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Approximate formulas for the resonance frequency shift in cavities with big variations of parameters inside small regions

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The Müller-Bethe-Schwinger-Casimir formulas for the frequency shift in electromagnetic resonators are generalized to the case of big variations of electric permittivity inside small regions. These formulas are important, in particular, for the studies of the dynamical Casimir effect.

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I. INTRODUCTION

The knowledge of the shift of resonance frequencies of electromagnetic cavities due to perturbations of geometry and/or material properties of walls or internal parts of the cavities is important for many applications. Simple approximate formulas for this shift were derived long ago [1–3] and some of them can be found in the books [4–8]. Different applications of these formulas and some generalizations can be found, e.g., in [9–17]. But it was assumed in all the cited references that the changes of the material properties (in particular, the electric permittivity) are small. However, there exist situations where variations of parameters can be very big, but the frequency shift is small due to the small size of the region where the parameters change their values.

An interesting example is a cavity containing a thin semiconductor slab illuminated by laser pulses. This setup was suggested in [18,19] to simulate the dynamical Casimir effect (DCE); see also [20,21] for recent reviews of different proposals. In this case a highly conducting thin layer is created near the surface of the semiconductor, so that the imaginary part of the electric permittivity changes from zero to values of the order of 10^5 or bigger. Nonetheless, the frequency shift remains very small due to the small thickness of the slab and the conducting layer. For the simplest rectangular or cylindrical geometries and homogeneously excited slabs, this shift can be calculated rather easily [22,23], but these special cases can be considered only as rough models of real experimental situations.

The aim of this paper is to obtain approximate formulas generalizing the Müller-Bethe-Schwinger-Casimir ones to the case of big changes of the complex electric permittivity in thin slabs. The plan is as follows. Section II reproduces exact (although not widely known) formulas for the difference of the eigenfrequencies in two cavities. Formulas are derived in Sec. III, and they are compared with the results of exact calculations in some simple cases in Sec. IV. Arbitrarily thin inhomogeneous slabs attached to the plain boundary of a cavity are considered in Sec. V, and the specific features related to the DCE experiments are discussed in Sec. VI. The final section contains conclusions.

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II. EXACT FORMULA FOR THE FREQUENCY SHIFT

Let us consider monochromatic electric and magnetic fields of the form $\mathcal{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\exp(-i\omega t)$ and $\mathcal{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})\exp(-i\omega t)$ inside a cavity with *ideal* walls. If the cavity is filled in with a *linear* medium described by means of the electric permittivity $\varepsilon_1(\mathbf{r})$ and magnetic permeability $\mu_1(\mathbf{r})$, then the Maxwell equations determining the eigenfrequency ω_1 have the form (I use the Gauss system of units)

$$\text{rot}\,\mathbf{E}_1 = \frac{i\omega_1}{c}\mu_1\mathbf{H}_1, \tag{1}$$

$$\text{rot}\,\mathbf{H}_1 = -\frac{i\omega_1}{c}\varepsilon_1\mathbf{E}_1. \tag{2}$$

For a cavity filled in with a medium described by means of functions $\varepsilon_2(\mathbf{r})$ and $\mu_2(\mathbf{r})$, we have similar equations

$$\text{rot}\,\mathbf{E}_2 = \frac{i\omega_2}{c}\mu_2\mathbf{H}_2, \tag{3}$$

$$\text{rot}\,\mathbf{H}_2 = -\frac{i\omega_2}{c}\varepsilon_2\mathbf{E}_2. \tag{4}$$

Let us multiply (forming the scalar products) both sides of Eq. (1) by the function $\mathbf{H}_2$, Eq. (2) by $\mathbf{E}_2$, Eq. (3) by $-\mathbf{H}_1$, and Eq. (4) by $-\mathbf{E}_1$. Taking the sum of the four new equations thus obtained, one has

$$\mathbf{H}_2\,\text{rot}\,\mathbf{E}_1 - \mathbf{E}_1\,\text{rot}\,\mathbf{H}_2 - \mathbf{H}_1\,\text{rot}\,\mathbf{E}_2 + \mathbf{E}_2\,\text{rot}\,\mathbf{H}_1 = \frac{i}{c}[\mathbf{E}_2\mathbf{E}_1(\omega_2\varepsilon_2 - \omega_1\varepsilon_1) - \mathbf{H}_2\mathbf{H}_1(\omega_2\mu_2 - \omega_1\mu_1)].$$

Now we integrate both sides of this equation over the total volume of the cavity, taking into account the identity

$$\text{div}[\mathbf{a}\times\mathbf{b}] \equiv \mathbf{b}\,\text{rot}\,\mathbf{a} - \mathbf{a}\,\text{rot}\,\mathbf{b}.$$

Due to the Gauss theorem, the volume integral in the left-hand side can be transformed into the surface integral over the total surface of the cavity,

$$\iint_{\text{walls}} ([\mathbf{H}_1\times\mathbf{E}_2] + [\mathbf{E}_1\times\mathbf{H}_2])\,d\mathbf{s}.$$

But this integral equals zero, because the scalar product of the vector integrand and the vector surface differential $d\mathbf{s}$ depends only on tangential components of the vectors $\mathbf{E}_2$ and $\mathbf{E}_1$, which become zero on the surface of an ideal cavity. Thus we arrive at the identity

$$\int [\mathbf{E}_2\mathbf{E}_1(\omega_2\varepsilon_2 - \omega_1\varepsilon_1) - \mathbf{H}_2\mathbf{H}_1(\omega_2\mu_2 - \omega_1\mu_1)]\,dV \equiv 0,$$

which can be rewritten as two equivalent forms,

$$\frac{\omega_2-\omega_1}{\omega_2} = -\frac{\iiint(\delta\varepsilon \mathbf{E}_2\mathbf{E}_1 - \delta\mu \mathbf{H}_2\mathbf{H}_1)\,dV}{\iiint(\varepsilon_1 \mathbf{E}_2\mathbf{E}_1 - \mu_1 \mathbf{H}_2\mathbf{H}_1)\,dV}, \quad (5)$$

$$\frac{\omega_2-\omega_1}{\omega_1} = -\frac{\iiint(\delta\varepsilon \mathbf{E}_2\mathbf{E}_1 - \delta\mu \mathbf{H}_2\mathbf{H}_1)\,dV}{\iiint(\varepsilon_2 \mathbf{E}_2\mathbf{E}_1 - \mu_2 \mathbf{H}_2\mathbf{H}_1)\,dV}, \quad (6)$$

where

$$\delta\varepsilon(\mathbf{r}) = \varepsilon_2(\mathbf{r}) - \varepsilon_1(\mathbf{r}), \quad \delta\mu(\mathbf{r}) = \mu_2(\mathbf{r}) - \mu_1(\mathbf{r}). \quad (7)$$

Formulas (5) and (6) are *exact*, and they hold for arbitrary complex functions $\varepsilon_{1,2}$ and $\mu_{1,2}$ (so that the eigenfrequencies $\omega_{1,2}$ are also complex in the most general case). A discussion of formula (5) and its applications in the special case of $\varepsilon_1 = \mu_1 = 1$ can be found in [10].

If the functions $\varepsilon_1(\mathbf{r})$ and $\mu_1(\mathbf{r})$ are *real* (in such a case, ω_1 is also a real number, whereas ε_2, μ_2, and ω_2 can be complex quantities), then one can apply the same procedure as above to Eqs. (3) and (4) and the complex-conjugated equations (1) and (2), obtaining, e.g., the equation

$$\frac{\omega_2-\omega_1}{\omega_1} = -\frac{\iiint(\delta\varepsilon \mathbf{E}_2\mathbf{E}_1^* + \delta\mu \mathbf{H}_2\mathbf{H}_1^*)\,dV}{\iiint(\varepsilon_2 \mathbf{E}_2\mathbf{E}_1^* + \mu_2 \mathbf{H}_2\mathbf{H}_1^*)\,dV} \quad (8)$$

instead of (6). Note that the function $\mathbf{E}_1$ can be chosen real in this case. Then $\mathbf{H}_1$ is purely imaginary, and this fact explains the difference in signs of the "magnetic" terms in Eqs. (6) and (8). However, formula (8) is not valid for complex functions $\varepsilon_1(\mathbf{r})$ and $\mu_1(\mathbf{r})$, because in this case the complex-conjugated equations (1) and (2) contain the products $\omega_1^*\varepsilon_1^*$ instead of $\omega_1\varepsilon_1$.

III. GENERALIZATION OF THE STANDARD APPROXIMATE FORMULA

It is assumed usually that the unperturbed fields $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{H}_1(\mathbf{r})$ are known for any point $\mathbf{r}$ of the cavity. The problem is that the perturbed fields $\mathbf{E}_2(\mathbf{r})$ and $\mathbf{H}_2(\mathbf{r})$ are not known [except for the simplest cases when they can be obtained by scaling the fields $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{H}_1(\mathbf{r})$ in *all* points [10]]. If the variation $\delta\varepsilon(\mathbf{r})$ is small everywhere, then one may believe that the perturbed fields $\mathbf{E}_2(\mathbf{r})$ and $\mathbf{H}_2(\mathbf{r})$ are close to $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{H}_1(\mathbf{r})$, respectively. Then it is natural to replace $\mathbf{E}_2(\mathbf{r})$ by $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{H}_2(\mathbf{r})$ by $\mathbf{H}_1(\mathbf{r})$ in the right-hand side (RHS) of (5). Moreover, one can write $\omega_2 \approx \omega_1$, $\varepsilon_2 \approx \varepsilon_1$, and $\mu_2 \approx \mu_1$ in the denominators of the fractions in identities (5) or (6). Then either of these two identities results in a simple approximate formula for the frequency shift $\delta\omega = \omega_2 - \omega_1$, given in many textbooks (see, e.g., [5]),

$$\frac{\delta\omega}{\omega} \approx -\frac{\iiint\left[\delta\varepsilon(\mathbf{r})\mathbf{E}_1^2 - \delta\mu(\mathbf{r})\mathbf{H}_1^2\right]dV}{\iiint\left(\varepsilon_1\mathbf{E}_1^2 - \mu_1\mathbf{H}_1^2\right)dV} = -\frac{\iiint\left[\delta\varepsilon(\mathbf{r})\mathbf{E}_1^2 - \delta\mu(\mathbf{r})\mathbf{H}_1^2\right]dV}{2\iiint\varepsilon_1\mathbf{E}_1^2\,dV}. \quad (9)$$

The second equality holds due to the well-known identity

$$\int \varepsilon_1\mathbf{E}_1^2\,dV \equiv -\int \mu_1\mathbf{H}_1^2\,dV. \quad (10)$$

If ε_1 and μ_1 are real, then one can use equivalent formulas (see [8]) which can be obtained from (9) by the changes $E_1^2 \to |E_1|^2$ and $H_1^2 \to -|H_1|^2$. In this case identity (10) means the equality of electric and magnetic energies in ideal resonance cavities (if ε_1 and μ_1 are positive functions with negligible frequency dispersions). Hereafter I shall consider only the case of nonmagnetic media with $\mu_1 = \mu_2 = 1$ (having in mind concrete applications to the DCE and because a generalization to the magnetic case is obvious), so that $\delta\mu(\mathbf{r}) \equiv 0$ in the subsequent formulas.

Formula (9) gives a divergent result if $\delta\varepsilon(\mathbf{r}) \to \infty$. However, it seems obvious that, if the function $\delta\varepsilon(\mathbf{r})$ is different from zero only inside some small volume $\delta V \ll V$ (where V is the total cavity volume), then the ratio $\delta\omega/\omega$ remains small even for very big values of $\delta\varepsilon(\mathbf{r})$. My goal is to find such a generalization of formula (9), which can be used in the case when $\delta\varepsilon(\mathbf{r})$ is much bigger than unity inside some *thin flat* slab or film.

In this special case, it is convenient to divide the electric vector into two parts, $\mathbf{E}(\mathbf{r}) = (E_z,\mathbf{E}_t)$, where E_z is the component perpendicular to the surface of the slab and $\mathbf{E}_t$ is the two-dimensional vector parallel to this surface, and to rewrite Eq. (6) in terms of the components of vector $\mathbf{E}(\mathbf{r})$ and the electric displacement vector $\mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$ as follows:

$$\frac{\omega_2-\omega_1}{\omega_1} = -\frac{\iiint[D_{z2}E_{z1} - E_{z2}D_{z1} + (\varepsilon_2-\varepsilon_1)\mathbf{E}_{t2}\mathbf{E}_{t1}]\,dV}{\iiint(D_{z2}E_{z1} + \varepsilon_2\mathbf{E}_{t2}\mathbf{E}_{t1} - \mathbf{H}_2\mathbf{H}_1)\,dV}. \quad (11)$$

Then, taking into account the conditions of continuity of the normal component of vector $\mathbf{D}(\mathbf{r})$ on the surfaces separating media with different dielectric properties, one can assume that the normal component D_{z2} inside the thin slab is the same as it was for the medium with ε_1, but with the new frequency ω_2, i.e., $D_{z2} \approx D_{z1}(\omega_2)$. Similarly, considering the tangential components of vector $\mathbf{E}(\mathbf{r})$, one can suppose that function $\mathbf{E}_{t2}(\mathbf{r})$ at each point inside the thin slab is close to the value $\mathbf{E}_{t1}(\mathbf{r})$ at the same point, but taken for the new frequency ω_2.

The crucial point, which permits us to generalize formula (9), consists in the account of the fact that the fields depend not only on the coordinate vector $\mathbf{r}$, but also on the frequency ω which enters the Maxwell equations (3) and (4), so that one should write the electric field as $\mathbf{E}(\mathbf{r};\omega)$. Supposing that small variations of frequency are accompanied by *small* variations of functions $\mathbf{E}_{t1}$ and E_{z1} (describing the field components in the *unperturbed* cavity), we use the Taylor expansions of these functions with respect to the frequency change $\delta\omega$, taking into account only the first (linear) terms. Thus we arrive at the following approximate expressions for the electric field components inside the slab:

$$\mathbf{E}_{t2}(\mathbf{r};\omega_2) \approx \mathbf{E}_{t1}(\mathbf{r};\omega_1) + \mathbf{G}_t(\mathbf{r})\delta\tilde{\omega}, \quad (12)$$

$$D_{z2}(\mathbf{r};\omega_2) \approx D_{z1}(\mathbf{r};\omega_1) + \varepsilon_1 G_z(\mathbf{r})\delta\tilde{\omega}, \quad (13)$$

where $\tilde{\omega} = \delta\omega/\omega_1$ and

$$\mathbf{G}(\mathbf{r}) = \omega_1\partial\mathbf{E}_1(\mathbf{r};\omega)/\partial\omega|_{\omega=\omega_1}. \quad (14)$$

I suppose that the dependence of ε_1 on the frequency can be neglected. However, ε_2 can depend on frequency. For example, the semiconductor medium can be described by means of the function (in the Gauss system of cgs units)

$$\varepsilon_2(n;\omega) = \varepsilon_b + 4\pi i\sigma/\omega, \quad \sigma(\mathbf{r}) = n(\mathbf{r})|eb|, \tag{15}$$

where σ is the conductivity, $n(\mathbf{r})$ is the concentration of electron-hole pairs, $|e|$ is the electron charge, $|b| = |b_e| + |b_h|$ is the total mobility of the pair, and ε_b is the dielectric constant in the absence of free carriers. Since the frequency ω_2 is unknown, the function $\varepsilon_2(\mathbf{r})$ can be represented as follows:

$$\varepsilon_2(\mathbf{r};n;\omega_2) \approx \varepsilon_2(\mathbf{r};n;\omega_1) + \nu(\mathbf{r};n;\omega_1)\delta\tilde{\omega}, \tag{16}$$

$$\nu(\mathbf{r};n;\omega) = \omega\partial\varepsilon_2(\mathbf{r};n;\omega)/\partial\omega. \tag{17}$$

According to this scheme, function E_{z2} in the numerator of the RHS of (11) can be represented as

$$E_{z2} \approx \varepsilon_1 E_{z1}(\omega_2)/\varepsilon_2(\omega_2) \approx \varepsilon_1 E_{z1}(\omega_1)/\varepsilon_2(\omega_1) + \delta\tilde{\omega}\left[\varepsilon_1 G_z/\varepsilon_2(\omega_1) - \varepsilon_1 E_{z1}(\omega_1)\nu(\omega_1)/\varepsilon_2^2(\omega_1)\right].$$

Putting all these approximate expressions in the RHS of (6), neglecting terms containing $(\delta\omega)^2$ (that means that the terms containing $\delta\tilde{\omega}$ should be omitted in the denominator) and taking into account Eq. (10), we arrive at the following generalization of formula (9), which gives a finite frequency shift in the limit $\delta\varepsilon \to \infty$:

$$\frac{\delta\omega}{\omega_1} \approx -\frac{\iiint \delta\varepsilon(\mathbf{r})\left(\mathbf{E}_{t1}^2 + E_{z1}^2\varepsilon_1/\varepsilon_2\right)dV}{\iiint \left[2\varepsilon_1\mathbf{E}_1^2 + \delta\varepsilon(\mathbf{r})\left(\mathbf{E}_{t1}^2 + \mathbf{G}_t\mathbf{E}_{t1} + G_z E_{z1}\varepsilon_1/\varepsilon_2\right) + \nu(\mathbf{r})\left(\mathbf{E}_{t1}^2 + E_{z1}^2\varepsilon_1^2/\varepsilon_2^2\right)\right]dV}. \tag{18}$$

Here all the functions in the RHS, including $\varepsilon_2(\mathbf{r};n;\omega)$ and $\nu(\mathbf{r};n;\omega)$, are taken at the known frequency ω_1. Note that the denominator in (18) does not contain the term with $\delta\varepsilon(\mathbf{r})E_{z1}^2$.

IV. COMPARISON WITH EXACT SOLUTIONS

To evaluate the accuracy of formula (18) and to see how it works, let us consider a few examples in which exact solutions of the problem can be found.

A. TE mode in two thin homogeneous slabs

As the first example let us consider a cylindrical cavity of length L with an arbitrary cross section and the axis parallel to the z direction, supposing that the main volume of the cavity is empty, except for a thin slab of a thickness $D \ll L$, which consists of two parts: a "background" of thickness $D - D_1$ with a moderate real dielectric constant ε_b (which adjoins the cavity flat wall) and a film of thickness D_1 with a big dielectric constant $\varepsilon_s + \varepsilon_b$, where ε_s can be a complex number (this thin film separates the background from the empty part of the cavity). We assume that the dielectric properties of the slab do not depend on the frequency and the transverse coordinate $\mathbf{r}_\perp$. Thus $\varepsilon(\mathbf{r})$ depends on the longitudinal coordinate z as follows:

$$\varepsilon(z) = \begin{cases} 1 & \text{for } -L < z < 0, \\ \varepsilon_b + \varepsilon_s & \text{for } 0 < z < D_1, \\ \varepsilon_b & \text{for } D_1 < z < D. \end{cases} \tag{19}$$

We also suppose that the cavity walls are made from an ideal conductor, so that we can use the ideal boundary conditions $E_t|_{\text{wall}} = 0$ for the tangential components of the electric field.

We consider the fundamental TE mode with the only component of the electromagnetic field parallel to the slab surface. It satisfies the three-dimensional scalar Helmholtz equation (we assume that the magnetic permeability of the slab is the same as in the vacuum)

$$\Delta E + (\omega/c)^2\varepsilon(z)E = 0. \tag{20}$$

The solution to Eq. (20) can be factorized as

$$E(z,\mathbf{r}_\perp) = \psi(z)\Phi(\mathbf{r}_\perp), \tag{21}$$

where the function $\Phi(\mathbf{r}_\perp)$ obeys the two-dimensional Helmholtz equation

$$\Delta_\perp\Phi + k_\perp^2\Phi = 0, \quad \Phi|_{\text{wall}} = 0, \tag{22}$$

so the problem is reduced to solving the one-dimensional Helmholtz equation

$$\psi'' + [(\omega/c)^2\varepsilon(z) - k_\perp^2]\psi = 0 \tag{23}$$

with the boundary conditions

$$\psi(-L) = \psi(D) = 0. \tag{24}$$

In the case of the dielectric function (19), the function $\psi(z)$ can be written as follows:

$$\psi(z) = \begin{cases} F_1 \sin[k(z+L)] & \text{for } -L < z < 0, \\ F_2 \sin(k_2 z + \phi_2) & \text{for } 0 < z < D_1, \\ F_3 \sin[k_3(z-D)] & \text{for } D_1 < z < D, \end{cases} \tag{25}$$

where

$$k_2^2 = (k^2 + k_\perp^2)(\varepsilon_s + \varepsilon_b) - k_\perp^2, \tag{26}$$

$$k_3^2 = (k^2 + k_\perp^2)\varepsilon_b - k_\perp^2, \tag{27}$$

and the constant coefficient k is related to the field eigenfrequency ω as

$$\omega = c(k^2 + k_\perp^2)^{1/2}. \tag{28}$$

The value of the longitudinal wave number k can be found from the equations which are consequences of the continuity conditions for the functions $\psi(z)$ and $\psi'(z)$ at the surfaces $z = 0$ and $z = D_1$:

$$\tan(kL) = \frac{k}{k_2}\tan(\phi_2), \tag{29}$$

$$\tan(k_2 D_1 + \phi_2) = \frac{k_2}{k_3}\tan[k_3(D_1 - D)]. \tag{30}$$

Solving Eq. (30) with respect to phase ϕ_2 and putting the solution into Eq. (29), we arrive at the equation

$$\tan(kL) = \frac{k}{k_2}\,\frac{k_2\tan[k_3(D_1-D)] - k_3\tan(k_2D_1)}{k_3 + k_2\tan[k_3(D_1-D)]\tan(k_2D_1)}. \quad (31)$$

The absolute value of the product $k_3(D_1 - D)$ does not exceed the quantity $2\pi\sqrt{|\varepsilon_b|}D/\lambda$, where

$$\lambda = 2\pi[(\pi/L)^2 + k_\perp^2]^{-1/2} = 2\pi c/\omega_0 \quad (32)$$

is the wavelength corresponding to the fundamental eigenfrequency of an ideal cavity of length L. Assuming that $|\varepsilon_b| < 16$ (as for many dielectric materials or semiconductors at low temperatures), we see that the function $\tan[k_3(D_1 - D)]$ can be replaced by its argument, provided $D \ll \lambda/(8\pi)$. After this simplification, the coefficient k_3 drops out, together with parameter ε_b, and Eq. (31) goes to

$$\tan(kL) = \frac{k}{k_2}\,\frac{k_2(D_1-D) - \tan(k_2D_1)}{1 + k_2(D_1-D)\tan(k_2D_1)}. \quad (33)$$

Obviously, the longitudinal wave number k can be represented as

$$k = (\pi/L)(1+\xi) \quad (34)$$

with $|\xi| \ll 1$ if $D \ll \lambda$. Therefore we can write $\tan(\pi\xi) = \pi\xi$ in the left-hand side of Eq. (33), putting at the same time $\xi = 0$ in the right-hand side (which is small even for $\xi = 0$). This gives an immediate answer for the value of ξ. If $D_1 = 0$ (this case corresponds to the frequency ω_1), then Eq. (33) becomes $\tan(kL) = -kD$, with the solution $\xi_1 \approx -D/L$ (the corrections have an order of $(D/L)^3\varepsilon_b$).

The frequency ω_2 corresponds to a nonzero thickness D_1. A simple answer can be obtained if $|k_2D_1| \ll 1$ (despite that $|\varepsilon_s| \gg 1$), so that one can use the approximation $\tan(k_2D_1) \approx k_2D_1$. This can happen under quite realistic conditions. Indeed, in a conductive medium $|\varepsilon_s| \approx 4\pi\sigma/\omega$, where σ is the conductivity of the film in cgs units. For a good metal one has $|\varepsilon_s| \sim 10^8$ for $\omega \sim 10^{10}\ \mathrm{s}^{-1}$ (or $\lambda \sim 10$ cm). Then the limitation is $D_1 \ll 1\ \mu$m. For semiconductors with $|\varepsilon_s| \sim 10^5$ the upper limit can be shifted to much bigger values: $D_1 \ll 1$ mm. Making all these simplifications [in particular, neglecting the small term $k_\perp^2(\varepsilon_b - 1)$ in the definition of coefficient k_2, i.e., replacing this coefficient by $k_2(\xi = 0) = (2\pi/\lambda)\sqrt{\varepsilon_s}$] and assuming in addition that $D_1 \ll D$, one can arrive at the following simple expression:

$$\xi_2 \approx -\frac{D}{L(1-A_s)}, \quad (35)$$

where

$$A_s = 4\pi^2\varepsilon_s\frac{D_1D}{\lambda^2} = k_2^2D_1D. \quad (36)$$

There is no singularity in formula (35), because parameter A_s is complex, as a matter of fact, and its real part is much smaller than unity.

The relative shift between resonance frequencies can be written, in view of (28), as

$$(\omega_2 - \omega_1)/\omega_1 = \eta^2(\xi_2 - \xi_1), \quad (37)$$

where

$$\eta = \lambda/(2L) = [1 - (2\pi)^2/(k_\perp\lambda)^2]^{1/2}. \quad (38)$$

Thus the result of calculations based on the exact solution of the Maxwell equations is

$$\delta\omega/\omega_1 \approx -\frac{D\lambda^2}{4L^3}\,\frac{A_s}{1-A_s}. \quad (39)$$

Now let us see how formula (39) can be derived from (18). The only nonzero component E_{t1} of the electric field in the empty cavity is given by formulas (21) and (25) with $D_1 = 0$. We take $F_1 = 1$, since this constant factor is canceled in (18), as well as the integral of function $\Phi(\mathbf{r}_\perp)$ (which does not depend on the frequency ω_1). Equation (25) can be simplified as $\psi_1(z) \approx F_3k_3(z - D)$ for $0 < z < D$ if $|k_3D| \ll 1$. But the consequence of the continuity condition for the function $\psi_1(z)$ at $z = 0$ and Eq. (34) is the relation $-F_3k_3D = \sin(kL) = -\pi\xi_1$ (since $|\xi_1| \ll 1$). Thus one can write

$$\psi_1(z) \approx -\pi\xi_1(1 - z/D), \quad 0 < z < D. \quad (40)$$

Consequently,

$$2\int_{-L}^{D}\varepsilon_1(z)\psi_1^2(z)dz \approx L, \quad (41)$$

$$\int_0^{D_1}\delta\varepsilon(z)\psi_1^2(z)dz \approx (\pi\xi_1)^2D_1\varepsilon_s, \quad (42)$$

where the corrections of the order of $D/L \ll 1$ and $D_1/D \ll 1$ are neglected in (41) and (42), respectively.

The function (40) depends on the frequency ω_1 through the coefficient ξ_1. From (34) one obtains $\partial\xi_1/\partial\omega_1 = k_0^{-1}\partial k/\partial\omega_1$, where $k_0 = \pi/L$. The derivative $\partial k/\partial\omega_1$ can be found from Eq. (28): $\partial k/\partial\omega_1 = \omega_1/(c^2k)$. Neglecting the small difference between k and k_0, we find

$$\partial\xi_1/\partial\omega_1 \approx \omega_1/(ck_0)^2. \quad (43)$$

Consequently, the function (14) can be written as $G_t = \tilde{G}(z)\Phi(\mathbf{r}_\perp)$ with

$$\tilde{G}(z) \approx -\frac{\omega_1^2}{c^2k_0^2}\pi(1 - z/D), \quad 0 \leqslant z \leqslant D, \quad (44)$$

so that

$$\int_0^{D_1}\delta\varepsilon(z)\tilde{G}(z)\psi_1(z)dz \approx -DL(\omega_1/c)^2D_1\varepsilon_s \quad (45)$$

if $D_1 \ll D$ (remember that $\xi_1 \approx -D/L$). One can see that the term (42) is much smaller than (45) (the ratio is of the order of D/L), so that it can be neglected in the denominator of the fraction in the RHS of Eq. (18). Then formula (18) leads to (39) with the same coefficient A_s, defined in Eq. (36). Analyzing all the assumptions made during the derivation, one can conclude that formula (39) is justified if $\sqrt{|\varepsilon_s|}D_1/\lambda \ll 1$ and $D_1 \ll D$. Note, however, that the absolute value of parameter A_s can be much bigger than unity even under these restrictions. For example, taking $|\varepsilon_s| \sim 10^6$, $\lambda \sim 10$ cm, $D_1 \sim 10\ \mu$m and $D \sim 1$ mm, one gets $|A_s| \sim 40$, while $\sqrt{|\varepsilon_s|}D_1/\lambda \sim 0.1$ and $D_1/D \sim 0.01$.

B. TM mode in two thin homogeneous slabs

Now let us consider the TM mode in the same geometry as in the preceding subsection. In the exact approach it is convenient to solve the equation for the electric displacement vector **D**,

$$\Delta(\mathbf{D}/\varepsilon) + (\omega^2/c^2)\mathbf{D} = \text{grad div}(\mathbf{D}/\varepsilon). \tag{46}$$

If the dielectric function depends on the single variable z, then $\text{div}(\mathbf{D}/\varepsilon) = D_z\partial(1/\varepsilon)/\partial z$ due to the equation $\text{div}\mathbf{D} = 0$. Consequently, for the step-constant profile (19), Eq. (46) coincides with (20), and one can use the same factorization (21) for the function $D_z(\mathbf{r})$. The difference is in the continuity conditions at the interfaces of the slabs: one should require a continuity of functions $\psi(z)$ and $\varepsilon^{-1}\partial\psi/\partial z$ at $z = 0$ and $z = D_1$, whereas the derivative $\partial\psi/\partial z$ must go to zero at $z = -L$ and $z = D$. Thus instead of (25) one should write

$$\psi(z) = \begin{cases} F_1 \cos[k(z+L)] & \text{for } -L < z < 0, \\ F_2 \cos(k_2 z + \phi_2) & \text{for } 0 < z < D_1, \\ F_3 \cos[k_3(z-D)] & \text{for } D_1 < z < D, \end{cases} \tag{47}$$

with the same definitions of parameters as in (26), (27), and (28). Equations (29) and (30) should be replaced by

$$\tan(kL) = \frac{k_2}{k\varepsilon_s}\tan(\phi_2), \tag{48}$$

$$\tan(k_2 D_1 + \phi_2) = \frac{k_3\varepsilon_s}{k_2\varepsilon_b}\tan[k_3(D_1 - D)], \tag{49}$$

so that instead of (31) we have the equation

$$\tan(kL) = \frac{k_2}{k\varepsilon_s}\,\frac{k_3\varepsilon_s\tan[k_3(D_1-D)] - k_2\varepsilon_b\tan(k_2D_1)}{k_2\varepsilon_b + k_3\varepsilon_s\tan[k_3(D_1-D)]\tan(k_2D_1)}. \tag{50}$$

Using (34) and making the same simplifications as in the preceding section (replacing the tangent functions by their arguments, neglecting $k_\perp^2$ in k_2^2, and neglecting D_1 in the difference $D - D_1$), one can obtain from (50) the formula

$$\xi = \frac{k_3^2 L D}{\pi^2\left[k_3^2\varepsilon_s D D_1 - \varepsilon_b\right]}, \tag{51}$$

where the coefficient k in the definition (27) of k_3 should be replaced by π/L. If

$$\xi_1 = -k_3^2 L D/(\pi^2\varepsilon_b) \tag{52}$$

corresponds to the cavity with $D_1 = 0$ (but $D > 0$), then the additional frequency shift caused by the presence of a thin film with $|\varepsilon_s| \gg 1$ is given by formulas (37) and (38). The final result is similar to (39):

$$\frac{\omega_2 - \omega_1}{\omega_1} \approx -\frac{\zeta_m\tilde{A}_s}{1 - \tilde{A}_s}, \tag{53}$$

$$\zeta_m = \frac{k_3^2 D\lambda^2}{4L\pi^2\varepsilon_b}, \qquad \tilde{A}_s = k_3^2 D D_1\frac{\varepsilon_s}{\varepsilon_b}. \tag{54}$$

If ε_b is not too close to unity (say, $\varepsilon_b \sim 10$), then $\tilde{A}_s \approx A_s$ given in (36). Under this condition, the maximal relative frequency shift of the TM mode equals $\zeta_m \approx D/L$, i.e., it is $(2L/\lambda)^2 = \eta^{-2}$ times bigger than for the TE mode.

To see how formula (18) works, let us consider the fundamental TM mode in a rectangular cavity with $0 < x < L_x, 0 < y < L_y$, and $-L < z < D$. The components of the electric field in the empty part of the cavity, $-L < z < 0$, are as follows:

$$E_{z1} = \cos[k(z+L)]\sin(k_x x)\sin(k_y y), \tag{55}$$

$$E_{x1} = -\frac{kk_x}{k_\perp^2}\sin[k(z+L)]\cos(k_x x)\sin(k_y y), \tag{56}$$

$$E_{y1} = -\frac{kk_y}{k_\perp^2}\sin[k(z+L)]\sin(k_x x)\cos(k_y y), \tag{57}$$

$$k = \frac{\pi}{L}(1+\xi), \quad k_x = \frac{\pi}{L_x}, \quad k_y = \frac{\pi}{L_y}, \quad k_\perp^2 = k_x^2 + k_y^2.$$

These expressions are sufficient to calculate the first integral in the denominator of formula (18), because the contribution of the region $0 < z < D$ to this integral gives only a small correction of the order of $\varepsilon_b D/L$ with respect to the main term,

$$2\int \varepsilon_1\mathbf{E}_1^2\,dV \approx ML\omega_1^2/(ck_\perp)^2, \tag{58}$$

where $M = L_x L_y/4$. If the film thickness D_1 satisfies the conditions $D_1 \ll D$ and $k_2 D_1 \ll 1$, then integrals containing $\delta\varepsilon$ over the region $0 < z < D_1$ can be expressed as $M\varepsilon_s D_1 f(0+)$, where $f(0+)$ means the value of the function $f(z,x,y)$ in the integrand, divided by the related product of trigonometric functions of x and y and taken at the point $z = 0+$ inside the dielectric background slab. The components of the electric field $\mathbf{E}_1$ at this point can be obtained from the values of functions (55)–(57) at $z = 0$ using the continuity conditions. Replacing k by $k_0 = \pi/L$ in the amplitude factors and kL by $\pi + \pi\xi_1$ in the arguments of the sine and cosine functions, one finds (maintaining only the leading terms with respect to small parameters, such as D/L)

$$E_{z1}(0+) \approx -\varepsilon_b^{-1}\cos(\pi\xi_1), \tag{59}$$

$$E_{x1}(0+) \approx \frac{k_0 k_x\pi\xi_1}{k_\perp^2}, \quad E_{y1}(0+) \approx \frac{k_0 k_y\pi\xi_1}{k_\perp^2}. \tag{60}$$

Using (52) we obtain the following expressions:

$$\int \delta\varepsilon\mathbf{E}_{t1}^2\,dV \approx MD_1\varepsilon_s\left(\frac{k_3^2 D}{k_\perp\varepsilon_b}\right)^2, \tag{61}$$

$$\int \delta\varepsilon(\varepsilon_1/\varepsilon_2)E_{z1}^2\,dV \approx MD_1/\varepsilon_b. \tag{62}$$

The components of vector **G** (14) for $z = +0$ can be found from expressions (59) and (60) with the aid of (43). Thus we obtain

$$\int \delta\varepsilon(\varepsilon_1/\varepsilon_2)G_z E_{z1}\,dV \approx MLD_1D\left(\frac{\omega_1 k_3}{ck_0\varepsilon_b}\right)^2, \tag{63}$$

$$\int \delta\varepsilon\mathbf{G}_t\mathbf{E}_{t1}\,dV \approx -MLD_1D\frac{\varepsilon_s}{\varepsilon_b}\left(\frac{\omega_1 k_3}{ck_\perp}\right)^2. \tag{64}$$

One can see that the integral (63) is much smaller than (64) if $|\varepsilon_s\varepsilon_b| \gg 1$. The integral (61) can be neglected in the denominator of formula (18), because the ratio of this integral to (64) is of the order of D/L. On the other hand, the integral (61) gives the main contribution to the numerator of (18) under the condition

$$|\varepsilon_s\varepsilon_b|(D/L)^2 \gg 1. \tag{65}$$

Taking into account these relations, one can see that formula (18) goes to (53) with the same values of parameters ζ_m and $\tilde{A}_s$ as given by (54).

V. ARBITRARY THIN INHOMOGENEOUS SLABS

Now let us consider a more general case, when the dielectric function is different from the value $\varepsilon = 1$ only inside a flat thin slab, whose surface occupies some region $\mathcal{S}$ in the x-y plane and whose thickness D is much less than any other characteristic dimension L of the cavity. We assume that the plane $z = 0$ coincides with the surface of the slab. Moreover, we suppose that the background value ε_b is constant inside the slab, but the function $\varepsilon(\mathbf{r})$ rapidly changes from the maximal value ε_s on the surface of the slab to the value ε_b in a film of thickness $D_1 \ll D$. If the condition (65) is fulfilled, then, as was shown in the preceding section, formula (18) can be reduced to

$$\frac{\delta\omega}{\omega_1} \approx -\frac{\iiint \delta\varepsilon(\mathbf{r})\mathbf{E}_{t0}^2\,dV}{\iiint \left[2\mathbf{E}_0^2 + \delta\varepsilon(\mathbf{r})\mathbf{G}_{t0}\mathbf{E}_{t0}\right]dV}, \tag{66}$$

where the subscript "0" means the field in the *empty* cavity (because the tangential components of the electric field $\mathbf{E}_{t1}$ inside the film of thickness D_1 practically coincide with $\mathbf{E}_{t0}$). It is worth noting that formula (66) can be used even for dispersive media [with $\nu(\mathbf{r}) \neq 0$], except for very rare specific situations. Indeed, the term $\nu(\mathbf{r})E_{z1}^2\varepsilon_1^2/\varepsilon_2^2$ in the denominator of the fraction in the RHS of (18) is strongly suppressed if $|\varepsilon_1^2/\varepsilon_2^2| \ll 1$. The remaining term $\nu(\mathbf{r})\mathbf{E}_{t1}^2$ does not exceed in most of the cases the term $\delta\varepsilon(\mathbf{r})\mathbf{E}_{t1}^2$, whose contribution is small, as was shown in the examples considered in the preceding section. The exceptional case is $|\nu| \gg |\varepsilon_2|$, but this can happen only for specific frequencies near the resonances of the function $\varepsilon_2(\omega)$. In particular, for the model (15) with frequency-independent mobility (as for microwave frequencies) one has $\varepsilon_2(\omega) + \nu(\omega) \equiv \varepsilon_b$, meaning that $(\delta\varepsilon + \nu)\mathbf{E}_{t1}^2 \equiv 0$.

The problem is that it is difficult to calculate the vector $\mathbf{G}_{t0}$ in a general case, when the field $\mathbf{E}_0$ is calculated numerically. However, in some special (but realistic and important) cases this problem can be avoided in a rather simple way.

Namely, we suppose that the spatial dependence of $\delta\varepsilon(\mathbf{r})$ can be scaled as $\delta\varepsilon(\mathbf{r}) = \varepsilon_s g(\mathbf{r})$, where the "form factor" $g(\mathbf{r})$ does not depend on the maximal value of the dielectric function variation ε_s. Obviously, $g(\mathbf{r}) \equiv 0$ outside the slab (actually, it is close to zero even inside the slab, if $z > D_1$). If the characteristic scale D_1 of spatial variations of function $g(\mathbf{r})$ in the z direction is much smaller than that of the electric field (which is of the order of D or bigger), than one can replace functions $\mathbf{E}_{t0}$ and $\mathbf{G}_{t0}$ by their values on the surface $z = 0$ in the integrals containing $\delta\varepsilon$, so that (66) can be simplified as

$$\frac{\delta\omega}{\omega_1} \approx -\frac{\varepsilon_s \int g(\mathbf{r})\mathbf{E}_{t0}^2(x,y,0)\,dV}{\int \left[2\mathbf{E}_0^2(\mathbf{r}) + \varepsilon_s g(\mathbf{r})\mathbf{G}_{t0}(x,y,0)\mathbf{E}_{t0}(x,y,0)\right]dV}. \tag{67}$$

Taking the limit $\varepsilon_s \to \infty$, we obtain the maximal (by the absolute value) frequency shift $\zeta_m \equiv (\delta\omega/\omega_1)_{\max}$:

$$\zeta_m = -\frac{\iiint g(\mathbf{r})\mathbf{E}_{t0}^2(x,y,0)\,dV}{\iiint g(\mathbf{r})\mathbf{G}_{t0}(x,y,0)\mathbf{E}_{t0}(x,y,0)\,dV}. \tag{68}$$

Consequently, the integral containing $\mathbf{G}_{t0}$ can be expressed through the integral containing only $\mathbf{E}_{t0}$ and the parameter ζ_m. In turn, $\zeta_m = \zeta_{\rm id} - \zeta_b$, where $\zeta_{\rm id}$ is the relative frequency shift of the empty cavity, when the part of the cavity corresponding to the slab becomes an ideal conductor, and ζ_b is the relative frequency shift of the empty cavity, when the volume of the slab is filled in with the medium having the dielectric function ε_b. The coefficient ζ_b can be calculated by means of formula (18) with $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_b$. One can verify that only the term with E_{z0} in the numerator of (18) should be taken into account in this case, while the contribution of all other integrals containing $\delta\varepsilon$ can be neglected, so that

$$\zeta_b \approx \frac{1-\varepsilon_b}{\varepsilon_b}\Delta, \quad \Delta = \frac{\iiint_{\rm slab} E_{z0}^2\,dV}{2\iiint_{\rm cav}\mathbf{E}_0^2\,dV}. \tag{69}$$

Formula (69) was used, for example, in Refs. [12,13]. The coefficient Δ is called sometimes "the filling factor." On the other hand, the formula for $\zeta_{\rm id}$ is well known since the papers [1–3]:

$$\zeta_{\rm id} = \frac{\iiint_{\rm slab}(|\mathbf{H}_0|^2 - |\mathbf{E}_0|^2)\,dV}{2\iiint_{\rm cav}|\mathbf{E}_0|^2\,dV}. \tag{70}$$

Consequently,

$$\zeta_m = \frac{\iiint_{\rm slab}(|\mathbf{H}_0|^2 - |\mathbf{E}_{t0}|^2 - |E_{z0}|^2/\varepsilon_b)\,dV}{2\iiint_{\rm cav}|\mathbf{E}_0|^2\,dV}. \tag{71}$$

Thus formula (67) can be rewritten in the form analogous to (39) and (53):

$$\frac{\delta\omega}{\omega_1} \approx -\frac{\zeta_m A_g}{1-A_g}, \tag{72}$$

where ζ_m, given now by (71), does not depend on ε_s, whereas A_g is proportional to ε_s:

$$A_g = \frac{\varepsilon_s\iiint_{\rm slab} g(\mathbf{r})|\mathbf{E}_{t0}(x,y,0)|^2\,dV}{\iiint_{\rm slab}(|\mathbf{H}_0|^2 - |\mathbf{E}_{t0}|^2 - |E_{z0}|^2/\varepsilon_b)\,dV}. \tag{73}$$

For example, considering the TE mode in a rectangular empty cavity with the only nonzero component of the electric field $E_y(x,z) = \sin[k(z+L)]\sin(k_x x)$, we obtain, using (1), the following nonzero components of the magnetic field:

$$H_x = -\frac{ck}{i\omega}\cos[k(z+L)]\sin(k_x x),$$
$$H_z = \frac{ck_x}{i\omega}\sin[k(z+L)]\cos(k_x x).$$

The component H_x is much bigger than H_z and E_y in the slab region $0 < z < D$, so that Eqs. (71) and (73) [with $g(\mathbf{r}) = 1$ for $0 < z < D_1 \ll D$ and $g(\mathbf{r}) = 0$ otherwise] yield $\zeta_m \approx \lambda^2 D/(4L^3)$ and $A_g \approx (\omega/c)^2 D D_1\varepsilon_s$, in full agreement with (36) and (39).

In the TM case, Eqs. (1) and (55)–(57) give the following nonzero components of the magnetic field inside the empty rectangular cavity:

$$H_{x0} = \frac{\omega k_y}{ick_\perp^2}\cos[k(z+L)]\sin(k_x x)\cos(k_y y), \tag{74}$$

$$H_{y0} = -\frac{\omega k_x}{ick_\perp^2}\cos[k(z+L)]\cos(k_x x)\sin(k_y y). \tag{75}$$

Neglecting the contribution of $|\mathbf{E}_{t0}|^2$ in the integrals in formulas (71) and (73), one can verify again that these formulas give the same results as Eqs. (53) and (54). For the filling factor defined in Eq. (69), we find in this case the following expressions:

$$\Delta = \frac{D}{L}\left(\frac{ck_\perp}{\omega}\right)^2 = \frac{2D\eta}{\lambda(1-\eta^2)}, \tag{76}$$

where the coefficient η was defined in (38).

VI. APPLICATIONS TO THE DCE EXPERIMENTS

In the case of the DCE experiment described in [18,19], people use highly doped GaAs slabs ($\varepsilon_b \approx 13$) of thickness D about 600 μm, and a highly conducting layer nearby the surface $z = 0$ is created due to the photoabsorption of the laser radiation. The thickness of this layer D_1 is of the order of $l \approx \alpha^{-1}$, where α is the absorption coefficient. For example, if $\alpha = 10^4$ cm^{-1} (this is a realistic value), then $l/D \sim 2\times 10^{-3}$. Neglecting a possible nonuniformity of the illumination within the slab surface $\mathcal{S}$, we can assume that the function $g(\mathbf{r})$ inside the illuminated slab depends on the perpendicular coordinate z only, and $g(z)$ rapidly decreases with increase of z, due to the laser pulse attenuation. Then (73) goes to

$$A_g \approx \frac{\mathcal{J}\iint_S |\mathbf{E}_{t0}(x,y,0)|^2\,dS}{\iiint_{\text{slab}}(|\mathbf{H}_0|^2 - |\mathbf{E}_{t0}|^2 - |E_{z0}|^2/\varepsilon_b)\,dV}, \tag{77}$$

where

$$\mathcal{J} = \int_0^D \delta\varepsilon(z)dz. \tag{78}$$

It appears that using simple rectangular or cylindrical cavities (such as those considered in Sec. IV) requires too big energy of laser pulses, due to the big areas of slabs which must be illuminated. Therefore a more potentially useful geometry seems to be the so-called reentrant cylindrical cavity [24] of radius r_2 and height h_2, which has a central conducting cylindrical post of radius $r_1 < r_2$ and height $h_1 < h_2$. A thin dielectric (semiconductor) slab is attached to the post, so that $D \ll h_2 - h_1 = L$. Such cavities have been used in many devices, because their lowest resonance frequencies can be achieved for rather small dimensions [5,25–27].

Let us analyze the behavior of electric and magnetic fields near the flat surface of the central post. The magnetic field in this region is much smaller than the electric field component E_{z0} (contrary to the cases considered in Sec. IV). A rough evaluation of the predominant tangential component H_ϕ of the magnetic field near the surface, based on the model of a capacitor with time-dependent electric displacement flux (and $E_z \approx$ const nearby the post surface), gives $|H_\phi(\rho)| \approx \pi E_z\rho/\lambda$, where ρ is the radial distance from the cylinder axis in the x-y plane and λ is the resonance wavelength of the cavity. In this approximation, the ratio of the integrals in (77) containing $|\mathbf{H}_0|^2$ and $|E_{z0}|^2/\varepsilon_b$ is $(\varepsilon_b/2)(\pi r_1/\lambda)^2$. Consequently, the contribution of the magnetic field is not extremely small due to the factor ε_b, and it should be taken into account, if one wants to know the coefficient A_g with a sufficient accuracy.

The tangential component $|\mathbf{E}_{t0}|$ inside the slab is also much smaller than E_{z0}. In the first approximation one can think that $|\mathbf{E}_{t0}|$ is proportional to the distance $D - z$ from the surface of the metallic post (since this component turns into zero at this surface). Thus one can expect a rough evaluation of the ratio $|\mathbf{E}_{t0}|/|E_z|$ in this region as D/L. In such a case, the relative contributions of the integrals containing $|\mathbf{H}_0|^2$ and $|\mathbf{E}_{t0}|^2$ in (71) and (73) can hardly exceed a few percent, compared with the main contribution from the electric field component E_{z0}. In addition, E_{z0}^2 only slightly deviates from its values on the surface $z = 0$ inside the slab if $D \ll L$. Taking into account all these evaluations, as well as the inequality $l \ll D$, one arrives at the following expressions:

$$\zeta_m \approx -\frac{D\iint_S E_{z0}^2(x,y,0)\,dS}{2\varepsilon_b\iiint_{\text{cav}}\mathbf{E}_0^2(\mathbf{r})\,dV} \approx \zeta_{\text{id}}/\varepsilon_b. \tag{79}$$

Note that ζ_m is negative for the reentrant cavity. Moreover, this important parameter can be found experimentally from the measurement of ζ_{id} (by replacing the semiconductor slab with a metallic one of the same thickness and measuring the corresponding frequency shift), if the value of ε_b is known.

If the creation of electron-hole pairs in the semiconductor material changes only its conductivity, the function $\delta\varepsilon(z)$ is pure imaginary, $\delta\varepsilon = 4\pi i\sigma/\omega$. Immediately after the absorption of a very short laser pulse we have $N(0) \equiv \int_0^D n(z)dz = \kappa W/(SE_g)$, where W is the total pulse energy, S is the area of the slab surface, E_g is the energy gap of the semiconductor material, and $\kappa \leqslant 1$ is the quantum efficiency of photoabsorption. In the absence of significant surface recombination and diffusion, the total concentration per unit area depends on time exponentially [28] as $N(t) = N(0)\exp(-t/T_r)$, where T_r is the recombination time. Consequently, the time evolution of the real and imaginary parts of the complex frequency shift $\delta\omega = \omega_1(\chi - i\gamma)$ can be described approximately as follows:

$$\chi(t) \approx \frac{\zeta_m A^2(t)}{A^2(t)+1}, \quad \gamma(t) \approx \frac{|\zeta_m|A(t)}{A^2(t)+1}, \tag{80}$$

where $A(t) = iA_g(t) = A_0\exp(-t/T_r)$,

$$A_0 = \varepsilon_b Y K(D/\lambda), \quad Y = \frac{2|eb|\kappa W}{cE_gS}, \tag{81}$$

$$K = \left(\frac{\lambda}{D}\right)^2 \frac{\iint_S \mathbf{E}_{t0}^2(x,y,0)\,dS}{\iint_S E_{z0}^2(x,y,0)\,dS}. \tag{82}$$

Formula (81) clearly shows the influence of different factors on the parameter A_0. The ratio $(\lambda/D)^2$ is introduced in the definition of the dimensionless parameter K (82) in order to make it approximately independent of the slab thickness, since $|\mathbf{E}_{t0}(x,y,0)| \sim D$ for $D \ll L < \lambda$.

VII. CONCLUSION

The main results of this paper are given by the general formula (18) and its special case (66). One of their consequences is the universal interpolation formula given by Eqs. (72) and (73), which describes the resonance frequency shift caused by thin layers even for very big changes (by many orders of magnitude) of the complex dielectric function inside these layers. Note that the layer can be inhomogeneous and the dielectric function can be frequency-dependent. In particular, Eqs. (80)–(82) can be useful for the analysis of the experiments on the dynamical Casimir effects in cavities with photoexcited thin semiconductor slabs, which are now under way.

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