$$I^{A_1(A_N)_1}_{(N-1)-\text{way}} = \sum_{i_3,\ldots,i_{N-1}} (-1)^{i_3+,\ldots,+i_{N-1}} D^{00i_3,\ldots,i_{N-1}}_{(A_N)_1}, \tag{30}$$

are $(N-1)$-way invariants of local unitaries on $(N-1)$ qubits (even number of qubits). Similarly, one may construct $\tau^{A_1A_p}_{N-\text{odd}}$ for $2 \leqslant p \leqslant N$ and $N+1 \rightarrow 1$ (mod N). The entanglement monotone based on $I^{A_1A_p}_{\text{N-odd}}$ is $\tau^{A_1A_p}_{N-\text{odd}} = 4|I^{A_1Ap}_{N-\text{odd}}|$.

For $N = 3$, the three-qubit invariant of degree two determines three tangle [21] through $\tau_3 = 4|(D^{000} - D^{001})^2 -$

$4D^{00}_{(A_2)_0}D^{00}_{(A_2)_1}|$. For $N=5$, the five-way invariants of local unitaries on qubits A_1, A_2, A_3, and A_4 and corresponding four-way invariants combine to give

$$\begin{aligned} I_5^{A_1A_5} &= (D^{00000} - D^{00010} + D^{00110} - D^{00100} + D^{00001} \\ &\quad - D^{00011} + D^{00111} - D^{00101})^2 \\ &\quad - 4\left[D^{0000}_{(A_5)_0} - D^{0001}_{(A_5)_0} - D^{0010}_{(A_5)_0} + D^{0011}_{(A_5)_0}\right] \\ &\quad \times \left[D^{0000}_{(A_5)_1} - D^{0001}_{(A_5)_1} - D^{0010}_{(A_5)_1} + D^{0011}_{(A_5)_1}\right], \end{aligned} \quad (31)$$

which is a five-qubit invariant of degree four with fonts in five-way and four-way partial transpose with respect to qubit A_1. In general, one can construct $I_5^{A_pA_q}$ obtaining a five tangle $\tau_5^{A_pA_q} = 4|I_5^{A_pA_q}|$ for each choice of p and q value. Degree-four invariants to detect entanglement of two entangled qubits with their compliment in a five-qubit state are combinations of two-qubit invariants of five-way, four-way, three-way, and two-way fonts and can be obtained in a way analogous to that for five tangle.

V. CONCLUSION

To conclude, LU polynomial invariants for the N-qubit quantum state have been obtained from basic units of entanglement, referred to as negativity fonts. The method exploits the transformation properties of determinants of K-way negativity fonts under LU transformations. The entanglement monotone based on the square of the degree-two invariant for the N-even [Eq. (21)] and degree-four invariant of Eq. (28) for N odd is referred to as N tangle in analogy with three tangle [21]. The method aims at obtaining LU invariants that are relevant to classifying multiqubit entangled states. To illustrate the construction of entanglement monotones that detect the entanglement of specific parts of the composite system, degree-four invariants to detect entanglement of entangled pairs in a four-qubit state are reported. Our method can be used to generate the relevant invariants obtained by using different approaches in Refs. [11–15] and also to generate the additional invariants necessary to detect specific entanglement modes. Entanglement monotones constructed from invariants can identify the class to which a given state belongs. LU transformations redistribute the negativity fonts among K-way partial transposes and may also reduce the number of negativity fonts in a given partial transpose. To determine unitary transformations that relate two-unitary equivalent states is an important question in quantum information. The necessary and sufficient conditions for the equivalence of arbitrary N-qubit pure states under LU operations have been derived [24] and used to determine the different LU-equivalence classes of up to five-qubit states [25]. We find that the key to determine the unitary transformations relating two states belonging to the same class lies in the numerical value of the invariants, number and type of negativity fonts and transformation equations that the determinants of the negativity fonts for each state satisfy. Using the transformation equations for the determinants of negativity fonts to directly identify the unitaries that may equalize the number and type of negativity fonts in two states having the same values of LU invariants offers an alternate method to establish unitary equivalence. The method for obtaining LU invariants can be easily extended to qutrits and higher-dimensional systems.

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