Quantum correlations in a clusterlike system

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(Received 15 September 2010; published 18 November 2010)

We discuss a clusterlike one-dimensional system with triplet interaction. We study the topological properties of this system. We find that the degeneracy depends on the topology of the system and is well protected against external local perturbations. All these facts show that the system is topologically ordered. We also find a string order parameter to characterize the quantum phase transition. Besides, we investigate two-site correlations including entanglement, quantum discord, and mutual information. We study the different divergence behaviors of the correlations. The quantum correlation decays exponentially in both topological and magnetic phases, and diverges in reversed power law at the critical point. And we find that in topological order systems, the global difference of topology induced by dimension can be reflected in local quantum correlations.

DOI: 10.1103/PhysRevA.82.052320 PACS number(s): 64.70.Tg, 03.67.Mn

I. INTRODUCTION

Nowadays, quantum correlation has been attracting much attention since it plays a crucial role in quantum computation and quantum information. Entanglement, as an important quantum resource, takes responsibility for most quantum-information tasks such as quantum teleportation and computation [1]. But entanglement is fragile in open systems. Environment-induced decoherence destroys entanglement correlation in a short time, which makes quantum task difficult for implementation.

However, recent research shows that entanglement may not be the only worker carrying on quantum tasks. The quantum correlation without entanglement may also take effect in some scenes, e.g., the quantum computation with mixed states plus one pure qubit (DQC1) [2–4].

Quantum discord has been developed for the measure of "quantumness" of a pairwise correlation [5]. It makes it clear that entanglement is one kind of nonclassical correlation but not the only one. The quantum discord of some separable states is also nonzero. It may be used as a powerful tool to study quantum correlations.

Lots of work has been devoted to the study of correlations in different processes, such as decoherence and quantum phase transition [6–13]. The entanglement of formation [14] does not behave as smoothly as the correlation functions, and shows sudden death and rebirth in some scenes [15], which behavior has attracted more and more researchers. Quantum discord is pointed out to signal the quantum phase transition [8] such as fidelity [16], while our previous work also finds that in a topological quantum phase transition (TQPT) the local correlations are classical and the quantum correlation hides in the global system [9].

Topological order is a new kind of order beyond the conventional symmetry-breaking theory. In a topological order system, the degeneracy of the ground-state space depends on the topology of the system configuration, and the degenerate ground-state space is well protected against local perturbation. Such properties can be used for fault-tolerant computation [17–19]. Another idea is measurement-based computation, in which a *cluster state* is prepared and measured as the computation process [20]. There are deep relationships between these two methods of computation.

In this paper, we study pairwise correlations in a one-dimensional (1D) clusterlike system with triplet interaction, which can be implemented in optical lattice [21]. We discuss the properties of the topological order in the system, such as boundary-dependent degeneracy and topological protection. We find the *string order parameter* (SOP) by the method of duality mapping [22,23] to characterize TQPT.

Furthermore, the system can be decomposed as two independent chains of odd and even sites, namely, the spins on sites i are independent of spins on sites $i+(2n+1)$ where n is an integer, and we call this "bridge correlated."

The divergence of quantum discord with the distance of two sites is studied. We find that it behaves much like the correlation functions, i.e., it decays exponentially in both topological and magnetic phase areas and diverges in reversed power law at the critical points.

Moreover, the study of quantum discord and entanglement shows that the local quantum correlation of two sites is suppressed in the topological phase area. This is different from the study in two-dimensional (2D) TQPT [9], in which local quantum correlations vanish completely. And that means that in TQPT systems the global difference of the topology caused by dimension can be reflected in the local quantum correlations.

The paper is organized as follows. In Sec. II, we show the basic model of this clusterlike system. We discuss topological properties such as the degeneracy of the ground state space and topological protection. In Sec. III, we study the quantum correlation of this model. We investigate quantum discord, and mutual information and their divergence behaviors. Finally, we draw a summary in Sec. V.

II. TOPOLOGICAL PROPERTIES OF CLUSTERLIKE SYSTEM

In this section, we introduce the 1D clusterlike system originally proposed for quantum computation in optical lattice

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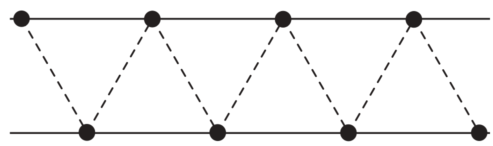


FIG. 1. Configuration of the system in optical lattice implementation. Tunneling between the nearest three sites (black points) in a triangle gives rise to the triplet interaction term.

[21]. We calculate the basic property of the low-energy spectrum. In a 1D world, there are only a few kinds of topologies that we are interested in, specifically, the open string and the closed loop, which correspond to open and periodic boundary conditions, respectively. We show that the degeneracy of the ground-state space is different in these two cases. Besides, the degeneracy is immune to local perturbation. These are the typical characteristics of topological order. And we find the SOP to characterize the quantum phase transition.

A. Model of clusterlike system

Here, we describe the model we discussed. The Hamiltonian of the system is described as follows:

$$H = -J\sum_i \left(\sigma_{i-1}^x\sigma_i^z\sigma_{i+1}^x + B\sigma_i^z\right) \equiv -J\sum_i \left(S_i + B\sigma_i^z\right), \tag{1}$$

where $J > 0$, σ_i^α are the Pauli matrices acting on the ith site and $S_i = \sigma_{i-1}^x\sigma_i^z\sigma_{i+1}^x$. Notice that S_i's commute with each other.

The model is originally proposed in Ref. [21] for quantum computation. It can be implemented in optical lattice. Atoms are arranged in a triangle lattice as shown in Fig. 1. Tunneling happens in the nearest three sites, which gives rise to the triplet interaction term. The one-body term can be adjusted by the Zeeman effect and appropriate laser field.

When $B = 0$, the ground-state space is the common eigenspace of S_i's that satisfies $S_i|\text{g.s.}\rangle = |\text{g.s.}\rangle$, which is a typically clusterlike space [24]. Cluster states are a kind of graph state, which plays an essential role in measurement-based quantum computation [20].

Here we analyze the low-energy spectrum of the system Eq. (1) in the stabilizer scheme [19,25]. All S_i's commute with each other, so we can treat the ground-state space of the system as the protected space of a set of independent *stabilizer generators* $\{S_i\}$.

In the stabilizer scheme, we have N qubits and K independent stabilizer generators which are the products of Pauli operator σ_i^α. The stabilizer generators commute with each other, and the common eigenspace of the stabilizers satisfying $S_i|\Phi\rangle = |\Phi\rangle$ is called the *protected space*, whose dimension is just 2^{N-K}, i.e., the stabilizers encode $N - K$ logical qubits in the protected space.

Assume we have N sites in the system Eq. (1); when $B = 0$, it is easy to see that the number of S_i's is N under the periodic boundary condition, and $N - 2$ under the open boundary condition. The dimensions of the protected space of $\{S_i\}$ are $2^0 = 1$ and $2^2 = 4$, respectively. That is to say, the ground-state space of the system Eq. (1) is nondegenerate under the loop boundary condition, and fourfold degenerate when it is opened, with the absence of an external field term.

Another important property of the system is that all the local correlation functions, except those composed of products of several S_i's, are zero. As an example, for $\hat{o} = \sigma_i^\alpha\sigma_j^\beta$, we can always find a certain S_k which anticommutes with $\hat{o}$, so that $\langle\hat{o}\rangle = \langle\hat{o}S_k\rangle = -\langle S_k\hat{o}\rangle = -\langle\hat{o}\rangle = 0$. And the system possesses Z_2 symmetry, i.e., $[H, \prod\sigma_i^z] = 0$. These are important properties as we will see below.

When $B \neq 0$, by means of the Jordan-Wigner transformation

$$\begin{aligned}\sigma_i^x &= (c_i^\dagger + c_i)\prod_{j<i}(1 - 2c_j^\dagger c_j),\\ \sigma_i^z &= 2c_i^\dagger c_i - 1,\end{aligned} \tag{2}$$

we can transform the system Eq. (1) into a fermion chain,

$$-H/J = \sum_i (c_{i-1} - c_{i-1}^\dagger)(c_{i+1}^\dagger + c_{i+1}) + B(2c_i^\dagger c_i - 1). \tag{3}$$

We can see that the system can be regarded as two independent chains containing odd and even sites, respectively.

Under the periodic boundary condition, the system possesses translational invariance. So it can be diagonalized in Fourier representation

$$-H/J = \sum_k e^{-2ik}(a_k - a_{-k}^\dagger)(a_{-k} + a_k^\dagger) + B(2a_k^\dagger a_k - 1), \tag{4}$$

where $c_n = \sum_k e^{ikn}a_k/\sqrt{N}$. By using Bogoliubov transformation, we get the diagonalized Hamiltonian

$$H/J = \sum_k \epsilon_k(2\gamma_k^\dagger\gamma_k - 1), \tag{5}$$

where $\epsilon_k = (1 + B^2 - 2B\cos 2k)^{\frac{1}{2}}$, $a_k = \cos\theta_k\gamma_k + i\sin\theta_k\gamma_{-k}^\dagger$, and $\tan 2\theta_k = \sin 2k/(B - \cos 2k)$.

When the string is opened, it is difficult to get the low-energy spectrum, and we will discuss the degeneracy of the ground-state space by the perturbation method in the following section.

B. Topologically protected degeneracy

When the string is opened, the ground space becomes fourfold degenerate when $B = 0$, as mentioned above. Actually each independent chain contributes two states. In this section, we show that this degeneracy is protected against external local perturbations. More precisely, the energy splitting caused by perturbation tends to zero in the thermodynamical limit.

As the string is opened, Fourier transformation does not take effect. We calculate the splitting of the ground-state energy. Assume the external field is absent at the time $t \to -\infty$, and adiabatically switched on. That is, we construct a new Hamiltonian with time-dependent external field $\lambda(t) = e^{-|\eta|t}$, where η is infinitely small. At $t = 0$, the system comes back to Eq. (1). That is,

$$H(t) = H_0 + \lambda(t)H'. \tag{6}$$

Since $\lambda(t)$ is switched on adiabatically, the system evolves from the clusterlike ground state $|\Phi_0\rangle$ at $t \to -\infty$ to an eigenstate $|\mathsf{G}\rangle$ of Eq. (1) at $t=0$, which should be one of the split ground states [26,27]. The average energy of the state $|\mathsf{G}\rangle$ is

$$\langle\mathsf{G}|H(t=0)|\mathsf{G}\rangle = \langle\Phi_0|U^\dagger(0,-\infty)(H_0+H')U(0,-\infty)|\Phi_0\rangle. \quad (7)$$

$U(0,-\infty)=\mathsf{T}\exp[-i\int_{-\infty}^{0}H'^I(t)\,dt]$ is the time-ordered evolution operator, expanded as

$$U(0,-\infty)=\mathbf{1}+(-i)\int_{-\infty}^{0}dt H'^I(t)+(-i)^2\int_{-\infty}^{0}dt_1 \times\int_{-\infty}^{t_1}dt_2 H'^I(t_1)H'^I(t_2)+\cdots, \quad (8)$$

where the perturbation term in the interaction picture is

$$H'^I(t)=\lambda(t)e^{iH_0t}H'e^{-iH_0t} = \lambda(t)\sum e^{-iJt(S_{i-1}+S_{i+1})}\sigma_i^z e^{iJt(S_{i-1}+S_{i+1})}, \quad (9)$$

ignoring the boundary terms without loss of generality. As $e^{-iJtS_i}=\cos Jt - i\sin JtS_i$, we can see that the inner product in Eq. (7) is composed of the sum of the multipoint correlation functions, which all vanish until the Nth order, according to what we saw in the last section. In the Nth term, global terms such as $\langle\prod\sigma_i^z\rangle$ appear and take effect. We can interpret it as a virtual particle running along the whole string. Therefore, the energy splitting of the ground-state space is $\sim\exp(-1/L)$, where L is the length scale of the system. In the thermodynamical limit, $L\to\infty$, the degeneracy is perfectly protected, like the case in toric code [17].

As we see, degeneracy emerges when the loop is opened. Besides, the degeneracy is immune against local perturbation when it is not too strong. These properties show that the system is a topologically ordered system. We can see that there is a close relationship between the clusterlike system and topological order. Here we regard both the topology-related degeneracy and topological protection as the essential character of topological order.

C. String order parameter

Topological order is an unconventional phase that cannot be described by the symmetry-breaking of local order parameters [28]. When $|B|\to\infty$, the system leaves the topological order and goes to a magnetized phase through quantum phase transition. We can find some global string order parameter to characterize the phase transition. Below, we show how to find the SOP by duality transformation [22,23].

Under the periodic boundary condition, we make such duality transformation below to represent the system by another self-consistent Pauli algebra $\{\mu_i^\alpha\}$,

$$\sigma_i^z=\mu_i^x\mu_{i+1}^x, \qquad \sigma_i^x=\prod_{j\leqslant i}\mu_j^z. \quad (10)$$

The system turns to be an XY model,

$$-H/J=\sum_i -\mu_i^y\mu_{i+1}^y+B\mu_i^x\mu_{i+1}^x. \quad (11)$$

Further, let $\mu_i^x=\tau_i^x\tau_{i+1}^x$ and $\mu_i^y=\prod_{j\leqslant i}\tau_j^z$, the system can be mapped to the Ising model in an external field,

$$-H/J=\sum_i -\tau_{i+1}^z+B\tau_i^x\tau_{i+2}^x. \quad (12)$$

We can also see that the system is actually composed of two independent chains. Combining the two transformations together, we can actually see it is

$$\sigma_i^z=\tau_i^x\tau_{i+2}^x, \quad \sigma_{i-1}^x\sigma_i^z\sigma_{i+1}^x=\tau_{i+1}^z. \quad (13)$$

The three nearest sites in a triangle (see Fig. 1) make up a new site in the dual lattice. The regular triangles and the inverted ones construct two independent Ising chains. τ_i^z can be seen as the observable that measures the "vortex" of the ith triangle site, clockwise or counterclockwise.

Lots of work has been devoted to discussing the quantum phase transition of the Ising model. There is a long-range order in the dual system [29]. When $|B|\geqslant 1$, we have

$$\lim_{j\to\infty}\langle\tau_0^x\tau_{2j}^x\rangle=\langle\tau_{2j}^x\rangle^2\sim[1-1/B^2]^{\frac{1}{4}}, \quad (14)$$

while it vanishes when $|B|<1$. τ_{2j}^x can be regarded as the order parameter characterizing the phase transition at $B=\pm1$. In the original spin representation, we can get the hidden SOP as

$$\Delta_{\text{even(odd)}}=\prod_i\sigma_{2i(+1)}^z. \quad (15)$$

Note that we are treating two independent Ising chains.

When the loop is opened, some boundary terms appear whose effects can be neglected in the thermodynamical limit. The physics does not change.

Here we emphasize that the existence of SOP is not the sufficient condition of topological order. As we see, we can also get SOP in the XY model of Eq. (11), i.e., $\tau_0^x\tau_{2j}^x=\prod_{i=0}^{2j-1}\mu_i^x$, by the duality map, which is a conventional symmetry-breaking system that has been studied so much. However, duality mapping is a useful tool to help us find the nonlocal order in a topological order system.

III. PAIRWISE CORRELATIONS

In this section, we study the pairwise correlations in our clusterlike system, such as quantum discord and entanglement of formation (EoF). Quantum discord is used as a measure for the "quantumness" of a pairwise state. Something interesting is found. We find that the quantum correlations are greatly suppressed in the topological order area compared with the magnetic polarized phase. The quantum discord decays exponentially as the distance of the two spins increases when $|B|\neq1$, and diverges in reverse power law at critical points, in the behavior exactly like the two-point correlation function. Only the EoF of the next-nearest spins is nontrivial, while that of spins farther from each other vanishes.

A. Entanglement, mutual information, and quantum discord

Entanglement, as the most important quantum resource, has been discussed a lot, and there are many different kinds of measures. One of the most sophisticated is the entanglement of formation [30], which is an entanglement measure defined for bipartite quantum states as

$$E(\rho) \equiv \min_{\{p_i,|\psi_i\rangle\}} \left[\sum_i p_i S^E(|\psi_i\rangle) \right], \tag{16}$$

where ρ is the density matrix of the bipartite states and $\{p_i,|\psi_i\rangle\}$ satisfies the condition that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. $|\psi_i\rangle$ is a bipartite pure state and $S^E(\cdot)$ gives the Von Neumann entropy of the reduced density matrix of $|\psi_i\rangle\langle\psi_i|$. For pure states, this quantity reduces to the entropy of entanglement. For a two-qubit system, fortunately, EoF can be express with concurrence C [14] as

$$E(\rho) = -f(C)\log_2 f(C) - [1 - f(C)]\log_2[1 - f(C)], \tag{17}$$

where $f(C) = (1 + \sqrt{1 - C^2})/2$. The concurrence $C = \max[0,\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4]$ and λ_i are the square roots of the eigenvalues of the matrix $\rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$.

Mutual information [25] quantifies the amount of common information shared by two subsystems. The classical mutual information is

$$I(A:B) = H(A) + H(B) - H(AB) = H(A) - H(A|B),$$

where $H(\cdot)$ is the Shannon information and $H(A|B)$ is the conditional information, which means the average information of A we gain when knowing the result of B. A natural generalized quantum version is obtained by changing the Shannon information to Von Neumann entropy,

$$\mathcal{I}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}). \tag{18}$$

Another generalization follows by giving the quantum measurement version of conditional entropy. The conditional entropy implies a measurement on B to get the information about A. So we impose projective measurement $\{\hat{\Pi}_i^B\}$ on B and collect the information,

$$\mathcal{J}\left(\rho^{AB} : \{\hat{\Pi}_i^B\}\right) = S(\rho^A) - \sum_i p_i S\left(\hat{\Pi}_i^B \rho^{AB} \hat{\Pi}_i^B / p_i\right), \tag{19}$$

where $p_i = \mathrm{Tr}[\hat{\Pi}_i^B \rho^{AB} \hat{\Pi}_i^B]$.

Quantum discord is defined as the minimum of the difference of $\mathcal{I}$ and $\mathcal{J}$ [5],

$$D(\rho^{AB}) = \min\left[\mathcal{I}(\rho^{AB}) - \mathcal{J}\left(\rho^{AB} : \{\hat{\Pi}_i^B\}\right)\right]. \tag{20}$$

Due to its power in mixed-state quantum computation [2–4], it has been discussed a lot recently.

Quantum discord can be used as a measurement for the quantumness of the bipartite correlation. It clarifies that entanglement is not the only "quantum" character of states. For example, for a separable state $\rho = |00\rangle\langle 00|/2 + |++\rangle\langle ++|/2$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, the quantum discord is not zero, which means ρ contains nonclassical correlation.

B. Pairwise state

To study the correlations in the system, we should first get the state of two spins, i.e., their reduced density matrix. The pairwise density matrix can be decomposed by a set of basis $\{\frac{1}{2}\sigma_i^\mu \sigma_j^\nu\}$, where μ and ν takes $0,\ldots,3$ and $\sigma_i^0 = \mathbf{1}$. It can be easily checked that $\{\frac{1}{2}\sigma_i^\mu \sigma_j^\nu\}$ is orthonormal under the Hilbert-Schmidt inner product $(A,B)_{\mathrm{H-S}} \equiv \mathrm{tr}(A^\dagger B)$ [9,25,31]. The reduced density matrix of two spins can be written as

$$\rho_{ij} = \frac{1}{4}\sum_{\mu\nu} \langle \sigma_i^\mu \sigma_j^\nu \rangle \sigma_i^\mu \sigma_j^\nu, \tag{21}$$

where $\langle \sigma_i^\mu \sigma_j^\nu \rangle = \mathrm{Tr}(\rho_{\mathrm{G}} \sigma_i^\mu \sigma_j^\nu) = \mathrm{tr}(\rho_{ij} \sigma_i^\mu \sigma_j^\nu)$ is the Hilbert-Schmidt inner product of ρ_{ij} and $\sigma_i^\mu \sigma_j^\nu$.

Because of the Z_2 symmetry of the system mentioned before, most terms in Eq. (21) can be eliminated except those of $\mathbf{1}_{ij}$, $\sigma_i^\mu \sigma_j^\mu$, $\sigma_i^z \otimes \mathbf{1}_j$, and $\mathbf{1}_i \otimes \sigma_j^z$. So we just need to calculate the expectation value of $\langle \sigma_i^\mu \sigma_{i+R}^\mu \rangle$ and $\langle \sigma^z \rangle$. Since the system can be treated as two independent fermion chains like Eq. (3), it can be seen that $\langle \sigma_i^\mu \sigma_{i+R}^\mu \rangle$ is zero when R is odd.

From the Jordan-Wigner transformation of Eq. (2), we can get $\langle \sigma^z \rangle$ and $\langle \sigma_0^\mu \sigma_R^\mu \rangle$ directly (we take $i = 0$ without loss of generality).

$$\begin{aligned}
\langle \sigma^z \rangle &= \langle (c_0 - c_0^\dagger)(c_0 + c_0^\dagger) \rangle = \langle A_0 B_0 \rangle, \\
\langle \sigma_0^z \sigma_R^z \rangle &= \langle (c_0 - c_0^\dagger)(c_0 + c_0^\dagger)(c_R - c_R^\dagger)(c_R + c_R^\dagger) \rangle \\
&= \langle A_0 B_0 A_R B_R \rangle, \\
\langle \sigma_0^x \sigma_R^x \rangle &= \langle (c_0 - c_0^\dagger)(c_1 + c_1^\dagger)(c_1 - c_1^\dagger)(c_2 + c_2^\dagger) \\
&\qquad \ldots (c_{R-1} - c_{R-1}^\dagger)(c_R + c_R^\dagger) \rangle \\
&= \langle A_0 B_1 A_1 B_2 \ldots A_{R-1} B_R \rangle, \\
\langle \sigma_0^y \sigma_R^y \rangle &= (-1)^{R-1} \langle B_0 A_1 B_1 A_2 \ldots B_{R-1} A_R \rangle,
\end{aligned}$$

where $A_i = c_i - c_i^\dagger$ and $B_i = c_i + c_i^\dagger$. We can check that $\langle A_0 A_i \rangle = \langle B_0 B_i \rangle = 0$ when $i \neq 0$, and the complicated expression in the above brackets can be handled with the help of the Wick theorem [10,32]. Let $G_{j-i} = \langle A_i B_j \rangle$, we have

$$\begin{aligned}
\langle \sigma^z \rangle &= G_0, \\
\langle \sigma_0^z \sigma_R^z \rangle &= G_0^2 - G_R G_{-R}, \\
\langle \sigma_0^x \sigma_R^x \rangle &= \begin{vmatrix} G_{-1} & G_{-2} & \ldots & G_{-R} \\ G_0 & G_{-1} & \ldots & G_{-(R-1)} \\ \vdots & \vdots & \ddots & \vdots \\ G_{R-2} & G_{R-3} & \ldots & G_{-1} \end{vmatrix}, \\
\langle \sigma_0^y \sigma_R^y \rangle &= \begin{vmatrix} G_1 & G_0 & \ldots & G_{-(R-2)} \\ G_2 & G_1 & \ldots & G_{-(R-3)} \\ \vdots & \vdots & \ddots & \vdots \\ G_R & G_{R-1} & \ldots & G_1 \end{vmatrix}.
\end{aligned}$$

And we have

$$\begin{aligned}
G_R &= \langle A_0 B_R \rangle = \langle (c_0 - c_0^\dagger)(c_R + c_R^\dagger) \rangle \\
&= \frac{1}{N} \sum_{p,q=-N/2}^{N/2} e^{i\frac{2\pi q}{N}R} e^{i(\theta_p + \theta_q)} \langle (\gamma_p^\dagger - \gamma_{-p})(\gamma_{-q}^\dagger + \gamma_q) \rangle \\
&= -\frac{1}{N} \sum_p e^{i\frac{2\pi p}{N}R} e^{2i\theta_p} \\
&\overset{N\to\infty}{=} -\frac{1}{4\pi} \int_{-2\pi}^{2\pi} dr \frac{e^{\frac{i}{2}Rr}(B - e^{-ir})}{(1 + B^2 - 2B\cos r)^{1/2}}.
\end{aligned}$$

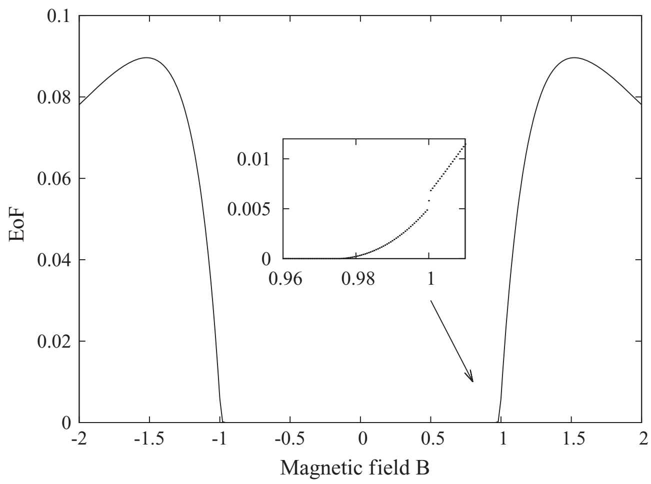


FIG. 2. EoF as a function of magnetic field strength B. Entanglement is born at $B_E \simeq 0.9767$, and a sudden change happens at the critical point.

Now we can substitute the expressions of correlation functions above into Eq. (21). Thus we have the reduced density matrix of any two spins in the system. Below, we will discuss the pairwise correlations in the system.

C. Local correlations in quantum phase transition

Now we discuss the correlations in the system. First, we calculate the EoF of two local spins. As we know, the nearest two spins are irrelevant. We give the EoF of the next-nearest spins shown in Fig. 2. In fact, numerical results show that the EoF of the two spins, whose distance is further than 2, is zero.

In Sec. II, we noted that the quantum phase transition can be characterized by SOP deduced from duality mapping, and the critical point lies at $B = \pm 1$. We can see that the EoF in "most" of the topological order area is zero and behaves like an order parameter, which is similar to the logarithmic negativity in previous work [21]. However, the EoF is born before reaching $B = \pm 1$, at the point around $|B| \simeq 0.9767$. There is a tiny "gap" at the critical point, which results from the finite scale, and it would decrease to a singular point in the thermodynamical limit.

We cannot treat EoF as an order parameter. However, it tells us that in the topological order area, local bipartite entanglement, as an important quantum correlation, is greatly suppressed. It invokes us to study the total quantum correlations in this area.

Second, we calculate the quantum discord of two spins with distance R in different magnetic field (Fig. 3), where R is even. Around the point $B = 1$, quantum discord has a tiny gap similar to that of EoF. These behaviors are both rooted in the property of correlation functions and would become a singular point in the thermodynamical limit.

It was mentioned in Ref. [9] that in a 2D TQPT, local correlations are always classical and the quantum correlations hide in the whole lattice, which also happens in many other 2D TQPT systems. But things are different in 1D systems (see Fig. 3).

In 2D topological order systems, there often exist many different conservative string operators whose paths are topologically equivalent, and we can always find one that anticommutes with certain local observables. Therefore, most local correlation functions would be eliminated and the density matrix Eq. (21) would become diagonalized.

However, in 1D systems, degrees of freedom are restricted. There are not so many topologically equivalent conservative string operators as in 2D. The 1D systems do not possess such high symmetry as in 2D systems, and many local correlation functions survive. The quantum discord, which measures the quantumness of pairwise correlations, gives zero only at the cluster state when $B = 0$. Nevertheless, qualitatively speaking, we can see that quantum discord is still quite small in most of the topological order area compared with that in the area $|B| > 1$. And we say that the local quantum correlation is greatly suppressed in the topological phase area.

On the other hand, this means that in TQPT systems the global difference of topology induced by dimension is reflected in the local quantum correlations. The dimension constrains the topology of the system, and also the types of global conservative quantities. In systems with higher dimension like that in Ref. [9], the external field term breaks some

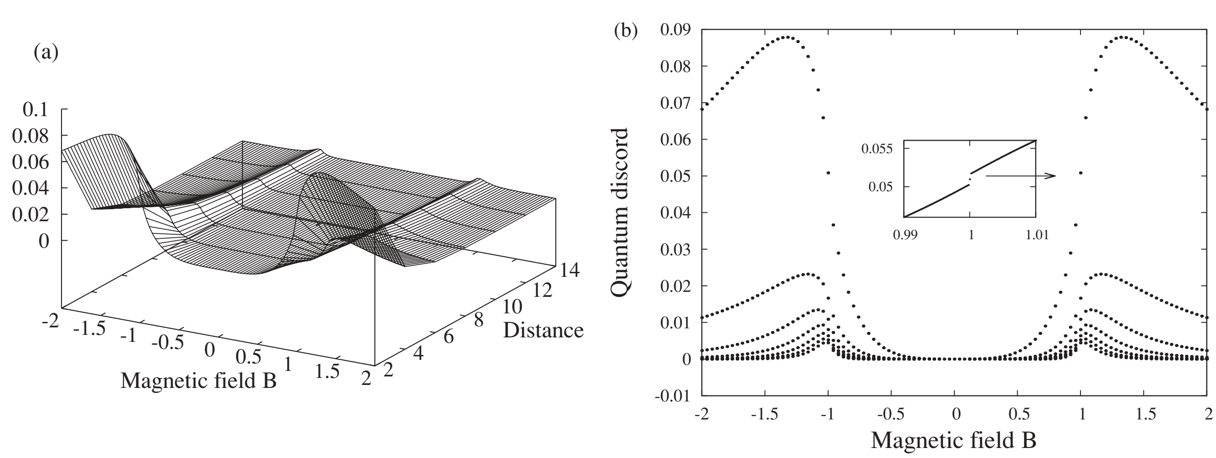
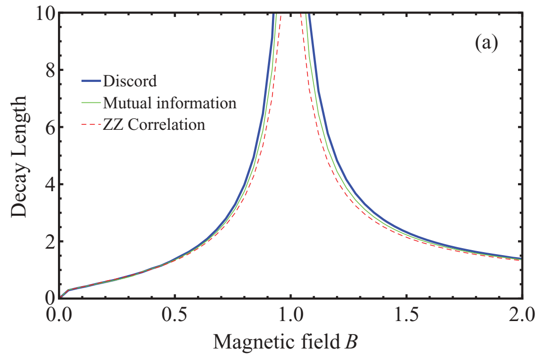


FIG. 3. Quantum discord vs B and R, where B is the magnetic field strength parameter and R is the site number which correlates with site 0.



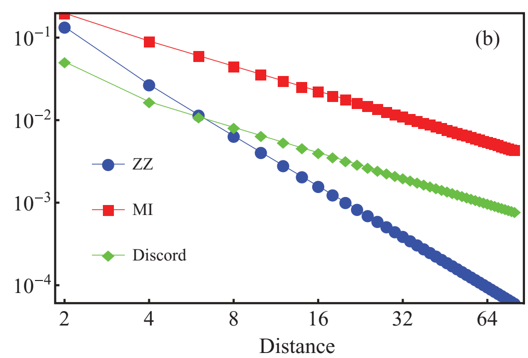


FIG. 4. (Color online) (a) Decay length of quantum correlations (quantum discord, mutual information, and the correlation function $\langle\sigma_0^z\sigma_R^z\rangle$) vs magnetic field strength B. (b) Decay behavior at $B = 1$. The correlations decay as $\sim R^{-\xi}$. We take R as even.

global conservative operators, while the survival ones are still capable to eliminate local quantum correlations. However, in 1D systems like what we study in this paper, there are not enough global conservative quantities left in the presence the magnetic field and the local quantum correlations are just suppressed. The survival of the local quantum correlation reflects the global restriction of the topology induced by dimension.

Besides, we are interested in the decay behavior of quantum discord along with the increase of the distance of the two spins we study. Numerical results show that the decay behaviors of quantum discord and total mutual information [Eq. (18)] are just similar to those of two-point correlation functions, i.e., they decay exponentially when $|B| \neq 1$ and with reversed power law at the critical points. This is different from the sudden change behavior of EoF, although EoF and quantum discord are defined in a similar way, namely, by finding the extreme. We show the exponential decay length of the correlations with the magnetic field strength B in Fig. 4.

At the critical points $B = \pm 1$, the correlations diverge as $\sim R^{-\xi}$. We show them in Fig. 4(b). For quantum discord, $\xi_D \simeq 1.0576$ and mutual information $\xi_M \simeq 1.0179$, and for the correlation function $\langle\sigma_0^z\sigma_R^z\rangle$, $\xi_{ZZ} \simeq 2.0464$. We guess this may relate to the universal scaling factor. When $B = 0$, the system is in the cluster state, and local quantum correlations vanish while quantum correlations still hide in the chain globally.

IV. SUMMARY

We investigate a special model whose Hamiltonian contains three-spin interactions. This model is composed of a cluster and a magnetic term, and we discuss the topological properties of this system. The degeneracy of the ground-state space differs in closed and open boundary conditions, and the degeneracy is topologically protected. We obtained the global SOP of this system by the method of duality mapping to characterize the TQPT.

Further, we discuss quantum correlations of this system. We calculate quantum discord, mutual information, and entanglement in this system. The EoF of two local spins is "dead" in most of the topological order area. Together with the study of quantum discord, we believe the quantum correlation is greatly suppressed in the topological order area. This is different from previous work in 2D TQPT [9], where local quantum correlations all vanish. We believe that is because 1D systems do not have the rich topology or high symmetry found in 2D systems.

On the other hand, in topological order systems, the dimension of the configuration constrains the topology of global conservative quantities. This global difference of topology induced by dimension can be reflected in the local quantum correlations. For example, the local quantum correlations survive in 1D TQPT systems, while they completely vanish in 2D, where there are more global conservative quantities left which are rooted in the richer topology of 2D systems.

Besides, we study the divergence behavior of the correlations. Quantum discord and mutual information diverge in reversed power law at the critical points and exponentially elsewhere. We believe more work can be done on the study of the universal scaling behavior of the divergence.

ACKNOWLEDGMENTS

The work is supported in part by the NSF of China Grants No. 10775116 and No. 11075138, and 973-Program Grant No. 2005CB724508.

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