# Optimal reconstruction of the states in qutrit systems

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(Received 16 February 2010; published 8 October 2010)

Based on mutually unbiased measurements, an optimal tomographic scheme for the multiqutrit states is presented explicitly. Because the reconstruction process of states based on mutually unbiased states is free of information waste, we refer to our scheme as the optimal scheme. By optimal we mean that the number of the required conditional operations reaches the minimum in this tomographic scheme for the states of qutrit systems. Special attention will be paid to how those different mutually unbiased measurements are realized; that is, how to decompose each transformation that connects each mutually unbiased basis with the standard computational basis. It is found that all those transformations can be decomposed into several basic implementable single- and two-qutrit unitary operations. For the three-qutrit system, there exist five different mutually unbiased-bases structures with different entanglement properties, so we introduce the concept of physical complexity to minimize the number of nonlocal operations needed over the five different structures. This scheme is helpful for experimental scientists to realize the most economical reconstruction of quantum states in qutrit systems.

DOI: 10.1103/PhysRevA.82.044102　　PACS number(s): 03.65.Wj, 03.65.Ta, 03.65.Ud, 03.67.Mn

## I. INTRODUCTION

The quantum state of a system is a fundamental concept in quantum mechanics, and a quantum state can be described by a density matrix, which contains all the information one can obtain about that system. A main task for implementing quantum computation is to reconstruct the density matrix of an unknown state, which is called quantum-state reconstruction or quantum-state tomography [1,2]. The technique was first developed by Stokes to determine the polarization state of a light beam [3]. Recently, a minimal qubit tomography process has been proposed by Řeháček *et al.* where only four measurement probabilities are needed to fully determine a single qubit state rather than the six probabilities needed in the standard procedure [4]. But the implementation of this tomography process requires measurements of $N$-particle correlations [5]. The statistical reconstruction of biphoton states based on mutually complementary measurements has been proposed by Bogdanov *et al.* [6,7]. Ivanov *et al.* proposed a method to determine an unknown mixed qutrit state from nine independent fluorescence signals [8]. Moreva *et al.* paid attention to the experimental problem of realizing the optimal protocol for polarization ququart-state tomography [9]. In 2009, Taguchi *et al.* developed the single-scan tomography of spatial three-dimensional (qutrit) states based on the effect of realistic measurement operators [10]. Allevi *et al.* studied the implementation of the reconstruction of the Wigner function and the density matrix for coherent and thermal states by switching on or off single-photon avalanche photodetectors [11].

In order to obtain the full information about the system, we need to perform a series of measurements on a large number of identically prepared copies of the system. These measurement results are not independent of each other, so there is redundancy in these results in the previously used quantum tomography processes [12], which causes resource waste. If we remove this redundancy completely, the reconstruction process will become optimal. Thus, designing an optimal set of measurements to remove the redundancy is of fundamental significance in quantum information processing.

Mutually unbiased bases (MUBs) have been used in a variety of topics in quantum mechanics [13–36]. MUBs are defined by the property that the squared overlap between a vector in one basis and all basis vectors in the other bases is equal. That is to say, detection over a particular basis state does not give any information about the state if it is measured in another basis. Ivanović first introduced the concept of MUBs to the problem of quantum-state determination [13] and proved the existence of such bases in the prime-dimension system by an explicit construction. It was then shown by Wootters and Fields that measurements in this special class of bases [i.e., mutually unbiased measurements (MUMs)] provide a minimal as well as optimal way to completely specify an unknown density matrix [14]. They proved that the maximal numbers of MUBs is $d+1$ in a prime-dimension system. This result also applies to the prime-power-dimension system.

MUBs play a special role in determining quantum states, such that they form a minimal set of measurement bases and provides an optimal way to determine a quantum state [13–16], etc. Recently, an optimal tomographic reconstruction scheme was proposed by Klimov *et al.* to determine a state of a multiqubit quantum system based on MUMs in trapped-ion systems [37]. However, the use of three-level systems instead of two-level systems has been proven to be more secure against a symmetric attack on a quantum key distribution protocol with MUMs than the currently existing measurement protocol [38,39]. Quantum tomography in high-dimensional (qudit) systems has been proposed and the number of required measurements is $d^{2n}-1$ with $d$ being the dimension of the qudit system and $n$ being the number of qudits [12]. This tomography process is not an optimal one, and there is a big redundancy among the measurement results there. To remove this redundancy, we propose an optimal tomography process

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for qutrit states. This optimal quantum tomography process is the MUB-based qutrit states tomography, and the number of required measurements is greatly reduced. A $d$-dimensional quantum system is represented by a positive semidefinite Hermitian matrix $\rho$ with unit trace in $d$-dimensional Hilbert space, which is specified by $d^2-1$ real parameters. A nondegenerate measurement performed on such a system provides $d-1$ independent probabilities. Thus, in general, one requires at least $d+1$ different orthogonal measurements to fully determine an unknown $\rho$. For $n$-qutrit tomography, only $3^n+1$ measurements are needed. Through analysis, one can find that the tomography process for qubit systems cannot be generalized to the qudit system case in a trivial way. The MUB-based tomography process for qubit systems proposed by Klimov *et al.* cannot be directly applied to qutrit systems [37]. This is because the entanglement feature of the MUBs of qubit systems is totally different from that of qutrit systems. Thus, we will study the physical implementation of an optimal tomographic scheme for the case of determining the states of multiqutrit systems based on MUMs.

From the experimental point of view, the physical complexity is a key point for the implementation of a scheme. Here, for multiqutrit quantum tomography, the physical complexity mainly comes from the entanglement bases; namely, the number of two-qutrit conditional operations needed to decompose these entangled bases. In addition, there exists many different MUBs with different entanglement properties in multiqutrit system. Thus, the physical complexity of the quantum tomography process here depends on the entanglement structure of the MUBs used, and it becomes very important to optimize the quantum tomography process over all possible MUB entanglement structures of the system.

This article is arranged as follows. In the next section we introduce the MUBs in a $d$-dimensional system [$d=p^n$ ($p\neq 2$)] and show how to reconstruct an unknown state by MUMs. Here, $p$ is a prime. Section III briefly reviews the general method of reconstructing a qutrit state, where the number of measurements is 8. However, the MUB-based qutrit tomography proposed here only needs 4 measurements, which means the number of measurements is reduced. Here, the measurement reduction for the single-qutrit case is not obvious, so in Sec. IV we extend the one-qutrit case to two-qutrit systems. For the two-qutrit system, the number of measurements is only 10 for determining all the elements of the density operator rather than $3^4-1=80$ measurements in the scheme proposed in Ref. [12]. This means a great reduction in experimental complexity. Thus, we conclude that the optimal measurements on the unknown qutrit states are the MUMs. In Sec. V, we discuss the physical complexity for implementing the MUMs in three-qutrit systems, and give the optimal MUB for the qutrit system quantum tomography process with minimized physical complexity. The last section is the conclusion.

## II. MUTUALLY UNBIASED BASES AND MUTUALLY UNBIASED MEASUREMENTS

As shown by Wootters and Fields [14] and Klappenecker and Rötteler [19], in finite-field language, the first MUB in a quantum system of dimension $d=p^n$ ($p\neq 2$) is the standard basis $B^0$ given by the vector $(a_k^{(0)})_l=\delta_{kl}, k,l\in\mathcal{F}_{p^n}$, where the superscript denotes the basis, $k$ the vector in the basis, $l$ the component, and $\mathcal{F}_{p^n}$ is the field with $p^n$ elements. The other $d$ MUBs are denoted by $B^r$ and consist of vectors $(a_k^{(r)})_l$ defined by [14] $(a_k^{(r)})_l=(1/\sqrt{d})\omega^{\mathcal{T}r(rl^2+kl)}, r,k,l\in\mathcal{F}_{p^n}, r\neq 0$. Here, $\omega=\exp(2\pi i/p)$ and $\mathcal{T}r\theta=\theta+\theta^p+\theta^{p^2}+\cdots+\theta^{p^{n-1}}$. The set of mutually unbiased projectors can be given by $P_k^{(r)}=|a_k^{(r)}\rangle\langle a_k^{(r)}|$. It is worth noticing that $|a_k^{(r)}\rangle$ contains the computational basis $B^0$. Here, $\mathrm{Tr}(P_j^{(s)}P_k^{(r)})=(1/d)(1-\delta_{sr}+d\delta_{sr}\delta_{jk})$. Then the measurement probabilities given by $\omega_k^{(r)}=\mathrm{Tr}(P_k^{(r)}\rho)$ completely determine the unknown density operator of a $d$-dimensional system [13]: $\rho=\Sigma_{r=0}^{d}\Sigma_{k=0}^{d-1}\omega_k^{(r)}P_k^{(r)}-I$.

For instance, in a qutrit system, there are three MUBs besides the computational basis $B^0=\{|0\rangle,|1\rangle,|2\rangle\}$ in the following form with $\omega=\exp(2\pi i/3)$:

$$\begin{aligned}B^1:\{&|a_0^{(1)}\rangle=(1/\sqrt{3})(|0\rangle+|1\rangle+|2\rangle),\\ &|a_1^{(1)}\rangle=(1/\sqrt{3})(|0\rangle+\omega|1\rangle+\omega^*|2\rangle),\\ &|a_2^{(1)}\rangle=(1/\sqrt{3})(|0\rangle+\omega^*|1\rangle+\omega|2\rangle)\};\end{aligned}\tag{1a}$$

$$\begin{aligned}B^2:\{&|a_0^{(2)}\rangle=(1/\sqrt{3})(\omega|0\rangle+|1\rangle+|2\rangle),\\ &|a_1^{(2)}\rangle=(1/\sqrt{3})(|0\rangle+\omega|1\rangle+|2\rangle),\\ &|a_2^{(2)}\rangle=(1/\sqrt{3})(|0\rangle+|1\rangle+\omega|2\rangle)\};\end{aligned}\tag{1b}$$

$$\begin{aligned}B^3:\{&|a_0^{(3)}\rangle=(1/\sqrt{3})(\omega^*|0\rangle+|1\rangle+|2\rangle),\\ &|a_1^{(3)}\rangle=(1/\sqrt{3})(|0\rangle+\omega^*|1\rangle+|2\rangle),\\ &|a_2^{(3)}\rangle=(1/\sqrt{3})(|0\rangle+|1\rangle+\omega^*|2\rangle)\}.\end{aligned}\tag{1c}$$

## III. RECONSTRUCTION PROCESS FOR AN ARBITRARY SINGLE QUTRIT STATE

An unknown single qutrit state can be expressed as [12,40] $\rho=(1/3)\sum_{j=0}^{8}r_j\lambda_j$, where $\lambda_0$ is an identity operator and the other $\lambda_j$ are the SU(3) generators [41]. The general method to reconstruct the qutrit state is to measure the expectation values of the $\lambda$ operators [12], where $r_j=\langle\lambda_j\rangle=\mathrm{Tr}[\rho\lambda_j]$. Thus, one will find that the number of required measurements is 8. However, if we choose the MUMs to determine the qutrit state, the number of MUMs needed is only 4 rather than the 8 of Ref. [12]. The four optimal set of MUBs have been presented by Eqs. (1a), (1b), and (1c) plus the standard computational basis in the preceding section. Each of the three MUBs in Eqs. (1a), (1b), and (1c) is related with the standard computational basis by a unitary transformation. These transformations have been listed in Table I. Here, $F$ denotes the Fourier transformation

$$F|j\rangle=(1/\sqrt{3})\sum_{l=0}^{2}\exp(2\pi ilj/3)|l\rangle;\quad j=0,1,2,\tag{2}$$

$R$ denotes a phase operation

$$R=|0\rangle\langle 0|+\omega|1\rangle\langle 1|+\omega|2\rangle\langle 2|,\tag{3}$$

and the controlled gate is

$$X|i\rangle|j\rangle=|i\rangle|j\ominus i\rangle,\tag{4}$$

TABLE I. Transformations connecting the MUBs with the standard computational basis for a qutrit system based on Fourier transforms and the phase operations.

| Basis | Transformation |
|---|---|
| 2 | $F^{-1}$ |
| 3 | $F^{-1}R$ |
| 4 | $F^{-1}R^{-1}$ |

where $\ominus$ denotes the difference $j-i$ modulo 3. If there are $n$ qutrits, the number of MUMs is $3^n+1$, which is far less than the $3^{2n}-1$ in Ref. [12]. In other words, the use of MUMs can represent a considerable reduction in the operations and time required for performing the full state determination [37].

## IV. RECONSTRUCTION PROCESS FOR AN ARBITRARY TWO-QUTRIT STATE

If we further extend the one-qutrit case to the two-qutrit case, the density matrix can be expressed as $\rho_{12} = (1/9)\sum_{j,k=0}^{8} r_{jk}\lambda_j \otimes \lambda_k$, where $r_{jk} = \langle \lambda_j \otimes \lambda_k \rangle$. If we use the general method in Ref. [12] to fully determine the state, $d^{2n}-1 = 3^4-1 = 80$ measurements will be needed. So many measurements will inevitably introduce redundant state information, which is obviously a resource waste. So here we will take advantage of the MUMs to reconstruct the two-qutrit state. It is easy to find that the nine elements of $\mathcal{F}_9$ (finite field) are $\{0,\alpha,2\alpha,1,1+\alpha,1+2\alpha,2,2+\alpha,2+2\alpha\}$ by using the irreducible polynomials $\theta^2+\theta+2=0$ [14]. Here we use the representation $\{|0\rangle,|\alpha\rangle,|2\alpha\rangle,\ldots,|2+2\alpha\rangle\}$ as the standard basis.

One can find that there will be only $d^2+1 = 3^2+1 = 10$ MUMs to be done, which is much less than the 80 of Ref. [12]. This means that the operations and time needed for the entire state determination is greatly reduced. The decompositions for all the MUBs of the two-qutrit system have been listed in Table II.

TABLE II. Decompositions of MUBs for the two-qutrit system based on Fourier transformations, phase operations, and controlled-NOT gates ($X_{12}$) [42] with the first particle as source and the second as target. The subscript denotes the $i$th particle; $i = 1,2$.

| Basis | Decompositions |
|---|---|
| 2 | $F_1^{-1}F_2^{-1}$ |
| 3 | $F_1^{-1}R_1F_2^{-1}R_2$ |
| 4 | $F_1^{-1}X_{12}F_2^{-1}R_2^{-1}$ |
| 5 | $F_1^{-1}X_{12}^{-1}R_1F_2^{-1}R_2^{-1}$ |
| 6 | $F_1^{-1}X_{12}^{-1}F_2^{-1}R_1^{-1}$ |
| 7 | $F_1^{-1}R_1^{-1}F_2^{-1}R_2^{-1}$ |
| 8 | $F_1^{-1}X_{12}^{-1}F_2^{-1}R_2$ |
| 9 | $F_1^{-1}R_1^{-1}X_{12}F_2^{-1}R_2$ |
| 10 | $F_1^{-1}R_1X_{12}F_2^{-1}$ |

TABLE III. Decompositions of MUBs (0,12,16) for three-qutrit system.

| Basis | Decompositions |
|---|---|
| 1 | $F_2^{-1}X_{23}F_1^{-1}F_3^{-1}$ |
| 2 | $F_1^{-1}R_1^{-1}F_2^{-1}R_2^{-1}X_{23}F_3^{-1}R_3^{-1}$ |
| 3 | $F_1^{-1}X_{13}R_3$ |
| 4 | $F_2^{-1}R_2X_{23}^{-1}F_1^{-1}X_{12}R_2F_3^{-1}$ |
| 5 | $F_1^{-1}X_{12}^{-1}F_3^{-1}$ |
| 6 | $F_2^{-1}R_2X_{23}F_1^{-1}R_1X_{13}^{-1}R_3F_3^{-1}R_3$ |
| 7 | $F_2^{-1}X_{23}^{-1}R_3^{-1}F_1^{-1}R_1^{-1}X_{12}^{-1}R_3^{-1}$ |
| 8 | $F_1^{-1}R_1^{-1}X_{13}^{-1}F_2^{-1}R_2F_3^{-1}R_3^{-1}$ |
| 9 | $F_1^{-1}X_{13}^{-1}F_2^{-1}R_2X_{23}^{-1}F_3^{-1}R_3$ |
| 10 | $F_1^{-1}R_2X_{12}X_{13}^{-1}F_2^{-1}$ |
| 11 | $F_1^{-1}R_1X_{12}^{-1}F_2^{-1}R_2X_{23}^{-1}F_3^{-1}$ |
| 12 | $F_1^{-1}R_1^{-1}X_{13}X_{12}^{-1}F_2^{-1}F_3^{-1}R_3^{-1}$ |
| 13 | $F_1^{-1}X_{12}F_2^{-1}R_2^{-1}X_{23}^{-1}F_3^{-1}R_3^{-1}$ |
| 14 | $F_1^{-1}R_1X_{12}F_2^{-1}X_{23}F_3^{-1}R_3$ |
| 15 | $F_2^{-1}R_2X_{23}$ |
| 16 | $F_1^{-1}R_1X_{13}F_2^{-1}R_2X_{23}F_3^{-1}R_3^{-1}$ |
| 17 | $F_1^{-1}R_1^{-1}X_{13}F_2^{-1}F_3^{-1}$ |
| 18 | $F_1^{-1}R_1^{-1}X_{13}^{-1}F_2^{-1}R_2^{-1}X_{23}^{-1}R_3F_3^{-1}R_3^{-1}$ |
| 19 | $F_1^{-1}R_1X_{13}^{-1}F_2^{-1}R_2^{-1}F_3^{-1}R_3$ |
| 20 | $F_1^{-1}X_{12}F_3^{-1}R_3^{-1}$ |
| 21 | $F_1^{-1}R_1^{-1}X_{12}^{-1}F_2^{-1}R_2F_3^{-1}R_3$ |
| 22 | $F_1^{-1}X_{12}^{-1}F_2^{-1}X_{23}^{-1}F_3^{-1}R_3$ |
| 23 | $F_1^{-1}R_1^{-1}X_{13}F_2^{-1}R_2^{-1}X_{23}^{-1}$ |
| 24 | $F_1^{-1}R_1^{-1}X_{13}^{-1}F_2^{-1}X_{23}R_3F_3^{-1}$ |
| 25 | $F_1^{-1}R_1X_{12}^{-1}X_{23}^{-1}F_2^{-1}R_2^{-1}$ |
| 26 | $F_1^{-1}R_1X_{12}X_{23}^{-1}F_2^{-1}R_2^{-1}F_3^{-1}$ |
| 27 | $F_1^{-1}R_1^{-1}X_{12}F_2^{-1}R_2$ |
| 28 | $F_1^{-1}R_1F_2^{-1}X_{23}^{-1}$ |

## V. THE PHYSICAL COMPLEXITY FOR IMPLEMENTING THE MUMS IN THE THREE-QUTRIT SYSTEM

In general, the fidelity of single logic gates can be greater than 99%, but nonlocal gates have a relatively lower fidelity. The fidelity of a practical controlled-NOT (CNOT) gate can reach a value up to 0.926 for trapped-ion systems in the laboratory [43]. Klimov *et al.* have introduced the concept of the physical complexity of each set of MUBs as a function of the number of nonlocal gates needed for implementing the MUMs [37]. Here, the fidelity value of the CNOT gates for qubit systems also can be used to evaluate the physical complexity of the MUBs of qutrit systems. The reason we can affirm this is because of the following point. Although the systems involved here are three-state systems, all the operations used in our reconstruction process can be decomposed into effective two-state operations. Thus, the complexity of the current tomography scheme is proportional to the number of the nonlocal gates used ($C \propto 6$) for two-qutrit systems. As shown in Ref. [44], the only MUB structure for a two-qutrit system is (4,6), where 4 is the number of the separable bases and 6 is the number of the bipartite entangled bases.

However, in the three-qutrit case there are five sets of MUBs with different structures; namely, {(0,12,16), (1,9,18), (2,6,20), (3,3,22), (4,0,24)} [21,44]. It is easy to see that the (0,12,16) set of MUBs has the minimum physical complexity. We say that the optimal set of the MUBs is (0,12,16). The decompositions of the MUBs in the three-qutrit case for (0,12,16) are listed in Table III. Thus, the set of MUBs (0,12,16) has a complexity $C \propto 44$, which is a very significant value for experimental realizations.

For the multiqutrit system ($n > 3$), it is not easy to get all the sets of MUBs, and it is even more difficult to get the explicit decompositions of the optimal set of MUBs. Nevertheless, the results for the two-qutrit and three-qutrit cases have provided the experimentalists valuable references.

## VI. CONCLUSION

We have explicitly presented an optimal tomographic scheme for single-qutrit states, two-qutrit states, and three-qutrit states based on MUMs. Because the MUB-based state reconstruction process is free of information waste, the minimal number of required conditional operations are needed. Thus, we call our qutrit tomographic scheme the optimal scheme. Here, we explicitly decompose each measurement into several basic single- and two-qutrit operations. Furthermore, all these basic operations have been proven implementable [42]. The physical complexity of a set of MUBs also has been calculated, which is an important threshold in experiments. We hope these decompositions can help experimental scientists to realize the most economical reconstruction of quantum states in qutrit systems in the laboratory.

## ACKNOWLEDGMENTS

This work is supported by National Natural Science Foundation of China (NSFC) under Grants No. 10704001 and No. 61073048, the Key Project of Chinese Ministry of Education (No. 210092), the Key Program of the Education Department of Anhui Province under Grants No. KJ2008A28ZC, No. KJ2009A048Z, No. 2010SQRL153ZD, and No. KJ2010A287, the Talent Foundation of Anhui University, the personnel department of Anhui province, and Anhui Key Laboratory of Information Materials and Devices (Anhui University).

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