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When different entanglement witnesses detect the same entangled states

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The question of under what conditions different witnesses (e.g., W_1, W_2) may detect some common entangled states [i.e., there exists some state ρ so that $\mathrm{Tr}(W_1\rho) < 0$ and $\mathrm{Tr}(W_2\rho) < 0$] is answered for both finite-dimensional and infinite-dimensional bipartite systems. Finitely many different witnesses $W_1, W_2, \ldots, W_n$ can detect some common entangled states if and only if $\sum_{i=1}^n d_i W_i$ is still a witness for any nonnegative numbers $d_1, d_2, \ldots, d_n$ with $\sum_{i=1}^n d_i = 1$; they cannot detect any common entangled state if and only if $\sum_{i=1}^n c_i W_i$ is a positive operator for some nonnegative numbers $c_1, c_2, \ldots, c_n$ with $\sum_{i=1}^n c_i = 1$. For two witnesses W_1 and W_2 more can be said. First, W_1 and W_2 can detect the same set of entangled states if and only if $W_1 = aW_2$ for some number $a > 0$. Second, W_2 can detect more entangled states than W_1 can if and only if $W_1 = aW_2 + D$ for some number $a > 0$ and a positive operator D. As an application, some characterizations of the optimal witnesses are given and some structural properties of the decomposable optimal witnesses are presented.

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I. INTRODUCTION

Entanglement plays a crucial role in quantum theory since it can be used as an essential resource in quantum information processing [1]. The detection of entanglement has attracted much attention in recent years. However, despite the remarkable progress in this field, there is no general qualitative or quantitative characterizing of entanglement [2–12].

Recall that the quantum states of a quantum system are described by density operators which are trace-one positive operators acting on the associated separable complex Hilbert space. A bipartite composite quantum system is associated with a tensor product of two separable complex Hilbert spaces H_i (i.e., $H = H_1 \otimes H_2$). By $\mathcal{S}^{(1)} = \mathcal{S}(H_1)$, $\mathcal{S}^{(2)} = \mathcal{S}(H_2)$, and $\mathcal{S} = \mathcal{S}(H_1 \otimes H_2)$ we denote the sets of all states on H_1, H_2, and $H_1 \otimes H_2$, respectively. A state $\rho \in \mathcal{S}$ is said to be *separable* if it is a trace-norm limit of the states of the form

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}, \quad \sum_i p_i = 1, \quad p_i \geqslant 0,$$

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ are states in $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$, respectively [13,14]. Otherwise ρ is said to be *entangled*. The set of all separable states will be denoted by $\mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2)$.

Among the multitudinous criteria for deciding whether a given state is entangled or not, the well-known one is the *entanglement witness criterion* [6]. This criterion provides a sufficient and necessary condition for the separability of a given state in a bipartite quantum system. In Ref. [6], it was shown that a given state is entangled if and only if there exists at least one entanglement witness detecting it. An *entanglement witness* (or *witness* for short) W is a self-adjoint operator (also called a Hermitian operator) acting on $H_1 \otimes H_2$ that satisfies $\mathrm{Tr}(W\sigma) \geqslant 0$ for all separable states $\sigma \in \mathcal{S}_{\mathrm{sep}}$ and $\mathrm{Tr}(W\rho) < 0$ for at least one entangled state ρ (in this case, we say that ρ is detected by W, or equivalently, W is a witness for ρ).

Although any entangled state can be detected by some specific choice of witness, there is no universal witness (i.e., there is no witness which can detect all entangled states). From the entanglement witness criterion, the task is reduced to find out all witnesses. However, constructing the witnesses for an entangled state is a hard task, and the determination of witnesses for all entangled states is a nondeterministic polynomial-time (NP) hard problem [15].

Witnesses cannot only be used to detect any entangled states, but also are directly measurable quantities. This makes the entanglement witnesses one of the main methods of detecting entanglement experimentally and a very useful tool for analyzing entanglement in the experiment. So it is important to know more about the features of the witnesses. Concerning this topic, much work has been done for finite-dimensional systems (for example, Refs. [16,17]). However, few results are known for infinite-dimensional systems. Generally, the structure of witnesses for infinite-dimensional systems are complicated. However, it was proved in Ref. [14] that for any entangled state a witness can be chosen so that it has a simple form of "nonnegative constant times the identity + a self-adjoint operator of finite rank." These kinds of witnesses are special Fredholm operators that are easily handled in mathematics. For example, such witnesses W have the spectrum consisting of finitely many eigenvalues. The goal of the present paper is to solve the question of when different witnesses can detect some common entangled states for mainly infinite-dimensional systems. Particularly, we deduce a sufficient and necessary condition that different witnesses can detect the same entangled states.

For simplicity, we introduce some notations. Let H_1, H_2 be complex Hilbert spaces and let $\mathcal{W} = \mathcal{W}(H_1 \otimes H_2)$ be the set

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of all entanglement witnesses of the system $H_1 \otimes H_2$

$$\mathcal{W} = \mathcal{W}(H_1 \otimes H_2) = \{W : W \in \mathcal{B}(H_1 \otimes H_2), W^\dagger = W, \\ \mathrm{Tr}(W\sigma) \geqslant 0 \quad \text{for all} \quad \sigma \in \mathcal{S}_{\rm sep} \quad \text{and} \quad W \text{ is not positive}\}.$$

For $W \in \mathcal{W}$ and $\Gamma \subset \mathcal{W}$ define

$$\mathcal{D}_W = \{\rho : \rho \in \mathcal{S}(H_1 \otimes H_2), \mathrm{Tr}(W\rho) < 0\},$$

and $\mathcal{D}_\Gamma = \bigcap_{W\in\Gamma} \mathcal{D}_W$. Then $\mathcal{D}_W$ and $\mathcal{D}_\Gamma$ are convex sets. Thus the witnesses in Γ can detect some common entangled states if and only if $\mathcal{D}_\Gamma \neq \emptyset$.

For $W_1, W_2 \in \mathcal{W}$, we say that W_1 is *finer* than W_2 if $\mathcal{D}_{W_2} \subset \mathcal{D}_{W_1}$, denoted by

$$W_2 \prec W_1,$$

which means that W_1 can recognize more entangled states than W_2 can. Thus, W_1 and W_2 can detect the same set of entangled states if and only if W_1 is finer than W_2 and W_2 is also finer than W_1, or equivalently, $\mathcal{D}_{W_2} = \mathcal{D}_{W_1}$. While a witness W is an *optimal* witness if there exists no other witness finer than W.

The finer relation "$\prec$" above is a partial order and $\mathcal{W}$ becomes a partially ordered set (poset) with respect to "$\prec$". If $W_1 \prec W_2$ or $W_2 \prec W_1$, we say that W_1 and W_2 are comparable. Otherwise, they are not comparable. Particularly, we say that W_1 and W_2 are equivalent if $W_1 \prec W_2$ and $W_2 \prec W_1$ hold simultaneously. Generally speaking, for two given witnesses W_1 and W_2 there are three different situations that may occur. (i) $W_1 \prec W_2$ or $W_2 \prec W_1$, and in particular, $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$; (ii) $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$ and $\mathcal{D}_{W_i}\mathcal{D}_{W_j}, i, j = 1{,}2$; and (iii) $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$.

Case (i) means that W_1 and W_2 are comparable, that is, W_2 (or W_1) can detect more entangled states than W_1 (or W_2) can, and in particular, W_1 and W_2 detect exactly the same entangled states if they are equivalent ($\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$). Case (ii) means that W_1 and W_2 are not comparable, but there exists some entangled state so that both W_1 and W_2 can recognize it. While case (iii) says that W_1, W_2 are not comparable, and there is no entangled state so that W_1 and W_2 can be detected simultaneously. Thus W_1 and W_2 can detect a common entangled state, that is, there exists a state ρ such that both $\mathrm{Tr}(W_1\rho) < 0$ and $\mathrm{Tr}(W_2\rho) < 0$, if and only if case (i) or (ii) holds.

For the finite-dimensional case, case (i) above was studied in Ref. [16] and cases (ii) and (iii) were studied in Ref. [17]. Suppose that $\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$, then the following conclusions are true. (1) $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$ if and only if $W_1 = (1-\varepsilon)W_2 + \varepsilon D$ for some $D \geqslant 0$ and $0 \leqslant \varepsilon < 1$; in particular, $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$ if and only if $W_1 = W_2$ [16]. (2) If there are no inclusion relations between $\mathcal{D}_{W_1}$ and $\mathcal{D}_{W_2}$, then $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$ if and only if $W = \epsilon W_1 + (1-\epsilon)W_2$ is not positive for any $0 \leqslant \epsilon \leqslant 1$ [17]. However, we remark that, though the result (2) above is true as we will show, there are some mistakes in the proof of it in Ref. [17]. Then an interesting and natural question arises: What can we say for the infinite-dimensional case? The main purpose of the present paper is to show that the similar results hold for the infinite-dimensional systems. Note that one of the difficulties of extending the above results to the infinite-dimensional systems is that the condition $\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$ makes no sense, in general, for the infinite-dimensional case. So, we have to discuss the question without the trace-equal assumption.

This paper is organized as follows. In Sec. II, we propose a sufficient and necessary condition for any two given general witnesses W_1 and W_2 to be comparable. Let H_1, H_2 be complex Hilbert spaces. Assume that $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$. We show that (1) $W_1 \prec W_2$ if and only if $W_1 = aW_2 + D$ for some operator $D \geqslant 0$ and some real number $a > 0$; (2) $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$ if and only if there exists a positive number $a > 0$ such that $W_1 = aW_2$. Then these results are applied in Sec. III to obtain a sufficient and necessary condition for a witness to be optimal. We show that $W \in \mathcal{W}(H_1 \otimes H_2)$ is optimal if and only if $W' = aW - D \notin \mathcal{W}(H_1 \otimes H_2)$ for any nonzero operator $D \geqslant 0$ and scalar $a > 0$. Some structural properties of the optimal decomposable witnesses are also presented. In Sec. IV, we discuss the question of when two witnesses that are not comparable can detect a common entangled state. Combining with the results in Sec. II, the question of when finitely many witnesses can detect a common entangled state is answered. We show that $\cap_{k=1}^n \mathcal{D}_{W_k} \neq \emptyset$ if and only if every convex combination of $W_1, \ldots, W_n$, that is, $\sum_{i=1}^n d_i W_i$ with $d_i \geqslant 0$ and $\sum_{i=1}^n d_i = 1$, is still a witness; $\cap_{k=1}^n \mathcal{D}_{W_k} = \emptyset$ if and only if there exists at least one convex combination W of $W_1, \ldots, W_n$ such that $W \geqslant 0$.

Throughout this paper we call an operator $A \in \mathcal{B}(H)$ as positive, denoted by $A \geqslant 0$, if $\langle x|A|x\rangle \geqslant 0$ for all $|x\rangle \in H$. $\|\cdot\|_{\rm Tr}$ denotes the trace norm and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm, that is, $\|A\|_{\rm Tr} = \mathrm{Tr}[(A^\dagger A)^{\frac{1}{2}}]$ and $\|A\|_2 = [\mathrm{Tr}(A^\dagger A)]^{\frac{1}{2}}$. For an operator A, A^T stands for the transpose of A with respect to some given orthonormal basis. By A^{T_2}, we denote the partial transpose of A with respect to the second subsystem H_2 if $A \in \mathcal{B}(H_1 \otimes H_2)$ [i.e., $A^{T_2} = (I_1 \otimes \tau)A$] where τ is the transpose operation. $\mathcal{T}(H_1 \otimes H_2)$ denotes the set of all trace class operators (the operators with finite trace norm) in $\mathcal{B}(H_1 \otimes H_2)$ while $\mathcal{T}^+(H_1 \otimes H_2)$ stands for the set of all positive elements in $\mathcal{T}(H_1 \otimes H_2)$.

II. COMPARABLE WITNESSES, WITNESSES DETECTING THE SAME ENTANGLED STATES

In this section, we mainly highlight the *finer* relation between two given witnesses W_1, W_2 of an *infinite-dimensional* bipartite system. Namely, we study the question of under what conditions W_2 detects more (or the same) entangled states than (as) W_1 does.

For the finite-dimensional bipartite quantum system, it is known that if W_1, $W_2 \in \mathcal{W}$ with $\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$, then $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$ if and only if $W_1 = (1-\varepsilon)W_2 + \varepsilon D$ for some $D \geqslant 0$ and $0 \leqslant \varepsilon < 1$; $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$ if and only if $W_1 = W_2$ [16]. Since the condition $\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$ does not make sense, in general, for the infinite-dimensional case, we have to consider the question without the trace-equal assumption.

The following is the main result of this section which answers the above question for both infinite-dimensional systems and finite-dimensional cases.

Theorem 2.1. Let H_1, H_2 be complex Hilbert spaces. Assume that $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$. Then (1) $W_1 \prec W_2$ if and only if $W_1 = aW_2 + D$ for some operator $D \geqslant 0$ and some real number $a > 0$. (2) $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$ if and only if there exists a number $a > 0$ such that $W_1 = aW_2$.

To prove Theorem 2.1, we need several lemmas.

A useful technical result in Ref. [16] asserts that, for finite-dimensional systems, every entanglement witness has a positive trace. This result does not make any sense for the infinite-dimensional cases. However, we can generalize it to the infinite-dimensional case by showing that the restriction of any entanglement witness is nonzero as a linear functional to the convex set consisting of separable states. This result is useful for our purpose.

Lemma 2.2. Let H_1, H_2 be complex Hilbert spaces. For any $W \in \mathcal{W}(H_1 \otimes H_2)$, there is a separable pure state $\sigma \in \mathcal{S}_{\rm sep}(H_1 \otimes H_2)$ such that $\mathrm{Tr}(W\sigma) > 0$.

Proof. Let $\{|i\rangle\}$ and $\{|j\rangle\}$ be any orthonormal bases of H_1 and H_2, respectively. Then, $\{|i\rangle|j\rangle\}$ is an orthonormal basis of $H_1 \otimes H_2$. It turns out $\langle i|\langle j|W|i\rangle|j\rangle \geqslant 0$ since $\langle i|\langle j|W|i\rangle|j\rangle = \mathrm{Tr}(W|i\rangle\langle i| \otimes |j\rangle\langle j|) \geqslant 0$ for any i, j.

To prove the lemma, it suffices to show that there exist orthonormal bases $\{|i\rangle\}$ and $\{|j\rangle\}$ such that $\mathrm{Tr}(W|i\rangle\langle i| \otimes |j\rangle\langle j|) \neq 0$ for some i, j. To get a contradiction assume that this is not true. Then

$$\langle\psi_1|\langle\psi_2|W|\psi_1\rangle|\psi_2\rangle = 0,$$

for all product vectors $|\psi_1\rangle|\psi_2\rangle \in H_1 \otimes H_2$. For any pure state $|\psi\rangle \in H_1 \otimes H_2$, let $|\psi\rangle = \sum_{k=1}^n \lambda_k|k\rangle|k'\rangle$ be the Schmidt decomposition of $|\psi\rangle$, where $\lambda_k > 0$, $\sum_{k=1}^n \lambda_k^2 = 1$ and $\{|k\rangle\}_{k=1}^n$, $\{|k'\rangle\}_{k'=1}^n$ are the orthonormal sets, respectively, in H_1, H_2; here n is called the Schmidt number of $|\psi\rangle$. Then

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \left(\sum_k \lambda_k|k\rangle|k'\rangle\right)\left(\sum_l \lambda_l\langle l|\langle l'|\right)\\ &= \sum_{k,l} \lambda_k\lambda_l|k\rangle\langle l| \otimes |k'\rangle\langle l'|\\ &= \sum_{k=l} \lambda_k^2|k\rangle\langle k| \otimes |k'\rangle\langle k'|\\ &\quad + \sum_{k<l} \lambda_k\lambda_l(|k\rangle\langle l| \otimes |k'\rangle\langle l'| + |l\rangle\langle k| \otimes |l'\rangle\langle k'|).\end{aligned}$$

For a given pair (k, l) with $k \neq l$, define $|\psi_{k,l}\rangle = \frac{1}{\sqrt{2}}(|k\rangle|k'\rangle + |l\rangle|l'\rangle)$. We have

$$\begin{aligned}&|k\rangle\langle l| \otimes |k'\rangle\langle l'| + |l\rangle\langle k| \otimes |l'\rangle\langle k'|\\ &\quad = 2|\psi_{k,l}\rangle\langle\psi_{k,l}| - |k\rangle\langle k| \otimes |k'\rangle\langle k'| - |l\rangle\langle l| \otimes |l'\rangle\langle l'|.\end{aligned}$$

This indicates that if $n < \infty$ then $\langle\psi|W|\psi\rangle = 0$. As the set of all unit vectors with the finite Schmidt number is dense in the set of all unit vectors in $H_1 \otimes H_2$, we see that $\langle\psi|W|\psi\rangle = 0$ holds for all unit vector $|\psi\rangle$ and hence $W = 0$, a contradiction. ■

Analogous to the finite-dimensional case [16], the following lemma is obvious.

Lemma 2.3. Let H_1, H_2 be complex Hilbert spaces. For a given $W \in \mathcal{W}(H_1 \otimes H_2)$, if $\rho \in \mathcal{D}_W$ and $\varrho_W \in \mathcal{T}^+(H_1 \otimes H_2)$ satisfying $\mathrm{Tr}(W\varrho_W) = 0$, then $(\rho + \varrho_W)/\mathrm{Tr}(\rho + \varrho_W) \in \mathcal{D}_W$.

The next lemma is crucial to prove Theorem 2.1. Note that the counterpart lemma in Ref. [16] for the finite-dimensional case is not valid for the infinite-dimensional case.

Lemma 2.4. Let H_1, H_2 be complex Hilbert spaces and W_1, $W_2 \in \mathcal{W}(H_1 \otimes H_2)$. Assume that $W_1 \prec W_2$ and let

$$\lambda := \inf_{\rho_1 \in \mathcal{D}_{W_1}} \frac{|\mathrm{Tr}(W_2\rho_1)|}{|\mathrm{Tr}(W_1\rho_1)|}.$$

Then the following statements are true:

1. If $\rho \in \mathcal{S}(H_1 \otimes H_2)$ satisfies $\mathrm{Tr}(W_1\rho) = 0$, then $\mathrm{Tr}(W_2\rho) \leqslant 0$.
2. $\lambda > 0$.
3. If $\rho \in \mathcal{S}(H_1 \otimes H_2)$ satisfies $\mathrm{Tr}(W_1\rho) > 0$, then $\mathrm{Tr}(W_2\rho) \leqslant \lambda\mathrm{Tr}(W_1\rho)$.

Proof. (1) Let us assume, to reach a contradiction, that $\mathrm{Tr}(W_2\rho) > 0$. Then, for any $\rho_1 \in \mathcal{D}_{W_1}$ and $a \geqslant 0$, we have $\rho(a) = (\rho_1 + a\rho)/(1 + a) \in \mathcal{D}_{W_1}$. On the other hand, there exists a positive number a_0 such that $\mathrm{Tr}[W_2\rho(a)] > 0$ holds for all $a \geqslant a_0$, which is impossible since it leads to $\rho(a) \notin \mathcal{D}_{W_2}$.

(2) Assume that, on the contrary, $\lambda = 0$. Then there exists a sequence $\{\rho_n\} \subset \mathcal{D}_{W_1}$ such that

$$\varepsilon_n = \frac{\mathrm{Tr}(W_2\rho_n)}{\mathrm{Tr}(W_1\rho_n)} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.1}$$

Note that there exists $\sigma \in \mathcal{S}_{\rm sep} = \mathcal{S}_{\rm sep}(H_1 \otimes H_2)$ such that both $\mathrm{Tr}(W_1\sigma)$ and $\mathrm{Tr}(W_2\sigma)$ are nonzero. If not, then for any $\sigma \in \mathcal{S}_{\rm sep}$, either $\mathrm{Tr}(W_1\sigma) = 0$ or $\mathrm{Tr}(W_2\sigma) = 0$. Thus, by Lemma 2.2, there exist $\sigma_1, \sigma_2 \in \mathcal{S}_{\rm sep}$ so that $\mathrm{Tr}(W_1\sigma_1) = t > 0$, $\mathrm{Tr}(W_1\sigma_2) = 0$, $\mathrm{Tr}(W_2\sigma_1) = 0$ and $\mathrm{Tr}(W_2\sigma_2) = s > 0$. Let $\sigma = \frac{s}{t+s}\sigma_1 + \frac{t}{t+s}\sigma_2 \in \mathcal{S}_{\rm sep}$. Then $\mathrm{Tr}(W_1\sigma) = \mathrm{Tr}(W_2\sigma) = \frac{ts}{t+s} \neq 0$, contradicting to the assumption.

Now we can take $\sigma \in \mathcal{S}_{\rm sep}$ so that both $\mathrm{Tr}(W_1\sigma)$ and $\mathrm{Tr}(W_2\sigma)$ are nonzero. Let

$$\tilde{\rho}_n = \frac{1}{1 - \frac{\mathrm{Tr}(W_1\rho_n)}{\mathrm{Tr}(W_1\sigma)}}\left[\rho_n - \frac{\mathrm{Tr}(W_1\rho_n)}{\mathrm{Tr}(W_1\sigma)}\sigma\right] \in \mathcal{S},$$

with ρ_n satisfying Eq. (2.1). Then $\mathrm{Tr}(W_1\tilde{\rho}_n) = 0$ and by (1), we have $\mathrm{Tr}(W_2\tilde{\rho}_n) \leqslant 0$ for every n. However,

$$\begin{aligned}\mathrm{Tr}(W_2\tilde{\rho}_n) &= \frac{1}{1 - \frac{\mathrm{Tr}(W_1\rho_n)}{\mathrm{Tr}(W_1\sigma)}}\left[\mathrm{Tr}(W_2\rho_n) - \frac{\mathrm{Tr}(W_1\rho_n)}{\mathrm{Tr}(W_1\sigma)}\mathrm{Tr}(W_2\sigma)\right]\\ &= \frac{1}{1 - \frac{\mathrm{Tr}(W_1\rho_n)}{\mathrm{Tr}(W_1\sigma)}}\left[\varepsilon_n - \frac{\mathrm{Tr}(W_2\sigma)}{\mathrm{Tr}(W_1\sigma)}\right]\mathrm{Tr}(W_1\rho_n),\end{aligned}$$

and $\varepsilon_n \to 0$, which implies that for sufficiently large n, we have $\varepsilon_n - \frac{\mathrm{Tr}(W_2\sigma)}{\mathrm{Tr}(W_1\sigma)} < 0$ and hence $\mathrm{Tr}(W_2\tilde{\rho}_n) > 0$, a contradiction. This completes the proof of (2).

(3) Assume that $\mathrm{Tr}(W_1\rho) > 0$. Take $\rho_1 \in \mathcal{D}_{W_1}$ and let $\tilde{\rho} = \frac{1}{\mathrm{Tr}(W_1\rho) - \mathrm{Tr}(W_1\rho_1)}[\mathrm{Tr}(W_1\rho)\rho_1 - \mathrm{Tr}(W_1\rho_1)\rho]$. Then we have $\mathrm{Tr}(W_1\tilde{\rho}) = 0$. By (1), we obtain that $\mathrm{Tr}(W_2\tilde{\rho}) \leqslant 0$. Thus we have $\mathrm{Tr}(W_1\rho)\mathrm{Tr}(W_2\rho_1) \leqslant \mathrm{Tr}(W_1\rho_1)\mathrm{Tr}(W_2\rho)$. It follows that

$$\frac{\mathrm{Tr}(W_2\rho)}{\mathrm{Tr}(W_1\rho)} \leqslant \frac{|\mathrm{Tr}(W_2\rho_1)|}{|\mathrm{Tr}(W_1\rho_1)|}.$$

Taking the infimum with respect to $\rho_1 \in \mathcal{D}_{W_1}$ on the right side of the above equation, we get $\mathrm{Tr}(W_2\rho) \leqslant \lambda\mathrm{Tr}(W_1\rho)$. ■

Now we are in a position to give our proof of Theorem 2.1. Note that the proof of statement (2) is also different from that of the counterpart result in Ref. [16] for the finite-dimensional case. The approach in Ref. [16] does not work for the infinite-dimensional case.

Proof of Theorem 2.1. (1) If $W_1 = aW_2 + D$ for some positive operator D and some scalar $a > 0$, then for any $\rho \in \mathcal{D}_{W_1}$, we have $a\mathrm{Tr}(W_2\rho) + \mathrm{Tr}(D\rho) = \mathrm{Tr}(W_1\rho) < 0$, which implies that $\mathrm{Tr}(W_2\rho) < 0$. Hence $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$. Conversely, assume that $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$. Then, by Lemma 2.4,

$$\mathrm{Tr}(W_2\rho) \leqslant \lambda \mathrm{Tr}(W_1\rho), \tag{2.2}$$

holds for all $\rho \in \mathcal{S}$, where $\lambda = \inf_{\rho_1 \in \mathcal{D}_{W_1}} \frac{|\mathrm{Tr}(W_2\rho_1)|}{|\mathrm{Tr}(W_1\rho_1)|} > 0$. This implies that $D_1 = \lambda W_1 - W_2 \geqslant 0$ and hence, with $D = \lambda^{-1}D_1$, $W_1 = \lambda^{-1}W_2 + D$, as desired.

(2) We only need to prove the "only if" part. Assume that $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$. Then, by the statement (1) just proved above, there exist operators $D_i \geqslant 0$ and scalars $a_i > 0$, $i = 1,2$, such that $W_1 = a_1 W_2 + D_1$ and $W_2 = a_2 W_1 + D_2$. It follows that $W_1 = a_1(a_2W_1 + D_2) + D_1 = a_1a_2W_1 + a_1D_2 + D_1$. Thus $(1 - a_1a_2)W_1 = a_1D_2 + D_1 \geqslant 0$. Since $W_1 \in \mathcal{W}$, we must have $a_1a_2 = 1$. Hence $D_1 = D_2 = 0$ and $W_2 = a_2W_1$, completing the proof. ■

III. OPTIMIZATION OF ENTANGLEMENT WITNESSES

In this section we discuss the optimization of entanglement witnesses, especially for infinite-dimensional systems by applying Theorem 2.1.

The following result states that a witness is optimal if and only if any negative permutation will break the witness. For the finite-dimensional case, a similar result was obtained in Ref. [16].

Theorem 3.1. Let H_1, H_2 be complex Hilbert spaces. Then $W \in \mathcal{W}(H_1 \otimes H_2)$ is optimal if and only if $W' = aW - D \notin \mathcal{W}(H_1 \otimes H_2)$ for any nonzero operator $D \geqslant 0$ and scalar $a > 0$.

Proof. To prove the "if" part assume, on the contrary, that W is not optimal, then $W \prec W'$ for some $W' \in \mathcal{W}(H_1 \otimes H_2)$ with W and W' are linearly independent. It follows from Theorem 2.1(1) that $W = aW' + D$ for some $D \geqslant 0$ and $a > 0$, which reveals that $W' = \frac{1}{a}W - \frac{1}{a}D$.

To prove the "only if" part assume that W is optimal, but there exists a nonzero operator $D \geqslant 0$, scalar $a > 0$ so that $W' = aW - D \in \mathcal{W}(H_1 \otimes H_2)$. Then $W = \frac{1}{a}W' + \frac{1}{a}D$ and W' is linearly independent to W. But by Theorem 2.1, $W \prec W'$, a contradiction. ■

In the following, we discuss the condition for an entanglement witness that cannot subtract some positive operators. For convenience, we define

$$\mathcal{P}_W = \{|\psi\rangle|\phi\rangle \in H_1 \otimes H_2 : \langle\psi|\langle\phi|W|\psi\rangle|\phi\rangle = 0\}. \tag{3.1}$$

Proposition 3.2. Let H_1, H_2 be complex Hilbert spaces and $W \in \mathcal{W}(H_1 \otimes H_2)$. Let $\mathcal{P}_W$ be as in Eq. (3.1). If $D \in \mathcal{B}(H_1 \otimes H_2)$ is positive and $D\mathcal{P}_W \neq \{0\}$, then $W - aD \notin \mathcal{W}(H_1 \otimes H_2)$ for any $a > 0$.

Proof. If $D\mathcal{P}_W \neq \{0\}$, then there exists a product vector $|\psi_0\rangle|\phi_0\rangle \in \mathcal{P}_W$ such that

$$\langle\psi_0|\langle\phi_0|D|\psi_0\rangle|\phi_0\rangle > 0.$$

Let $\rho_0 = |\psi_0\rangle\langle\psi_0| \otimes |\phi_0\rangle\langle\phi_0|$. It is clear that ρ_0 is separable and $\mathrm{Tr}[(W - aD)\rho_0] = -a\mathrm{Tr}(D\rho_0) < 0$, which leads to $W - aD \notin \mathcal{W}(H_1 \otimes H_2)$ for all $a > 0$. ■

The following corollary is obvious.

Corollary 3.3. Let H_1, H_2 be complex Hilbert spaces and $W \in \mathcal{W}(H_1 \otimes H_2)$. Let $\mathcal{P}_W$ be as in Eq. (3.1). If $\mathcal{P}_W$ spans $H_1 \otimes H_2$, then W is optimal.

Next we give some structural properties of optimal decomposable witnesses. Recall that a self-adjoint operator $A \in \mathcal{B}(H_1 \otimes H_2)$ is said to be *decomposable* if

$$A = P + Q^{T_2},$$

for some operators $P \geqslant 0$, $Q \geqslant 0$, where Q^{T_2} denotes the partial transpose of Q with respect to the second subsystem H_2. Otherwise, A is said to be *indecomposable*. For example, in the $n \times n$ system, the Hermitian swap operator $V = \sum_{i,j=0}^{n-1} |i\rangle\langle j| \otimes |j\rangle\langle i|$ is a decomposable witness since (1) $\mathrm{Tr}(V\sigma) \geqslant 0$ for all separable pure states σ, (2) V has a negative eigenvalue -1, and (3) $V = nQ^{T_2}$ with $Q = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}|i\rangle|i\rangle$ (Ref. [18]). The examples of indecomposable witnesses can be found in Refs. [14,19,20]. It is easy to show that the decomposable witnesses cannot detect any positive partial transpose (PPT) entangled states [21].

By applying Theorem 2.1, one can get a simple structural property of optimal decomposable entanglement witnesses for both the finite-dimensional systems and infinite-dimensional systems.

Theorem 3.4. Let H_1, H_2 be complex Hilbert spaces and $W \in \mathcal{W}(H_1 \otimes H_2)$ be a decomposable entanglement witness. If W is optimal, then $W = Q^{T_2}$ for some positive operator Q, and Q contains no product vectors in its range.

Proof. Since W is decomposable, $W = P + Q^{T_2}$ for some positive operators P, Q. Assume that $P \neq 0$. As $\mathrm{Tr}(Q^{T_2}\sigma) = \mathrm{Tr}(Q\sigma^{T_2}) \geqslant 0$ for all $\sigma \in \mathcal{S}_{\mathrm{sep}}$ and $W \in \mathcal{W}$, we must have $Q^{T_2} \in \mathcal{W}$. Thus, by Theorem 2.1 (1), one sees that $W \prec Q^{T_2}$, that is, W is not optimal. Hence, W is optimal implies that $P = 0$ and $W = Q^{T_2}$. Moreover, the range of Q contains no product vectors. In fact, if $|\psi\rangle|\phi\rangle \in R(Q)$ for some unit vectors $|\psi\rangle \in H_1$ and $|\phi\rangle \in H_2$, then there exists a vector $|\omega\rangle \in H_1 \otimes H_2$ such that $Q|\omega\rangle = |\psi\rangle \otimes |\phi\rangle$. Observe that $Q(I - \lambda|\omega\rangle\langle\omega|)Q = Q^2 - \lambda|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| \geqslant 0$ if and only if $I - \lambda|\omega\rangle\langle\omega| \geqslant 0$. It turns out that, for any $0 < \lambda < \||\omega\rangle\|^{-2}$ we have $[Q - \lambda|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|]^{T_2} \in \mathcal{W}$, which implies that $[Q - \lambda|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|]^{T_2}$ is finer than W, contradicting to the optimality of W. ■

Theorem 3.4 can be strengthened a little.

Theorem 3.5. Let H_1, H_2 be complex Hilbert spaces and $W \in \mathcal{W}(H_1 \otimes H_2)$ be a decomposable entanglement witness. If W is optimal, then $W = Q^{T_2}$ for some positive operator Q and there exists no positive operator A with $R(A) \subseteq R(Q)$ such that $A^{T_2} \geqslant 0$.

Proof. By Theorem 3.4, $W = Q^{T_2}$ as W is optimal. If there exists a positive operator A such that $R(A) \subseteq R(Q)$ and $A^{T_2} \geqslant 0$, then, by a well-known result from operator theory, there exists an operator $T \in \mathcal{B}(H_1 \otimes H_2)$ such that $A = QT$. It follows that $A^2 = QTT^\dagger Q \leqslant tQ^2$, where $t = \|T\|^2$. Thus, $A \leqslant \sqrt{t}Q$, which implies $Q - \lambda A \geqslant 0$ whenever $0 < \lambda < \frac{1}{\sqrt{t}}$. Thus we get $(Q - \lambda A)^{T_2} \in \mathcal{W}$. Now it follows from Theorem 2.1 (1) that $(Q - \lambda A)^{T_2}$ is finer than W, a contradiction. ■

Corollary 3.6. Let H_1, H_2 be complex Hilbert spaces and $W \in \mathcal{W}(H_1 \otimes H_2)$ be a decomposable entanglement witness. If W is optimal, then $W^{T_2} \notin \mathcal{W}$.

Proof. By Theorem 3.4, we know that $W = Q^{T_2}$ for some $Q \geqslant 0$. Therefore, $W^{T_2} = Q \geqslant 0$. ■

For low-dimensional systems, the optimal witnesses are easily constructed. For example, the optimal witnesses for two qubits (i.e., the 2×2 system) are of the form

$$W = |\psi\rangle\langle\psi|^{T_2},$$

where $|\psi\rangle$ is an entangled state vector [22]. In fact, an optimal witness detecting the state ρ can be constructed as $W = |\psi\rangle\langle\psi|^{T_2}$ from the eigenvector $|\psi\rangle$ of ρ^{T_2} with negative eigenvalue λ since $\mathrm{Tr}(|\psi\rangle\langle\psi|^{T_2}\rho) = \mathrm{Tr}(|\psi\rangle\langle\psi|\rho^{T_2}) = \lambda < 0$ [22]. This method can be generalized to the infinite-dimensional case, but the resulting witness may be not an optimal one.

IV. INCOMPARABLE WITNESSES THAT DETECT A COMMON STATE

Now we turn back to the question of when different entanglement witnesses that are incomparable can detect some common entangled states. This question was studied in Ref. [17] for the finite-dimensional cases, there the authors of Ref. [17, Theorem 4] asserted that in finite-dimensional systems, under the condition $\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$, if there exists no inclusion relation between $\mathcal{D}_{W_1}$ and $\mathcal{D}_{W_2}$, then $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$ if and only if $W = \lambda W_1 + (1-\lambda)W_2$ is not a positive operator for all $0 \leqslant \lambda \leqslant 1$. We point out, though this result is true, the proof of it in Ref. [17] is not correct.

Our attention is mainly focused on the infinite-dimensional cases. We establish a similar result without the assumption "$\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$" and provide a proof that is valid for both finite-dimensional systems and infinite-dimensional systems.

The following two lemmas are obvious.

Lemma 4.1. Let H_1, H_2 be complex Hilbert spaces and let W_1, $W_2 \in \mathcal{W}(H_1 \otimes H_2)$ with $W_1 \prec W_2$. If $W(a,b) = aW_1 + bW_2$, where a and b are positive numbers, then $W_1 \prec W(a,b) \prec W_2$.

Particularly, if $W_1 \prec W_2$, then all convex combinations of them are still witnesses.

Lemma 4.2. Let H_1, H_2 be complex Hilbert spaces. For W_1, $W_2 \in \mathcal{W}(H_1 \otimes H_2)$, let $W = aW_1 + bW_2 \neq 0$ with $a \geqslant 0$ and $b \geqslant 0$, then $\mathcal{D}_W \subset \mathcal{D}_{W_1} \cup \mathcal{D}_{W_2}$ and $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \subset \mathcal{D}_W$.

The following is our key lemma which is obtained for the finite-dimensional cases in Ref. [17] with a different and longer proof.

Lemma 4.3. Let H_1, H_2 be complex Hilbert spaces. For $W, W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$, if $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$ and if $\mathcal{D}_W \subset \mathcal{D}_{W_1} \cup \mathcal{D}_{W_2}$, then either $\mathcal{D}_W \subset \mathcal{D}_{W_1}$ or $\mathcal{D}_W \subset \mathcal{D}_{W_2}$.

Proof. Assume, on the contrary, that both $\mathcal{D}_{W_1} \cap \mathcal{D}_W$ and $\mathcal{D}_{W_2} \cap \mathcal{D}_W$ are nonempty. Take $\rho_i \in \mathcal{D}_{W_i} \cap \mathcal{D}_W$, $i = 1,2$. Consider the segment $[\rho_1,\rho_2] = \{\rho_t = (1-t)\rho_1 + t\rho_2 : 0 \leqslant t \leqslant 1\}$. As $\mathcal{D}_W$ is convex, we have

$$[\rho_1,\rho_2] \subseteq \mathcal{D}_W \subseteq \mathcal{D}_{W_1} \cup \mathcal{D}_{W_2}.$$

Thus we get

$$[\rho_1,\rho_2] = \big(\mathcal{D}_{W_1} \cap [\rho_1,\rho_2]\big) \cup \big(\mathcal{D}_{W_2} \cap [\rho_1,\rho_2]\big),$$

that is, $[\rho_1,\rho_2]$ is divided into two convex parts. It follows that there is $0 < t_0 < 1$ such that $\{\rho_t : 0 \leqslant t < t_0\} \subseteq \mathcal{D}_{W_1}$, $\{\rho_t : t_0 < t \leqslant 1\} \subseteq \mathcal{D}_{W_2}$, and either $\rho_{t_0} \in \mathcal{D}_{W_1}$ or $\rho_{t_0} \in \mathcal{D}_{W_2}$. Assume that $\rho_{t_0} \in \mathcal{D}_{W_1}$; then $\mathrm{Tr}(W_1\rho_{t_0}) < 0$. Thus, for sufficiently small $\varepsilon > 0$ with $t_0 + \varepsilon \leqslant 1$, we have

$$\begin{aligned} 0 &\leqslant \mathrm{Tr}\big(W_1\rho_{t_0+\varepsilon}\big) \\ &= \mathrm{Tr}\big(W_1\rho_{t_0}\big) + \varepsilon[\mathrm{Tr}(W_1\rho_2) - \mathrm{Tr}(W_1\rho_1)] < 0, \end{aligned}$$

a contradiction. Similarly, $\rho_{t_0} \in \mathcal{D}_{W_2}$ leads to a contradiction as well. This completes the proof. ■

Now we are ready to state and prove the main result in this section, which asserts that two entanglement witnesses can detect no common entangled states if and only if at least one convex combinations of them is positive, that is, breaks the witness.

Theorem 4.4. Let H_1, H_2 be complex Hilbert spaces and $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$. Then $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$ if and only if there exists $0 < \lambda < 1$ such that $W = \lambda W_1 + (1-\lambda)W_2$ is positive.

By Lemma 4.1, Lemma 4.2, and Theorem 4.4, the following result is immediate, which states that two witnesses can detect some common entangled states if and only if their convex combination does not break the witness.

Theorem 4.5. Let H_1, H_2 be complex Hilbert spaces and $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$. Then $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$ if and only if $W_\lambda = \lambda W_1 + (1-\lambda)W_2 \in \mathcal{W}(H_1 \otimes H_2)$ for all $0 \leqslant \lambda \leqslant 1$.

Proof of Theorem 4.4. If $W = \lambda W_1 + (1-\lambda)W_2 \geqslant 0$ for some $\lambda \in (0,1)$, then, by Lemma 4.2, $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \subseteq \mathcal{D}_W = \emptyset$.

Assume that $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$. Let $W(\lambda) = \lambda W_1 + (1-\lambda)W_2$, $0 \leqslant \lambda \leqslant 1$. Then, by Lemma 4.3, for all $\lambda \in [0,1]$, we have

$$\mathcal{D}_{W(\lambda)} \subset \mathcal{D}_{W_1}, \quad \text{or} \quad \mathcal{D}_{W(\lambda)} \subset \mathcal{D}_{W_2}.$$

When λ varies from 0 to 1 continuously, $\mathcal{D}_{W(\lambda)}$ also varies from $\mathcal{D}_{W_2}$ to $\mathcal{D}_{W_1}$ continuously. Take $\lambda_0 = \sup\{\lambda : \mathcal{D}_{W(\lambda)} \subset \mathcal{D}_{W_2}\}$.

We claim that if $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_2}$ then there exist $0 < \varepsilon < 1 - \lambda_0$ such that $W(\lambda_0 + \varepsilon)$ is a positive operator. Otherwise, if for all $0 < \varepsilon < 1 - \lambda_0$, $\mathcal{D}_{W(\lambda_0+\varepsilon)} \neq \emptyset$, then we have

$$\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_2}, \quad \mathcal{D}_{W(\lambda_0+\varepsilon)} \subset \mathcal{D}_{W_1},$$

and for all $\rho \in \mathcal{D}_{W(\lambda_0)}$, we have

$$\begin{gathered} \mathrm{Tr}[W(\lambda_0)\rho] < 0, \\ \mathrm{Tr}[W(\lambda_0)\rho] + \varepsilon[\mathrm{Tr}(W_1\rho) - \mathrm{Tr}(W_2\rho)] \geqslant 0. \end{gathered}$$

Noticing that $\mathrm{Tr}(W_1\rho) \geqslant 0$ and $\mathrm{Tr}(W_2\rho) < 0$, the second part of the last inequality is positive, and ε is an arbitrarily small positive number, hence the last inequality is impossible. (We remark that there is a mistake in the proof found in Ref. [17, Theorem 4]. In Ref. [17], the argument is "for all $\rho \in \mathcal{D}_{W(\lambda_0+\varepsilon)}$, we have

$$\begin{gathered} \mathrm{Tr}[W(\lambda_0)\rho] \geqslant 0, \\ \mathrm{Tr}[W(\lambda_0)\rho] + \varepsilon[\mathrm{Tr}(W_1\rho) - \mathrm{Tr}(W_2\rho)] = \mathrm{Tr}[W(\lambda_0+\varepsilon)\rho] < 0. \end{gathered}$$

Noticing that $\mathrm{Tr}(W_1\rho) < 0$ and $\mathrm{Tr}(W_2\rho) \geqslant 0$, the second part of the last inequality is negative, and ε is an arbitrarily small positive number, hence the last inequality is impossible." However, $\mathrm{Tr}[W(\lambda_0)\rho]$ maybe equals 0 for all possible ρ and the above argument is invalid.)

On the other hand, if $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_1}$ then there exist $0 < \varepsilon < \lambda_0$ such that $W(\lambda_0 - \varepsilon)$ is a positive operator. Otherwise, if for all $0 < \varepsilon < \lambda_0$, $\mathcal{D}_{W(\lambda_0-\varepsilon)} \neq \emptyset$, then we have

$$\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_1}, \quad \mathcal{D}_{W(\lambda_0-\varepsilon)} \subset \mathcal{D}_{W_2},$$

and for all $\rho \in \mathcal{D}_{W(\lambda_0)}$, we have

$$\mathrm{Tr}[W(\lambda_0)\rho] < 0, \quad \mathrm{Tr}[W(\lambda_0)\rho] + \varepsilon[\mathrm{Tr}(W_2\rho) - \mathrm{Tr}(W_1\rho)] \geqslant 0.$$

Noticing that $\mathrm{Tr}(W_2\rho) \geqslant 0$ and $\mathrm{Tr}(W_1\rho) < 0$, the second part of the last inequality is positive and ε is an arbitrarily small positive number, hence the last inequality is impossible. (We remark that there is a mistake similar to that pointed out above in the proof of Ref. [17, Theorem 4] here as well.)

To sum up the previous discussion, no matter $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_1}$ or $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_2}$ there exists $\lambda \in [0, 1]$ such that $W(\lambda)$ is a positive operator, which completes the proof of the theorem. ■

Based on Theorem 4.4 and Theorem 4.5, we can make a little generalization by the allowing of finitely many witnesses. The idea of the proof of the statement (1) is similar to that in Ref. [17] for the finite-dimensional cases, and we provide it here for the reader's convenience.

Denote by $\mathrm{cov}(\Gamma)$ the convex hull of Γ, that is, $\mathrm{cov}(\Gamma) = \{\sum_{i=1}^{k} d_i W_i : W_i \in \Gamma, d_i \geqslant 0, \sum_{i=1}^{k} d_i = 1, k \in \mathbb{N}\}$.

Theorem 4.6. Let H_1, H_2 be complex Hilbert spaces. Let $\Gamma = \{W_i : 1 \leqslant i \leqslant n\} \subseteq \mathcal{W}(H_1 \otimes H_2)$ be a finite set of entanglement witnesses. Then (1) $\mathcal{D}_\Gamma = \emptyset$ if and only if $\mathrm{cov}(\Gamma)$ contains some positive operators and (2) $\mathcal{D}_\Gamma \neq \emptyset$ if and only if $\mathrm{cov}(\Gamma) \subseteq \mathcal{W}(H_1 \otimes H_2)$.

Proof. (1) The sufficient part is clear. In fact, if $W = \sum_{i=1}^{n} d_i W_i \geqslant 0$ for some numbers $d_i \geqslant 0$ with $\sum_i \lambda_i = 1$, then $\mathcal{D}_W = \emptyset$, which implies that $\mathcal{D}_\Gamma = \emptyset$ since $\mathcal{D}_\Gamma \subseteq \mathcal{D}_W$.

Conversely, if $\mathcal{D}_\Gamma = \emptyset$, we assume, without loss of generality that any subset of Γ can detect some entangled states simultaneously. If $n = 2$, the theorem becomes Theorem 4.4. Assume that the theorem holds for $k \leqslant n - 1$. By induction, we have to show that the theorem holds for n. Since the method is the same, we only need to show it for the case $n = 3$. By assumption, we have

$$\mathcal{D}_{W_1} \neq \emptyset, \quad \mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset, \quad \mathcal{D}_{W_1} \cap \mathcal{D}_{W_3} \neq \emptyset,$$

but

$$\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \cap \mathcal{D}_{W_3} = \emptyset,$$

namely,

$$(\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}) \cap (\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}) = \emptyset.$$

Let

$$W(\lambda) = \lambda W_2 + (1 - \lambda)W_3, \quad \lambda \in [0,1];$$

then

$$\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda)} \subset (\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}) \cup (\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}).$$

Since $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}$ and $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}$ are disjoint and $\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda)}$ is convex, we know that $\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda)}$ varies from $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}$ to $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}$ whenever λ varies from 0 to 1. Using the similar argument as that in the proof of Theorem 4.4, we can conclude that there exists $0 < \lambda_0 < 1$ such that

$$\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda_0)} = \emptyset.$$

Therefore,

$$\begin{aligned} W &= \mu W_1 + (1 - \mu)W(\lambda_0) \\ &= \mu W_1 + (1 - \mu)\lambda_0 W_2 + (1 - \mu)(1 - \lambda_0)W_3 \geqslant 0, \end{aligned}$$

for some $\mu \in (0, 1)$. By induction on n we complete the proof of (1).

(2) The "only if" part is obvious. To check the "if" part assume that $\mathrm{cov}(\Gamma) \subseteq \mathcal{W}(H_1 \otimes H_2)$. If, on the contrary, $\mathcal{D}_\Gamma = \emptyset$, then, by the statement (1) just proved above, there exists $W \in \mathrm{cov}(\Gamma)$ such that $W \geqslant 0$. It follows that $W \notin \mathcal{W}$, a contradiction. ■

By Theorem 4.6 it is clear that $W_1, \ldots, W_n \in \mathcal{W}(H_1 \otimes H_2)$ detect some entangled states simultaneously if and only if all convex combinations of them are witnesses.

V. CONCLUSION

To sum up, in this paper we answer the question under what conditions different witnesses may detect some common entangled states. Generally speaking, for bipartite quantum systems, finitely many different witnesses $W_1, W_2, \ldots, W_n$ can detect some common entangled states if and only if their convex combinations (i.e., $\sum_{i=1}^{n} d_i W_i$ with numbers $d_i \geqslant 0$ and $\sum_{i=1}^{n} d_i = 1$) are still witnesses; they cannot detect any common entangled state if and only if one of their convex combinations is a positive operator. For two witnesses W_1 and W_2, more can be said: W_1 and W_2 can detect the same set of entangled states if and only if $W_1 = \alpha W_2$ for some positive number α; W_2 can detect more entangled states than W_1 can if and only if $W_1 = \alpha W_2 + D$ for some number $\alpha > 0$ and some positive operator D. As an application of the above results, we show that a witness is optimal if and only if any negative permutation of it will break the witness, that is, a witness W is optimal if and only if $W - D$ is not a witness for any positive operator D; W is decomposable optimal implies that W is the partial transpose of some positive operator.

Finally, we would like to stress that our results hold for both the infinite-dimensional and finite-dimensional cases. Though some of them are known for finite-dimensional systems under the additional assumption $\mathrm{Tr}(W_1) = \mathrm{Tr}(W_2)$, the proof of our main results for the infinite-dimensional case needs new methods.

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