Universal quantum computation in a semiconductor quantum wire network

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Universal quantum computation (UQC) using Majorana fermions on a two-dimensional topological superconducting (TS) medium remains an outstanding open problem. This is because the quantum gate set that can be generated by braiding of the Majorana fermions does not include *any* two-qubit gate and also no single-qubit $\pi/8$ phase gate. In principle, it is possible to create these crucial extra gates using quantum interference of Majorana fermion currents. However, it is not clear if the motion of the various order parameter defects (vortices, domain walls, etc.), to which the Majorana fermions are bound in a TS medium, can be quantum coherent. We show that these obstacles can be overcome using a semiconductor quantum wire network in the vicinity of an s-wave superconductor, by constructing topologically protected two-qubit gates and any arbitrary single-qubit phase gate in a topologically unprotected manner, which can be error corrected using magic-state distillation. Thus our strategy, using a judicious combination of topologically protected and unprotected gate operations, realizes UQC on a quantum wire network with a remarkably high error threshold of 0.14 as compared to 10^{-3} to 10^{-4} in ordinary unprotected quantum computation.

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I. INTRODUCTION

The decoherence of quantum states by the environment is the nemesis of any proposed quantum computation scheme. Topological quantum computation (TQC) proposes [1,2] an elegant way to solve this environmental decoherence problem by encoding quantum information in an intrinsically nonlocal way. Quantum information thus stored is expected to be essentially immune to any local perturbation due to the environment. A class of quantum many-body states, characterized by excitations with non-Abelian statistics (non-Abelian anyons), allows such nonlocal encoding of quantum information. In principle, the non-Abelian anyons can be moved (braided) around each other to exploit their statistics, which can be used to manipulate the stored quantum information and build quantum gates [3–6]. Therefore, TQC using non-Abelian excitations is intrinsically fault tolerant, which holds considerable promise in overcoming the environmental decoherence problem.

Statistics [7] is defined as the unitary transformations on many-body wave functions by the pair-wise exchange of the particles' quantum numbers. In $(2+1)$-dimensions, if the many-body ground state wave function happens to be a linear combination of states from a degenerate subspace, a pair-wise exchange of the particles can unitarily *rotate* the wave function in the ground-state subspace. In this case, the statistics is non-Abelian [1,2] and the system of such quantum particles is a non-Abelian system. Non-Abelian quantum systems in the so-called Ising topological class [2] are characterized by topological excitations called Majorana fermions. In some topological superconducting (TS) systems [8], Majorana fermions arise as nondegenerate zero-energy excitations bound to vortices of the superconducting order parameter. These topological excitations are protected from the higher-energy, nontopological, Bogoliubov excitations at the vortex cores [9] by the so-called mini-gap $\sim \frac{\Delta^2}{\epsilon_F}$, where Δ is the superconducting pair potential and ϵ_F is the Fermi energy. The second quantized operators γ_i corresponding to the Majorana excitations are self-Hermitian, $\gamma_i^\dagger = \gamma_i$, which is in sharp contrast to ordinary fermionic (or bosonic) operators for which $c_i \neq c_i^\dagger$. Therefore, each Majorana particle is its own antiparticle [10] unlike Dirac fermions where electrons and positrons (or holes) are distinct. Majorana particles have been predicted to occur in some exotic many-body states such as the proposed Pfaffian states in the filling fraction $\nu = 5/2$ fractional quantum Hall (FQH) system [11], spinless chiral p-wave superconductors or superfluids [12,13], the surface of three-dimensional strong topological insulators (TIs) [14], and noncentrosymmetric superconductors [15].

II. SEMICONDUCTOR AS A NON-ABELIAN SYSTEM

Recently, a semiconductor thin film with Rashba-type spin-orbit (SO) coupling was proposed to be a suitable platform for realizing a Majorana-carrying TS state by the proximity effect [16,17]. It was shown that in the presence of an s-wave superconducting pair potential Δ and a Zeeman splitting V_z, both of which can be proximity-induced (V_z can also be induced by a parallel magnetic field when the SO coupling includes a Dresselhaus component [18]), the appropriate TS state is realized when the parameters satisfy $V_z^2 > \Delta^2 + \mu^2$ where μ is the chemical potential in the semiconductor. Following this, it was quickly realized [19] that the one-dimensional (1D) version of the same setup, a semiconducting quantum wire with zero-energy Majorana states trapped at the two ends, would be an easier system in which to explore the physics of Majorana fermions, since the relevant mini-gap at the wire ends is of order $\Delta \gg \frac{\Delta^2}{\epsilon_F}$ (there are no other subgap states localized near the ends other than the Majorana states). It is important to note that s-wave proximity effect on an InAs quantum wire (which has a sizable SO coupling) has possibly been already realized in

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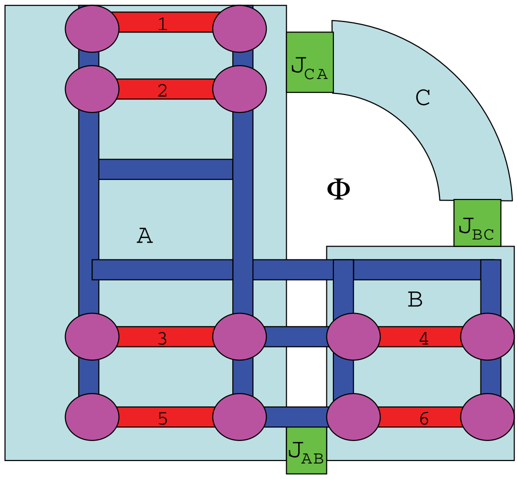

FIG. 1. (Color online) Schematic of entanglement generation in quantum wire topological qubits using superconductor Josephson junctions. The light blue background represents superconducting islands labeled A, B and C. The dark blue unnumbered segments are semiconductor quantum wires in the nontopological superconducting state, while the red numbered segments are semiconductor quantum wires in the topological superconducting state. The purple circles represent the Majorana fermions at the end of topological segments. The topological segments 1 and 2 form a representative topologically protected qubit. Entanglement is generated in the qubits formed by the topological segments (3,4) and (5,6) by transferring segments 4 and 6 to island B and then performing a quantum nondemolition measurement of the number of neutral fermions on island B using microwaves. Φ is the bias flux in the central hole. See text for details.

experiments [20]. Moreover, the required Zeeman splitting V in the wire can be introduced more easily than in the two-dimensional (2D) case by a magnetic field parallel to the superconductor [21,22]. For all these reasons, it seems that a Majorana-carrying TS state in a semiconductor quantum wire may be within experimental reach. A discussion of the SO coupled semiconductor as a non-Abelian platform in both 2D and 1D, along with scanning tunneling microscopy (STM) signatures of Majorana modes from the wire ends, can be found in Ref. [23].

III. TOPOLOGICAL QUBIT USING QUANTUM WIRES

Let us consider a semiconductor quantum wire in the TS state ($V^2 > \Delta^2 + \mu^2$). Each wire i (shown as the red, numbered segments in Fig. 1) has a pair of Majorana modes $\gamma_i^{(L,R)}$ (shown as circles at the wire ends) at the left (L) and right (R) ends. With wire i we can associate a regular fermion state represented by the operator $d_i^\dagger = \frac{\gamma_i^{(L)} + \imath\gamma_i^{(R)}}{2}$. Thus, the wire naturally forms a two-state system consisting of states $|0\rangle$ and $|1\rangle = d_i^\dagger|0\rangle$, where $d_i|0\rangle = 0$. Since the wave function for d_i is composed of a pair of nonoverlapping Majorana states, it is unaffected by all local changes in the Hamiltonian. Thus, the wire in the TS state constitutes a decoherence-free two-state system which can be used to build a topologically protected qubit. However, such a two-state system does not allow the superposition of the basis states, i.e., the states $(|0\rangle \pm |1\rangle/\sqrt{2})$ do not exist, because they violate the conservation of fermion parity [24]. To remedy this, a topological logical qubit can be defined [24] via a *pair* of quantum wires in the TS state, i.e., with the states $|\bar{0}\rangle = |00\rangle$ (d states in both quantum wires unoccupied), and $|\bar{1}\rangle = |11\rangle$ (d states in both quantum wires occupied). The superposition states, $(|\bar{0}\rangle \pm |\bar{1}\rangle)/\sqrt{2}$, are now allowed because the superconducting condensate only conserves fermion number modulo 2. Note also that these two states do not mix with the other two states $(|10\rangle, |01\rangle)$ of the two-wire system by any unitary operation that conserves fermion parity.

IV. QUANTUM WIRE NETWORK AND NON-ABELIAN STATISTICS

Recently, a network of 1D semiconductor quantum wires has been proposed [25] as a suitable platform to create, transport, and fuse Majorana fermions at the wire ends. The wire network consists of wire segments in the TS state (shown in red with numbers in Fig. 1) connected by segments in the non-topological superconducting (NTS) state (shown in blue without numbers in Fig. 1). The Majorana fermion states are transported by shifting the end points of the TS segments by applying locally tunable external gate potentials (which control μ). That pair-wise exchange [25] of these Majorana fermions leads to the familiar non-Abelian statistics (i.e., $\gamma_i \to \gamma_j$ but $\gamma_j \to -\gamma_i$), which follows most simply from fermion parity conservation. Suppose U is the unitary operator for exchange of Majorana fermions. Suppose also that γ_i, γ_j do not pick up a (relative) negative sign under U. U then transforms the neutral fermion operator $d^\dagger = \gamma_i + \imath\gamma_j$ into $Ud^\dagger U^\dagger = \imath d$. Applying $d^\dagger$ to $U|0\rangle$, where $|0\rangle$ is the empty state, it is easy to see that $U|0\rangle = \lambda|1\rangle = \lambda d^\dagger|0\rangle$, where λ is a proportionality constant. This contradicts fermion parity since U is even and $d^\dagger$ is odd under fermion parity. It then follows that there must be a relative negative sign whenever two Majorana fermions are exchanged. Note also that the fact that the Majorana fermions in the present case are situated at the ends of 1D wires (and not in a 2D system like a 2D chiral p-wave superconductor) does not make any difference, since they are essentially zero-dimensional objects. In the wire network, these zero-dimensional objects are being moved (braided) on the 2D substrate of the superconductor. A more microscopic derivation of this non-Abelian statistics using Kitaev's 1D construction for a TS state [26] has been given in Ref. [25].

The exchange and braiding operations on the Majorana fermions lead to some of the quantum gates such as the single-qubit $\pi/4$ phase gate and the single-qubit Hadamard gate. However, it is well known [24] that for a system of Majorana fermions, the exchange or braiding operations alone fail to provide any two-qubit gate: the topological braiding operations allowed in a quantum wire network, as in its 2D FQH or chiral p-wave Pfaffian counterpart, are not computationally sufficient. A system of Majorana fermions can be made computationally sufficient if the braiding-generated gate set is supplemented by a single-qubit $\pi/8$ phase gate and a two-qubit controlled-NOT, or CNOT, gate [24,27].

V. UNIVERSAL QUANTUM COMPUTATION WITH MAJORANA FERMIONS

A system of Majorana fermions can be made computationally sufficient in one of two ways [24,28]: (1) by dynamically changing the topology of the platform, which allows the crucial extra gates to be obtained in a topologically protected manner, or (2) by implementing these gates in a topologically unprotected manner, which, provided the other gates are topologically protected, can also lead to universal quantum computation (UQC) with the aid of certain error-correction protocols. At present, it is not clear how the topologically protected route can be implemented in any proposed TQC architecture including the quantum wire network. Therefore, in this paper, we take the second route to UQC as described above. In the proposal we will consider, only the $\pi/8$ phase gate will be implemented in a topologically unprotected way. The topologically protected single-qubit gates implemented through the braiding operations can then be used to perform "magic-state distillation" [24,29] to produce error-corrected $\pi/8$ phase gates from noisy ones. This purification protocol (which has polylogarithmic overhead) consumes several copies of a magic state, e.g., $(|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$, and outputs a single qubit with higher polarization along a magic direction. Once a sufficiently pure magic state is produced, it may then be consumed to generate a $\pi/8$ phase gate. This protocol permits a remarkably high error threshold of over 0.14 for the noisy gates, as compared to 10^{-3} to 10^{-4} in ordinary unprotected quantum computation. The simpler strategy of adopting the topologically unprotected route to UQC using a TS system still leaves us with a nontrivial problem. The principal reason why any system of Majorana fermions is not computationally sufficient is that two qubits cannot be entangled using the braiding operations alone. Any logical state of the two qubits, accessible by braiding one Majorana fermion around another, can always be written as a product of the logical states of the individual qubits. It has been shown [24] that a two-qubit CNOT gate can be created with Majorana fermions provided there is a supply of entangled pairs of two topological qubits. In the FQH context, it has been proposed [24,28,30] that quantum interference of Majorana currents can be used to generate the two-qubit entanglement. However, in a TS system, in which the Majorana fermions are trapped in the order parameter defects (vortices or domain walls), it is not clear that the motion of these defects is a quantum process that would lead to the desired quantum interference. Further, creating a single-qubit $\pi/8$ phase gate in a TS system is also problematic. A simple method for this could be moving a pair of Majorana fermions in a given qubit near each other. Because of the overlap of the Majorana wave functions, the energy degeneracy of the $|0\rangle$ and the $|1\rangle$ states are then split. The dynamic phase in the resulting time evolution could then be used to produce arbitrary single-qubit phase gates, were it not for the fact that the Majorana wave functions in the TS medium are oscillatory in space, which results in corresponding oscillations in the energy splitting as a function of inter-Majorana distance [31]. We show in this paper that both of these obstacles can be overcome in the quantum wire network, which therefore allows a concrete realization of a UQC architecture.

VI. UQC IN THE QUANTUM WIRE NETWORK

We first show that entangled pairs of qubits can be generated by the setup shown in Fig. 1. The two superconducting islands B and C in Fig. 1, together with the main superconductor A (which holds the wire network), constitute a three-island Josephson junction flux qubit, which when biased with half a flux quantum, has a degenerate pair of states composed of clock-wise supercurrent (CW) and counter-clock-wise supercurrent (CCW). The charging energy of the islands leads to tunneling between these two states, leading to a splitting of the degeneracy with the new eigenstates $\frac{1}{\sqrt{2}}(\mathrm{CW}\pm\mathrm{CCW})$. This splitting between the energies is also sensitive to a Berry phase contribution which can be controlled by gate electrodes in the vicinity of the islands [32]. In addition, as we show later, the splitting also depends on the parity of the number of neutral fermions shared between the pairs of Majorana fermions at the ends of TS segments on island B (here we have assumed that the capacitance of the main island A is large enough so we can ignore its charging energy). Using this, the system can be tuned such that an even number of neutral fermions on island B leads to an exact degeneracy, while an odd number of neutral fermions leads to a splitting between the states $\frac{1}{\sqrt{2}}(\mathrm{CW}\pm\mathrm{CCW})$. This splitting can be measured by coupling the system to an rf circuit [32], which can be used [33] to perform a nondemolition measurement of the state $(|0\rangle,|1\rangle)$ of a pair of Majorana fermions on island B. As has been emphasized in Ref. [33], this provides an explicitly quantum mechanical method for the charge measurement of a pair of Majorana fermions. Note that in the analogous method of charge measurement using quantum interference of currents in a TS state, it is not clear if the motion of order parameter defects (vortices, domain walls) is a quantum mechanical process.

Let us now show how to use the quantum superposition states of the flux qubit to also create quantum entanglement between two topological qubits. The entangled state between the two qubits can then be used [24] as the ancillary two-qubit states to construct a two-qubit quantum gate. To generate an entangled pair of qubits, we first create a pair of qubits composed of TS segments (3,4) and (5,6) both in the state $|\bar{0}\rangle \equiv |0,0\rangle$ on the main island A. By applying a Hadamard gate to both, we then transform the states of both qubits to $|\bar{0}\rangle + |\bar{1}\rangle$. The combined state of the two-qubit system is now $(|\bar{0}\rangle + |\bar{1}\rangle)_{3,4} \otimes (|\bar{0}\rangle + |\bar{1}\rangle)_{5,6}$. We then transfer the TS segments 4 and 6 to island B by applying external gate potentials. If the parity of neutral fermions on segments 4 and 6 is even (odd), the degeneracy of the states $\frac{1}{\sqrt{2}}(\mathrm{CW}\pm\mathrm{CCW})$ is split (not split). By an rf measurement, one can then collapse the quantum states of the two qubits as

$$(|\bar{0}\rangle + |\bar{1}\rangle)_{3,4} \otimes (|\bar{0}\rangle + |\bar{1}\rangle)_{5,6} \rightarrow |\bar{0}\rangle_{3,4} \otimes |\bar{0}\rangle_{5,6} + |\bar{1}\rangle_{3,4} \otimes |\bar{1}\rangle_{5,6}, \quad (1)$$

which is the desired entangled pair. If in the rf measurement, no splitting is observed (50% chance), the process has to be repeated until a splitting *is* observed, producing the desired entangled pair. Therefore, this method provides entangled pairs of qubits with a 50% success rate *deterministically*.

In addition to two-qubit entanglement and a CNOT gate, for UQC, one needs a single-qubit $\pi/8$ phase gate. As discussed earlier [24,28], a simple way to create such a gate could be to bring a pair of Majorana fermions from a topological qubit near each other and let the microscopic physics split the degeneracy between the $|0\rangle$ and $|1\rangle$ states. This has also been discussed in the context of general anyons [34]. Any arbitrary single-qubit phase gate can then be created in principle by accumulating the relative dynamic phase between the $|\bar{0}\rangle$ and $|\bar{1}\rangle$ states over a finite period of time. However, it now appears that such a scheme does not work in both TS and FQH systems, because the splitting between the $|\bar{0}\rangle$ and $|\bar{1}\rangle$ states oscillates with distance between the Majorana fermions in a given pair because of the oscillatory nature of the wave functions [31,35]. Recently an interferometric proposal has been suggested to avoid these oscillations in the FQH system [6]. At first glance, it appears that the same problem would arise in our quantum wire network, because the Majorana wave functions oscillate in the TS segments as a function of distance from the domain wall [oscillating solid black lines in Fig. 2(a)] just like in the case of a chiral p-wave superconductor. However, it is important to note that these functions *do not oscillate* and in fact monotonically decay with a decay length inversely proportional to the gap (which is essentially proportional to the gate voltage V_{gate} for $V_{\text{gate}} \gg \mu$) in the nontopological segments of the wire network [decaying solid black lines in Fig. 2(a)]. Further, in the wire network, to induce splitting between the states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ between a pair of Majorana end states, one does not need to physically move these states near each other (a process which is prone to errors). Instead, one could simply reduce V_{gate} in the nontopological segments to generate the required overlap of the Majorana fermion wave functions on the two ends.

The presence of oscillations in the wave functions in the TS and the absence thereof in the NTS can be understood from simple considerations of the asymptotic wave functions. The TS segment consists of a superconducting state on a Fermi-liquid-like state with Fermi wave vector k_F. Therefore, the wave function of a zero-energy Majorana state has an asymptotic form $\psi(r) \propto e^{-r/\xi} e^{ik_F r}$ where the coherence length $\xi = \Delta/v_F$. [23] The overlaps of this wave function ψ clearly has oscillations at the wave vector k_F as is the case with p-wave superconductors [31]. In contrast, the NTS part of the system is depleted and therefore has a vanishing Fermi wave vector ($k_F = 0$). Thus the overlaps across the NTS have a purely decaying form as is clear from Fig. 2. Below, with the help of Fig. 2, we explicitly show how an arbitrary single-qubit gate can be constructed.

From the plot of the wave function (solid black line) in Fig. 2(a), it is clear that the wave functions no longer oscillate in the NTS segment. To generate a phase shift, we first distort the TS segment 1 in the qubit in Fig. 2(b) into a U shape. This step is necessary to bring the two Majorana modes at the two ends of segment 1 separated by a NTS region. The gate voltage V_{gate} in the NTS segment is still kept large such that the overlap of the wave functions is still negligible. We now lower the gate voltage in the NTS segment to increase the decay lengths of the wave functions. This causes overlap between the Majorana fermion wave functions at the ends of the NTS segment. As seen in Fig. 2(b), this leads to a clear nonoscillatory dependence of the energy splitting between the $|\bar{0}\rangle$ and $|\bar{1}\rangle$ states as a function of the applied gate voltage.

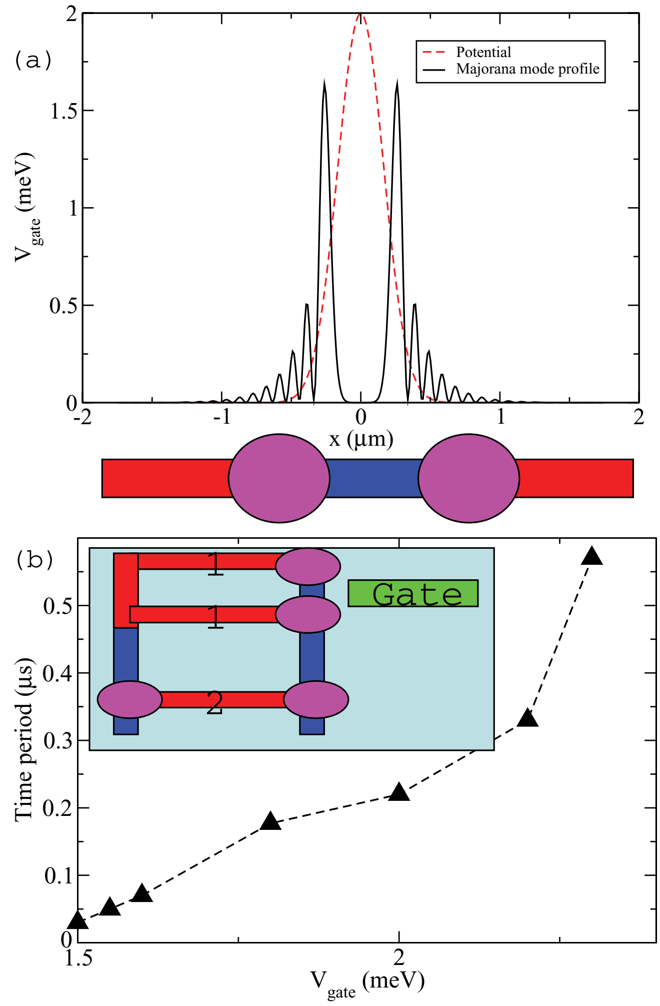

FIG. 2. (Color online) (a) Majorana wave functions (black lines) are fused across a nontopological wire segment (blue middle segment) with a Gaussian potential (red dashed line) with peak height 2.0 meV for an InAs nanowire. Note that there are no oscillations of the wave functions in the nontopological blue segment. The wave function, on the other hand, rapidly oscillates in the red (left and right) topological segments. The purely decaying wave functions in the nontopological segment lead to a well-controlled phase shift as a function of time. (b) Inset shows arbitrary single-qubit phase gate constructed by controlled overlap of Majorana fermion wave functions through the nontopological segment. The plot shows the dependence of the phase rotation period on the barrier height.

A $\pi/8$ phase gate is obtained by applying a V_{gate} pulse over an appropriate length of time. The appropriate length of time can be estimated by *experimentally* determining the energy splitting between the $|\bar{0}\rangle$ and $|\bar{1}\rangle$ states. The rate of phase shift ΔE can be determined by creating a test state $|\bar{0}\rangle$ and applying the sequence (Hadamard → phase gate → inverse Hadamard) in order. After this process, the probability of making a transition to the $|\bar{1}\rangle$ state is given by

$$P(t) = \frac{1 - \cos \Delta E t}{2}. \quad (2)$$

By experimentally determining this probability at time t, one gets ΔE for a given t and V_{gate}. Arbitrary single-qubit phase gates can then be constructed by calculating the required time and accumulating the correct phase shift for a given V_{gate}.

VII. MAJORANA BERRY PHASE IN THE FLUX QUBIT

To understand the origin of the Berry phase term generated by Majorana fermions, it is necessary to revert back to the fermion description of the system. This description is in terms of the fermion fields ψ_α for the electrons in both the semiconducting wires and on the superconducting island $\alpha = A,B,C$. The pairing interactions g_α leading to Cooper-pairing on the islands together with Coulomb interaction induced capacitance C_α and gate potential V_α^{ext} can be described in terms of an action

$$\begin{aligned} S = \int \imath\bar{\psi}_\alpha\partial_t\psi_\alpha - H_\alpha[\bar{\psi}_\alpha,\psi_\alpha] - \left(V_\alpha + V_\alpha^{\text{ext}}\right)\bar{\psi}_\alpha\psi_\alpha \\ - (\Delta_\alpha\bar{\psi}_\alpha\bar{\psi}_\alpha + \Delta_\alpha^*\psi_\alpha\psi_\alpha) + C_\alpha V_\alpha^2/2 \\ + \left|\Delta_\alpha^2\right|/2g_\alpha + t_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta, \end{aligned} \tag{3}$$

where V_α and Δ_α are the effective Hubbard-Stratonovich fields which may be interpreted as the time-dependent mean-field electrostatic potential and pairing potential, respectively. The relatively weak tunnelings $t_{\alpha\beta}$ between the islands and also the wires on the different islands will give rise to the Josephson couplings between the islands. For our calculation, we will consider a phase model such that $|\Delta_\alpha| = |\Delta|$. We will also assume that the capacitances of the islands is large enough so that we can assume the quantum fluctuations in V_α and ϕ_α to be slowly varying. In this limit, the fermionic part of the fields that contribute to the partition function may be assumed to follow the ground state of the Hamiltonian

$$\begin{aligned} H_{mf}[V_\alpha,\phi_\alpha] = H_\alpha[\bar{\psi}_\alpha,\psi_\alpha] + \left(V_\alpha + V_\alpha^{\text{ext}}\right)\bar{\psi}_\alpha\psi_\alpha \\ + (\Delta_\alpha\bar{\psi}_\alpha\bar{\psi}_\alpha + \Delta_\alpha^*\psi_\alpha\psi_\alpha) + t_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta \end{aligned} \tag{4}$$

as ϕ_α and V_α vary in time adiabatically. For systems containing Majorana fermions the ground state is degenerate. The Majorana fermion sector of the Hamiltonian may be described in terms of pairs of Majorana fermions combined into regular zero-energy fermions $d_{m\alpha}^\dagger$ that are localized on the island α. The relevant states may then be characterized by $Q^\dagger|\phi_\alpha\rangle$ where $|\phi_\alpha\rangle$ denotes the ground state with all Majorana fermion states empty, and $Q^\dagger = \prod d_{m\alpha}^{\dagger n_{m\alpha}}$ is the operator that accounts for the Majorana state occupation. The transition matrix element between the various phase states $Q_i^\dagger|\phi_{\alpha,i}\rangle$ and $Q_f^\dagger|\phi_{\alpha,f}\rangle$ is given by

$$\mathcal{T} = \sum_{w_\alpha}\int_{\phi_{\alpha,i}}^{\phi_{\alpha,f}+2w_\alpha\pi}\mathcal{D}\phi_\alpha\langle\phi_{\alpha,f}|Q_f U(t_f,t_i)Q_i^\dagger|\phi_{\alpha,i}\rangle, \tag{5}$$

where $dU(t,t_i)/dt = H_{mf}(t)U(t,t_i)$ is the unitary time evolution matrix for the fermionic state over a particular phase trajectory. The states $Q_i^\dagger|\phi_{\alpha,i}\rangle$ and $Q_f^\dagger|\phi_{\alpha,f}\rangle$ are ground states of $H_{mf}(t_i)$ and $H_{mf}(t_f)$, respectively, with appropriate Majorana state occupancy.

Since the zero-energy fermion operators $d_{m\alpha}^\dagger$ are spatially separated, and localized on each island α, they evolve with phase ϕ_α according to

$$d_{m\alpha}^\dagger = \int dr\, u_{m\alpha}(r)\psi_\alpha^\dagger(r)e^{\imath\int\dot{\phi}_\alpha/2} + u_{m\alpha}(r)\psi_\alpha(r)e^{-\imath\int\dot{\phi}_\alpha/2}. \tag{6}$$

Moreover, because of the absence of tunneling between the Majorana fermions, the occupation of each of these zero-energy modes is conserved during the evolution of the Hamiltonian. Therefore under the time evolution $U(t_f,t_i)$, $Q_i^\dagger$ evolves into $U(t_f,t_i)Q_iU(t_f,t_i)^\dagger = Q_f^\dagger(-1)^{\sum_\alpha w_\alpha n_\alpha}$ where $n_\alpha = \sum_m n_{m\alpha}$ and w_α is the winding number of the phase trajectory along a particular phase ϕ_α. Furthermore $Q_f Q_f^\dagger|\phi_{\alpha,f}\rangle = |\phi_{\alpha,f}\rangle$. Therefore the transition amplitude for a given fermion occupation on each island n_α is

$$\mathcal{T}(n_\alpha) = \sum_{w_\alpha}\int_{\phi_{\alpha,i}}^{\phi_{\alpha,f}+2w_\alpha\pi}\mathcal{D}\phi_\alpha(-1)^{w_\alpha n_\alpha}\langle\phi_{\alpha,f}|U(t_f,t_i)|\phi_{\alpha,i}\rangle. \tag{7}$$

The factor $(-1)^{\sum_\alpha w_\alpha n_\alpha}$ is precisely the Berry phase term associated with Majorana fermions and can be accounted for by adding a term $\dot{\phi}_\alpha n_\alpha/2$ to the phase action of the system. Adding this to the phase action considered by Tiwari and Stroud [32], which can be obtained by performing the V_α integral, is

$$S[n_\alpha,\phi_\alpha] \approx \dot{\phi}_\alpha^2/2C_\alpha + \dot{\phi}_\alpha\left[\frac{V_\alpha^{\text{ext}}}{C_\alpha} + n_\alpha\right] - J_{\alpha\beta}\cos(\phi_\alpha - \phi_\beta). \tag{8}$$

In the presence of an externally applied flux Φ as shown in Fig. 1, we replace the phase differences in the above equation by the gauge-invariant phase differences $\phi_1 = \phi_B - \phi_A \to \phi + a_1$, $\phi_C - \phi_B \to (-\phi + \phi' + a_2)$, and $\phi_2 = \phi_A - \phi_C \to -\phi' + a_3$. The Josephson couplings are taken to be $J_{AB} = J_{CA} = J$ and $J_{BC} = \alpha J$. The gauge potentials are chosen such that $a_1 = a_2 = a_3 = 2\pi\Phi/(3\Phi_0)$. Here the phase of island B, $\phi_B = \phi$ and $\phi_C = \phi'$.

The lowest Josephson energy configurations of this system are given by $(\phi_1,\phi_2) = (\phi^* + 2m\pi, -\phi^* + 2n\pi), (-\phi^* + 2m\pi, \phi^* + 2n\pi)$ where $\phi^* = \cos^{-1}\frac{1}{2\alpha}$ [32]. In the geometry considered, ignoring the large capacitances $C_A^{-1} = C_C^{-1} = 0$, the capacitance term $\dot{\phi}^2/2C$ can cause tunneling between the two minima [36]. For the case $\alpha > 1$, and starting at $(\phi^*, -\phi^*)$, there are two equivalent-in-energy low-barrier tunneling paths to two equivalent points $(2\pi - \phi^*, \phi^*)$ and $(-\phi^*, -2\pi + \phi^*)$ which are related to each other by the symmetry $\phi_1 \leftrightarrow -\phi_2$.

The total tunneling matrix element using the instanton approach [36] is now given by the imaginary time action $S_{E,j}$ for tunneling path j as

$$\Gamma = \sum_{j=1,2}\omega_j e^{-S_{E,j}}, \tag{9}$$

where ω_j is the attempt frequency. The contributions to the tunneling amplitudes from the two paths are identical by symmetry except for the term $\dot{\phi}(V_B^{\text{ext}} - \bar{d}_{mB}d_{mB}/2)$ which creates a difference in the two paths of

$$S_{E,1} - S_{E,2} = 2\imath\pi(Q - n_B/2), \tag{10}$$

where $n_B = \sum_m \bar{d}_{mB} d_{mB}$ is the total number of fermions on island B. This leads to an interferometric dependence

$$|\Gamma| = \sqrt{2}\Gamma_0 \cos \pi(Q - n_B/2), \tag{11}$$

and Γ_0 is the single path tunneling amplitude. The interference effect between the two paths may be interpreted as a flux tunneling around the charge and has been referred to as an Aharanov-Casher effect [36]. Tuning $Q = 0$, this leads to a splitting for $n_0 \pmod 2 = 1$ and no splitting for $n_0 \pmod 2 = 0$.

The magnitude of the splitting Δ, which contains information about the Aharanov-Casher phase, can be measured by applying a flux of $\Phi = \Phi_0/2$. In this case the two minimum Josephson energy configurations are degenerate, and any splitting that is measured is a result of the Aharanov-Casher phase.

VIII. CONCLUSION

We show that a network of semiconductor quantum wires in the vicinity of an s-wave superconductor allows universal quantum computation. To do this, we propose a scheme to generate entanglement between two topological qubits in the wire network with the assistance of a flux qubit. We also show that the wave functions of the Majorana fermion states at the end points of the topological segments *do not oscillate* in the adjoining nontopological segments, even though they have the familiar oscillatory behavior [31,35] in the topological segments. This fact can be used to create arbitrary single-qubit phase gates by controlled overlap of the Majorana wave functions *via the nontopological segments* of the wire network. Our schemes for deterministically generating two-qubit entanglement and arbitrary single-qubit phase gates establish the semiconductor wire network as a viable platform for universal quantum computation.

Note added in proof. Recently, we became aware of the appendix in Ref. [33] where some related issues have been discussed.

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