Self-trapping of two-dimensional vector dipole solitons in nonlocal media

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(Received 30 August 2010; published 12 October 2010)

We study the self-trapping of the superposition of two-dimensional vector dipole solitons in nonlocal media with an arbitrary degree of nonlocality. We apply the variational approach to find the exact solution of such vector dipole solitons and investigate their stability by using directly numerical simulations. The dynamics of such vector solitons are also compared with a scalar vortex. We show the nonlocality induces an attractive force which can completely stabilize the vector dipole solitons.

DOI: 10.1103/PhysRevA.82.043815 PACS number(s): 42.65.Tg, 42.65.Jx

I. INTRODUCTION

Spatial solitons, localized waves balanced between linear diffraction and nonlinear self-trapping, have drawn considerable attention in a range of physical settings [1]. In the case of nonlinear optics, the nonlinear response of the medium can be described in terms of the induced change in the refractive index which is often approximated as a local function of the wave intensity [1]. However, the nonlocal nonlinear response has been a strong interest subject recently [2]. This so-called nonlocal model represents the fact that the nonlinear change of the refractive index depends on the beam intensity in the neighborhood of each spatial point [2]. For other physical systems, the cause of the nonlocal nonlinear model can be interpreted with different physical mechanisms. This occurs, for example, in media with the nonlinearity caused by various transport processes such as heat conduction in media with thermal response [3,4], diffusion of charge carriers or atoms or molecules in atomic vapors [5], and drift or diffusion of photoexcited charges in photorefractive materials [6]. Propagation of electromagnetic waves in plasma [7] and parametric wave mixing [8] also belong to the nonlocal nonlinear domain. It is also the case of systems exhibiting a long-range interaction of constituent molecules or particles such as in nematic liquid crystals [9] or Bose-Einstein condensates with dipole [10] and gravity-like interactions [11].

The propagation of solitons in nonlocal medium is an interesting and important subject in that the nonlocality has profound impact on its physical dynamics and leads to novel phenomena of a generic nature [2]. Nonlocal nonlinearity has been shown to support a series of novel solitons, such as stable multipole solitons [12,13], discrete solitons [14,15], and azimuthons [16–19]. It also affects the interactions of out-of-phase bright [20,21] and dark solitons [22–24], provides attractive forces between the soliton components which always repel in local media. Nonlocality can also support high-dimensional stable, complex vortex soliton structures [25–28] because of collapse suppression of localized structures, including fundamental, vortex and rotating solitons in nonlocal nonlinear media [29]. Recently, experimental and theoretical studies have shown that the nonlocality also plays different important roles for the incoherent solitons in both instantaneous and noninstantaneous media [30–35], respectively.

Vector solitons, a localized envelope consisting of multicomponents, have also been investigated in nonlocal media, such as the stability of multipole vector solitons [36,37]. Alberucci *et al.* have investigated experimentally another class of vector solitons—namely, two-color, spatial solitons—in a nematic liquid crystal with highly nonlocal and anisotropic reorientational nonlinearity [38]. Theoretical properties of two-color vector fundamental solitons in nematic liquid crystals have also been studied in local [39] and highly nonlocal [40] limits. Lee *et al.* investigated theoretically the mutual trapping of bright and dark beams in nonlocal media [41]. In addition to stabilization of vector soliton pairs, nonlocal nonlinearity also helps to reduce the threshold power for forming a guided bright solitons [41]. We have investigated the incoherently coupled dipole soliton pairs in nonlocal media and shown that the fundamental beam can effectively improve the stability of the dipole beam [42]. Xu *et al.* studied the vector coupling between a fundamental Gaussian beam and a vortex beam in nematic liquid crystal [43].

Dipole-mode solitons, comprising two out-of-phase peaks packed together by the force acting between them, attracted much attentions in nonlocal nonlinear media, both experimentally [12] and theoretically [13,17,44–48]. Stationary two-dimensional (2D) dipole-mode solitons have been observed experimentally in media with the thermal nonlinearity [12]. It is also shown that the stability of the dipole-mode solitons is crucially dependent on the form of the nonlinear nonlocal response [13,17,44–47]. Recently, stable elliptic dipole-mode solitons in nonlocal nonlinear media with anisotropic semilocal nonlinearity has also been reported [48].

In this paper, we investigate the self-trapping of the superposition of two-dimensional vector dipole solitons in nonlocal media with an arbitrary degree of the nonlocality both analytically and numerically. We use the variational approach to derive analytical formulas for the vector dipole solitons. We further show the stability and the evolution of the vector dipole solitons by direct numerical simulations. Our results show that the nonlocality induces an attractive force, which can completely eliminate the split and the quasistable expanding of the solitons in purely local media, and stabilizes the vector dipole solitons. We also compare our results with the evolution of a scalar vortex.

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II. MODEL AND VARIATIONAL METHOD

We consider the propagation of a vector soliton consisting of N mutually incoherent optical components propagating in a nonlinear medium with a spatially nonlocal response. The propagation for the slowly varying beam envelopes $E_n(x,y,z)$ can be written in the form of the normalized coupled nonlocal nonlinear Schrodinger equations ($n = 1,2,\ldots,$ N) [49],

$$i\frac{\partial E_n}{\partial z}+\frac{\partial^2 E_n}{\partial x^2}+\frac{\partial^2 E_n}{\partial y^2}+E_n\int R(\mathbf{r}-\mathbf{r}')I(\mathbf{r}',z)d^2\mathbf{r}'=0, \tag{1}$$

where the propagation coordinate, z, is measured in the unit of the diffraction length L_D and the transverse coordinates, (x,y), are measured in the units of $(L_D/k)^{1/2}$. $I=\sum|E_n|^2$ is the total beam intensity and $R(r)$ is the normalized nonlocal response function with $\int_{-\infty}^{\infty}R(r)dr=1$. The width of the response function $R(r)$ determines the degree of nonlocality. For a singular response, $R(r)=\delta(r)$, the refractive index becomes a local function of the light intensity, $\delta n(I)=I(r,z)$ (i.e., the refractive index change at a given point is solely determined by the light intensity at that point). With increasing width of $R(r)$ the light intensity in the vicinity of the point r also contributes to the index change at that point. When the width of the response function trends to infinity, the nonlocal model can be treated as accessible solitons. In cylindrical coordinate system, Eq. (1) can be written as follows:

$$i\frac{\partial E_n}{\partial z}+\frac{1}{r}\frac{\partial E_n}{\partial r}+\frac{\partial^2 E_n}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 E_n}{\partial \varphi^2}+E_n\int R(\mathbf{r}-\mathbf{r}')I(\mathbf{r}',z)d^2\mathbf{r}'=0, \tag{2}$$

where $r=\sqrt{x^2+y^2}$ and $\varphi=\tan^{-1}(y/x)$. In this paper, we aim to investigate the propagation of vector localized solutions of Eq. (1) using an incoherent superposition of the dipole components. Similarly discussed in Ref. [49], we look for the solutions of the vector solitons in the following form:

$$E_n=U(r)\Phi_n(\varphi)e^{ikz}, \tag{3}$$

with the self-consistency condition $\Sigma|\Phi_n(\varphi)|^2=1$ and the propagation constant k. The total intensity of the nonlocal vector soliton is a function of the radial coordinate only, which is presented as $I=U^2(r)$.

Substitute Eq. (3) into (2), we can obtain that

$$\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{m^2}{r^2}U-kU+U\int R(\mathbf{r}-\mathbf{r}')|U(\mathbf{r}',z)|^2d^2\mathbf{r}'=0, \tag{4}$$

and

$$\frac{d^2\Phi_n}{d\varphi^2}+m^2\Phi_n=0. \tag{5}$$

Equation (5) has an analytical solution:

$$\Phi_n(\varphi)=a_n\cos(m\varphi)+b_n\sin(m\varphi), \tag{6}$$

where the m (integer) is the topological charge, a_n and b_n are the complex coefficients satisfying the conditions $\Sigma Re(a_n b_n^*)=0$ and $\Sigma|a_n|^2=\Sigma|b_n|^2=1$, which define exact solutions of the system (1) for any N, and, in the particular case $N=1$, they describe a scalar vortex with $a=1$ and $b=i$. In this paper, we consider a two-component model ($N=2$) to investigate the dynamics of the vector solitons in nonlocal media. The complex coefficients satisfy the following relation:

$$a_1=(1+p^2)^{-1/2}, b_1=ipa_1, \tag{7}$$

and

$$a_2=pa_1, b_2=\pm ia_1, \tag{8}$$

where $0\leqslant p\leqslant 1$ is a real parameter. For the vector dipole solitons, represented as an incoherent superposition of two dipole modes, the coefficients satisfy $m=1$ and $p=0$ [49]. The dipole components, thus, can be written as $E_1=U\cos(\varphi)$ and $E_2=iU\sin(\varphi)$.

In order to get some insight into possible propagation dynamics of the nonlocal vector solitons, we firstly employ the so-called Lagrangian (or variational) approach. It is easy to show that Eq. (1) can be derived from the following Lagrangian density:

$$l=\sum_{n=1,2}\frac{i}{2}r\left(E_n^*\frac{\partial E_n}{\partial z}-E_n\frac{\partial E_n^*}{\partial z}\right)-r\left(\left|\frac{\partial E_n}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial E_n}{\partial \varphi}\right|^2\right)+\frac{1}{2}r|E_n|^2\int R(\mathbf{r}-\mathbf{r}')I(\mathbf{r}',z)d^2\mathbf{r}'. \tag{9}$$

$R(\mathbf{r})$ is the normalized nonlocal response function with a characteristic width σ_0 which represents the degree of the nonlocality. In our variational process, we consider here the case of the Gaussian nonlocal response [17,30]:

$$R(r)=\left(\pi\sigma_0^2\right)^{-1}\exp\left(-r^2/\sigma_0^2\right), \tag{10}$$

which can describe the dynamics of solitons in nonlocal media physically. For convenience, we also choose a typical single ring vortex form for the amplitude of the vector solitons,

$$U(r)=Ar\exp(-r^2/2\sigma^2), \tag{11}$$

with the beam width σ [25]. The variational method is represented as $\delta\int Ldz=0$, where $L=\int_{-\infty}^{\infty}ldxdy$. Combining Eqs. (9)–(11), we can obtain the effective Lagrangian $L=L(A,\sigma_0)$ depending only on the parameters A and σ_0. From the Euler-Lagrange equations we then obtain

$$A^2=\frac{\left(2\sigma^2+\sigma_0^2\right)^4}{\sigma^{10}\sigma_0^2+\sigma^{12}+\sigma^8\sigma_0^4}. \tag{12}$$

The total power of the vector dipole solitons is $P=\pi A^2\sigma^4$. In Fig. 1, we show the total power as a function of the degree of nonlocality with different beam widths. We can see that the total power of the vector dipole solitons will decrease when the beam width increases, whereas the total power will increase when the degree of the nonlocality increases. This is because the nonlinearity depends on the power in nonlocal Kerr media; the larger the power, the larger the nonlinearity. As the degree of the nonlocality increases, the nonlinearity of the media will be weakened and then the diffraction of the beam will be strengthened which requires a larger nonlinearity induced by the power to balance the diffraction.

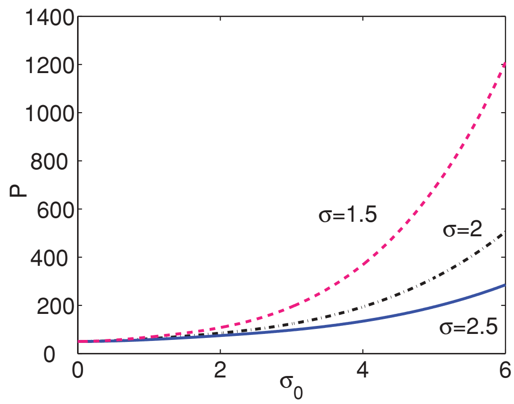

FIG. 1. (Color online) Total power of the vector dipole solitons versus degree of nonlocality with different beam widths.

We also plot the total power versus the propagation constant k in Fig. 2. The total power of the vector dipole solitons will increase when the propagation constant increases which is similar to the scalar dipole solitons in nonlocal media [17]. It is also obvious that the power will increase when the degree of nonlocality increases.

III. NUMERICAL RESULTS

In this paper, we are interested in the interactions between the incoherent dipole components and the propagation stability of the vector solitons in nonlocal media. In this section, the predictions of the variational approach will be confronted with direct numerical simulations by using the split-step Fourier transform method. The approximate solitons resulting from the variational approach will be used as an initial condition to our two-dimensional code to compute their evolution dynamics.

In Figs. 3–6 we show the stability dynamics of the vector dipole solitons in nonlocal media when degrees of the nonlocality are local, weakly nonlocal, general nonlocal, and strongly nonlocal, respectively. In all our simulations, we make the initial beam width $\sigma = 1$. For comparison with the vector case, we also present the evolution of a scalar vortex $E = U(r)\exp(im\varphi)$ with the topological charge $m = 1$ (the scalar vortex can be regarded as a coherent superposition of E_1 and E_2) in the nonlocal nonlinear media. The numerical results

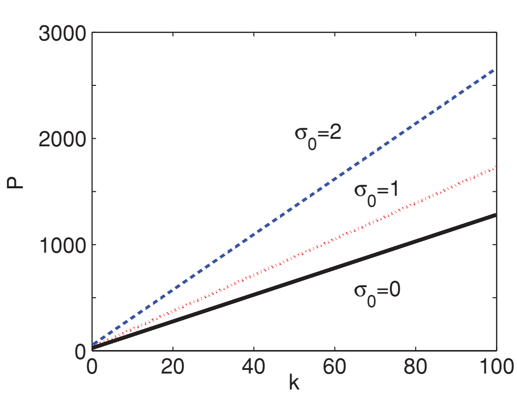

FIG. 2. (Color online) Total power of the vector dipole solitons versus k with different degree of nonlocality.

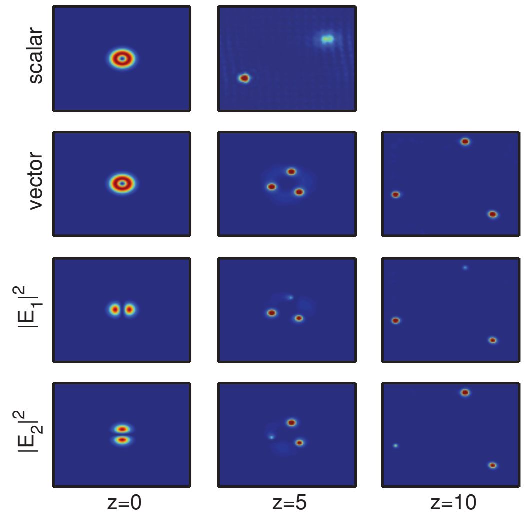

FIG. 3. (Color online) Dynamics and symmetry-breaking instability of the scalar vortex and vector dipole solitons with the degree of the nonlocality is $\sigma_0 = 0$ (local).

tell us the nonlocality and the mutual attraction between the necklace-ring components jointly stabilize the dynamics of the vector solitons in nonlocal media. For the scalar vortex beam, despite the fact it will decay into a pair of scalar solitons during its propagation [49], the nonlocality can effectively improve the stability of the beam (Figs. 3–5), and it will be stable only in the strongly nonlocal case (Fig. 6). Figure 5 tells us we can obtain the stationary vector dipole solitons in a general nonlocal case with $\sigma_0 = 1$, whereas the scalar vortex beam is still unstable with the same degree of the nonlocality. The result indicates that the mutual self-trapping between the vector components exhibit much more stability for the solitons than the scalar beam, which is also displayed in the local case (compare the scalar and the vector case in Fig. 3) and weakly nonlocal case (compare the scalar and the vector case in Fig. 4). The nonlocality also stabilize the vector dipole

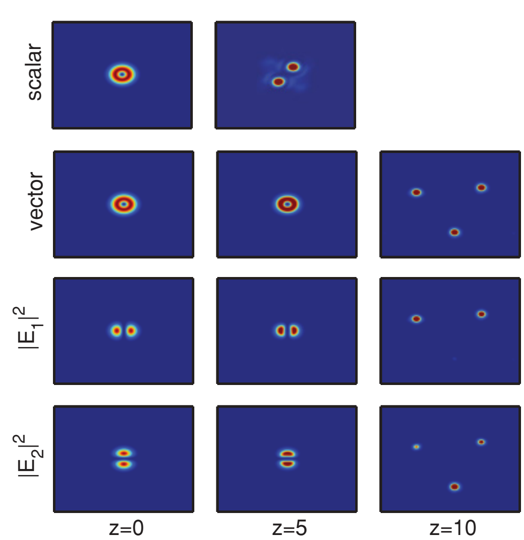

FIG. 4. (Color online) Dynamics and symmetry-breaking instability of the scalar vortex and vector dipole solitons with the degree of the nonlocality is $\sigma_0 = 0.1$ (weakly nonlocal).

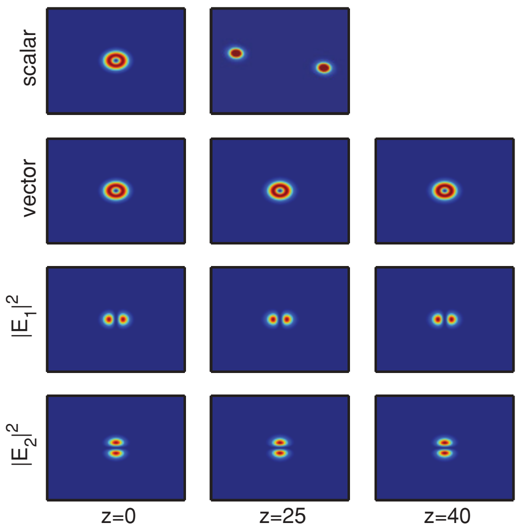

FIG. 5. (Color online) Dynamics and evolution of the scalar vortex and vector dipole solitons when the degree of the nonlocality is $\sigma_0 = 1$ (generally nonlocal).

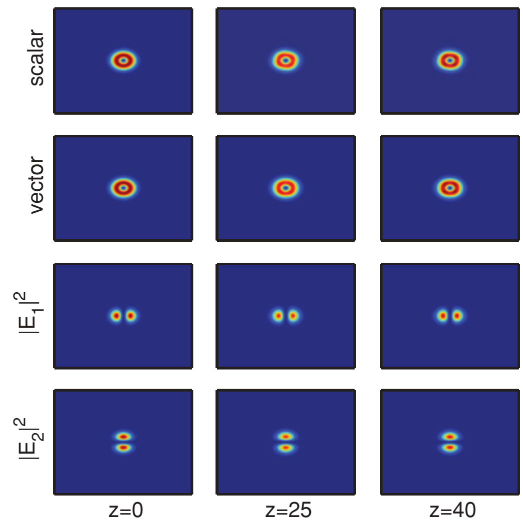

FIG. 6. (Color online) Dynamics and evolution of the scalar vortex and vector dipole solitons when the degree of the nonlocality is $\sigma_0 = 10$ (strongly nonlocal).

beam. For example, it will be stable when the propagation distance is five diffraction lengths in the weak nolocality with $\sigma_0 = 0.1$ (Fig. 4). However, the vector beam split into three components with the same propagation distance in the local case with $\sigma_0 = 0$ (Fig. 3).

In local and weakly nonlocal media, the vector property and the nonlocality cannot prevent the break-up of the high-dimensional beam; both the scalar vortex and the vector dipole beam will split into some fundamental solitons. However, the split dynamics of the vector solitons is very different from the scalar vortex. The scalar vortex will split into an even number of solitons produced by the symmetry-breaking instability (Figs. 3 and 4). The vector dipole beam will split into odd-number solitons with uniform intensity in spite of the fact that its component intensity is asymmetrically modulated (Figs. 3 and 4) [49]. In this paper, we find the vector beam can only propagate several diffraction lengths in the local and weakly nolocal media (Figs. 3 and 4), which is very different from the case of Fig. 1(b) discussed in Ref. [49], where the vector beam displays a long-term stable dynamics up to the propagation distances of almost 55 diffraction lengths. The result is due to the different types of the nonlinearity used in our work and in Ref. [49]. The model used in Ref. [49] is the saturable nonlinearity which can effectively eliminate the collapse of the beam and show a larger stationary propagation distance. In such saturable nonlinear media, the vector beam can only propagate quasistably; it will expand slowly with the propagation distance. However, the nonlocality induces an attractive force [24], which can completely suppress the repulsion between the neighbor beam "petals" with a π phase flip [49], and we can obtain the fully stationary vector dipole solitons, as shown in Figs. 5 and 6. In the strongly nonlocal media, the nonlocality will weaken the strength of the nonlinearity and induce a smoothly parabolic potential [42]. The solitons will be more stable, but weakly confined in such an optical-induced waveguide and its profile will become nearly rectangle (Fig. 6).

It should be indicated that we only consider the propagation dynamics of solitons without a perturbation in this paper. In fact, a perturbed scalar dipole soliton with a small perturbation in the input field distribution will oscillate unstably in nonlocal media with a thermal nonlinearity [12]. It will break up and decay into a ground-state solitons at a long propagation distance at moderate energy levels. However, we emphasize that we have obtained the reasonable and interesting results of the propagation dynamics of the scalar vortex and the vector dipole solitons without perturbation in nonlocal media. Vector dipole solitons with a perturbation is an important issue, which is also, indeed, our next aim [41].

IV. CONCLUSION

In conclusion, we have investigated the self-trapping of the superposition of two-dimensional vector dipole solitons in nonlocal media with an arbitrary degree of the nonlocality both analytically and numerically. We used the variational approach to derive analytical formulas for the vector dipole solitons. We further showed the stability of the vector dipole solitons by direct numerical simulations. The evolution dynamics of the vector solitons is compared with a scalar vortex. Our results showed that the vector property and the nonlocality play important roles in the dynamics of the vector dipole beam. The nonlocality induces an attractive force, which can completely eliminate the split and the quasistable expansion of the beam in local media, and stabilizes the vector dipole solitons.

ACKNOWLEDGMENTS

We are grateful to Professor W. Krolikowski for his useful help and discussion. This work is supported in part by the Chinese National Natural Science Foundation (Grant No. 60808002), and partly supported by Shanghai Leading Academic Discipline Project (Project No. S30105).

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