

# Magnetization effect in momentum conservation in partially coherent wavefields

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The law of momentum conservation and its applications for partially coherent electromagnetic fields has been recently studied with the emphasis on the electric polarization. In this work, we provide more complete expressions of the formalism by including magnetization. Even though polarization gives a dominant effect in momentum flow in electromagnetic wave systems in general, magnetization can also be important in optical metamaterials. Therefore, this more general formalism will be useful for some physical situations where magnetization can be nonnegligible.

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## I. INTRODUCTION

In 1986, researchers demonstrated that [1] microscopic particles may be trapped in high-intensity regions of focused optical fields, providing a very useful and popular tool for the microscopist known as “optical tweezing.” Variations of the technique have resulted in a number of strategies for manipulating particles with optical fields; for instance, the authors of [2–5] have demonstrated that particles may be rotated by a transfer of angular momentum. Furthermore, several researchers [6–8] have shown that the intensity distribution in the focal region may be altered by modifying the spatial coherence of the focused wave, potentially providing an additional degree of freedom for particle manipulation.

Discussions of optical trapping have brought renewed attention to the momentum of light and the conservation laws associated with it. In a recent paper [9], a law of momentum conservation for partially coherent electromagnetic waves in the space-frequency domain was demonstrated, along with an associated scattering formalism. These results were formulated for sources of electric polarization, or equivalently, scattering objects with unit permeability. This encompasses most ordinary sources and materials, which are nonmagnetic at optical frequencies.

The introduction of optical metamaterials [10], however, has raised the possibility of, if not the inevitability of, man-made materials with nontrivial magnetic properties. Metamaterials have been suggested for the development of a number of counterintuitive optical devices, such as “perfect lenses” [11] and “optical cloaks” [12]. It is therefore of interest to investigate how the momentum properties of light change when the source has a magnetization as well as a polarization, and furthermore to see what influence partial coherence has on the formalism.

In this paper we derive the Maxwell stress tensor for a partially coherent electromagnetic wave that is generated by a partially coherent source possessing both magnetization and polarization. In Sec. II, we introduce the basic expressions for partially coherent waves and sources in the space-frequency domain, while in Sec. III we derive the Maxwell stress tensor for such a mixed electric and magnetic source. In Sec. IV we provide some concluding remarks.

## II. PARTIALLY COHERENT ELECTROMAGNETIC FIELDS PRODUCED BY POLARIZATION AND MAGNETIZATION

We begin by considering the behavior of partially coherent electromagnetic fields produced by sources consisting of polarization and magnetization. For convenience, we work with the space-frequency representation of electromagnetic fields (see, for instance, [13,14]), in which a partially coherent wave field at frequency  $\omega$  can be represented as an average over an appropriately chosen ensemble of monochromatic wave fields [15].

We introduce a monochromatic ensemble of sources with polarization  $\mathbf{P}(\mathbf{r},\omega)$  and magnetization  $\mathbf{M}(\mathbf{r},\omega)$ , confined to a volume  $V$  in free space, and introduce the electric and magnetic Hertz vectors [16]  $\pi_e(\mathbf{r},\omega)$  and  $\pi_m(\mathbf{r},\omega)$ , respectively, which satisfy the equations

$$\nabla^2 \pi_e(\mathbf{r},\omega) + k^2 \pi_e(\mathbf{r},\omega) = -4\pi \mathbf{P}(\mathbf{r},\omega), \quad (1)$$

$$\nabla^2 \pi_m(\mathbf{r},\omega) + k^2 \pi_m(\mathbf{r},\omega) = -4\pi \mathbf{M}(\mathbf{r},\omega), \quad (2)$$

where  $k = \omega/c$ ,  $c$  being the speed of light. The solution to these equations may be represented in integral form

$$\pi_e(\mathbf{r},\omega) = \int_V \mathbf{P}(\mathbf{r}',\omega) G(\mathbf{r} - \mathbf{r}') d^3 r', \quad (3)$$

$$\pi_m(\mathbf{r},\omega) = \int_V \mathbf{M}(\mathbf{r}',\omega) G(\mathbf{r} - \mathbf{r}') d^3 r', \quad (4)$$

where

$$G(\mathbf{R}) = \frac{e^{ikR}}{R}, \quad (5)$$

is the free-space Green’s function for the Helmholtz equation.

It follows from the properties of the Hertz vectors that the electric and magnetic fields are related to them by the expressions

$$\mathbf{E}(\mathbf{r},\omega) = \nabla \times [\nabla \times \pi_e(\mathbf{r},\omega)] + ik \nabla \times \pi_m(\mathbf{r},\omega), \quad (6)$$

$$\mathbf{B}(\mathbf{r},\omega) = \nabla \times [\nabla \times \pi_m(\mathbf{r},\omega)] - ik \nabla \times \pi_e(\mathbf{r},\omega). \quad (7)$$

We will find it more convenient to write these fields in a tensor notation

$$E_i = \partial_i \partial_j \pi_j^e - \partial_j^2 \pi_i^e + ik \epsilon_{ijk} \partial_j \pi_k^m, \quad (8)$$

$$B_i = \partial_i \partial_j \pi_j^m - \partial_j^2 \pi_i^m - ik \epsilon_{ijk} \partial_j \pi_k^e, \quad (9)$$

where  $\pi_k^e$  is the  $k$ th component of the Hertz vector  $\boldsymbol{\pi}_e$ , and so forth, and  $\partial_i = \partial/\partial x_i$  is the partial derivative with respect to the  $i$ th Cartesian coordinate. The symbol  $\epsilon_{ijk}$  is the Levi-Civita symbol. We further simplify these expressions by introducing the operators

$$\mathcal{M}_{ij} \equiv \partial_i \partial_j - \partial_i^2 \delta_{ij}, \quad (10)$$

$$\mathcal{N}_{ij} \equiv ik \epsilon_{ilj} \partial_l, \quad (11)$$

where  $\delta_{ij}$  is the Kronecker delta symbol. We may then write

$$E_i = \mathcal{M}_{il} \pi_l^e + \mathcal{N}_{il} \pi_l^m, \quad (12)$$

$$B_i = \mathcal{M}_{il} \pi_l^m - \mathcal{N}_{il} \pi_l^e. \quad (13)$$

We now consider partially coherent fields. In the space-frequency representation, we may introduce the cross-spectral density tensor of the electric field as

$$W_{ij}^{EE}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle E_i^*(\mathbf{r}_1, \omega) E_j(\mathbf{r}_2, \omega) \rangle_\omega, \quad (14)$$

where the brackets  $\langle \rangle_\omega$  represent averaging over the ensemble of space-frequency realizations. Similarly, the cross-spectral density tensor of the magnetic field is given by

$$W_{ij}^{BB}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle B_i^*(\mathbf{r}_1, \omega) B_j(\mathbf{r}_2, \omega) \rangle_\omega. \quad (15)$$

On substituting from Eq. (12) into Eq. (14), we find that we may write the latter expression as

$$W_{ij}^{EE} = \mathcal{M}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)} W_{lm}^{ee} + \mathcal{N}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)} W_{lm}^{me} + \mathcal{M}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} W_{lm}^{em} + \mathcal{N}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} W_{lm}^{mm}. \quad (16)$$

Here we have introduced the functions

$$W_{lm}^{ee}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle \pi_l^{e*}(\mathbf{r}_1, \omega) \pi_m^e(\mathbf{r}_2, \omega) \rangle, \quad (17)$$

$$W_{lm}^{em}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle \pi_l^{e*}(\mathbf{r}_1, \omega) \pi_m^m(\mathbf{r}_2, \omega) \rangle, \quad (18)$$

and so forth, to represent the cross-spectral density tensors of products of Hertz vectors; the arguments of the tensors will be suppressed for brevity for the moment. The notation  $\mathcal{M}_{ij}^{(1)}$  is used for an operator that acts on the first variable  $\mathbf{r}_1$  of a cross-spectral density tensor, with a similar definition for the other operators.

The cross-spectral density tensor for the magnetic field may similarly be written as

$$W_{ij}^{BB} = \mathcal{M}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)} W_{lm}^{mm} - \mathcal{N}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)} W_{lm}^{em} - \mathcal{M}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} W_{lm}^{me} + \mathcal{N}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} W_{lm}^{ee}. \quad (19)$$

The individual tensors  $W_{lm}^{ee}$ ,  $W_{lm}^{em}$ , and so forth, may be written in terms of the cross-spectral density tensors of the individual sources, for instance,

$$W_{lm}^{ee}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \int_V \int_V G^*(\mathbf{r}_1 - \mathbf{r}'_1) G(\mathbf{r}_2 - \mathbf{r}'_2) W_{lm}^{PP} \times (\mathbf{r}'_1, \mathbf{r}'_2, \omega) d^3 r'_1 d^3 r'_2, \quad (20)$$

$$W_{lm}^{em}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \int_V \int_V G^*(\mathbf{r}_1 - \mathbf{r}'_1) G(\mathbf{r}_2 - \mathbf{r}'_2) W_{lm}^{PM} \times (\mathbf{r}'_1, \mathbf{r}'_2, \omega) d^3 r'_1 d^3 r'_2, \quad (21)$$

with

$$W_{lm}^{PP}(\mathbf{r}'_1, \mathbf{r}'_2, \omega) \equiv \langle P_l^*(\mathbf{r}'_1, \omega) P_m(\mathbf{r}'_2, \omega) \rangle, \quad (22)$$

$$W_{lm}^{PM}(\mathbf{r}'_1, \mathbf{r}'_2, \omega) \equiv \langle P_l^*(\mathbf{r}'_1, \omega) M_m(\mathbf{r}'_2, \omega) \rangle. \quad (23)$$

The correlation tensors represented by Eqs. (16) and (19) allow us to calculate most second-order electromagnetic phenomena of physical interest, namely the energy, power flow, and momentum flow of the electromagnetic field, including the effects from sources of both polarization and magnetization.

### III. MAXWELL STRESS TENSOR ARISING FROM POLARIZATION AND MAGNETIZATION

For monochromatic fields, the Maxwell stress tensor that characterizes momentum flow is given by [17]

$$T_{ij}(\mathbf{r}, \omega) = \frac{1}{4\pi} \left\{ E_i(\mathbf{r}, \omega) E_j(\mathbf{r}, \omega) + B_i(\mathbf{r}, \omega) B_j(\mathbf{r}, \omega) - \frac{1}{2} \delta_{ij} [E_l(\mathbf{r}, \omega) E_l(\mathbf{r}, \omega) + B_l(\mathbf{r}, \omega) B_l(\mathbf{r}, \omega)] \right\}. \quad (24)$$

It is to be noted that we restrict ourselves to the fields in free space, outside of dielectric and magnetic materials; we therefore avoid the ambiguity in the definition of the momentum of light in matter that is the center of the Abraham-Minkowski controversy [18].

The ensemble averaged version of the stress tensor is then given by the expression

$$\langle T_{ij}(\mathbf{r}, \omega) \rangle = \frac{1}{4\pi} \left\{ W_{ij}^{EE}(\mathbf{r}, \mathbf{r}, \omega) + W_{ij}^{BB}(\mathbf{r}, \mathbf{r}, \omega) - \frac{1}{2} \delta_{ij} [W_{ll}^{EE}(\mathbf{r}, \mathbf{r}, \omega) + W_{ll}^{BB}(\mathbf{r}, \mathbf{r}, \omega)] \right\}. \quad (25)$$

From the definitions of the Hertz vector correlation tensors [e.g., Eq. (17)] we may note the following relation between the mixed tensors

$$W_{lm}^{em}(\mathbf{r}_1, \mathbf{r}_2, \omega) = [W_{ml}^{me}(\mathbf{r}_2, \mathbf{r}_1, \omega)]^*. \quad (26)$$

With this result, we may write the sum of the electric and magnetic cross-spectral density tensors as

$$W_{ij}^{EE} + W_{ij}^{BB} = \mathcal{M}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)} [W_{lm}^{ee} + W_{lm}^{mm}] + \mathcal{N}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} [W_{lm}^{ee} + W_{lm}^{mm}] + [\mathcal{M}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} - \mathcal{N}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)}] W_{lm}^{em} + \{ [\mathcal{M}_{jl}^{(1)*} \mathcal{N}_{im}^{(2)} - \mathcal{N}_{jl}^{(1)*} \mathcal{M}_{im}^{(2)}] W_{lm}^{em} \}^*. \quad (27)$$

The first two terms of this expression combined represent the sum of contributions from the polarization and magnetization sources in the absence of interference; the latter two terms of the expression represent interference terms between the two types of sources. We may write the total stress tensor for a source of magnetization and polarization in the form

$$\langle T_{ij}(\mathbf{r}, \omega) \rangle = T_{ij}^e(\mathbf{r}, \omega) + T_{ij}^m(\mathbf{r}, \omega) + T_{ij}^c(\mathbf{r}, \omega), \quad (28)$$

where  $T_{ij}^e(\mathbf{r}, \omega)$  is the stress tensor derived from the electric polarization of the source alone,  $T_{ij}^m(\mathbf{r}, \omega)$  is the stress tensor derived from the magnetization of the source alone, and  $T_{ij}^c(\mathbf{r}, \omega)$  is the stress tensor derived from the interference between the electric polarization of the source and the magnetization of the source and can be rewritten as

$$T_{ij}^c(\mathbf{r}, \omega) = \frac{1}{4\pi} \left\{ C_{ij}(\mathbf{r}) - \frac{1}{2} \delta_{ij} [C_{ll}(\mathbf{r})] \right\}, \quad (29)$$

and

$$C_{ij}(\mathbf{r}) = [\mathcal{M}_{il}^{(1)*} \mathcal{N}_{jm}^{(2)} - \mathcal{N}_{il}^{(1)*} \mathcal{M}_{jm}^{(2)}] W_{lm}^{em} + \{ [\mathcal{M}_{jl}^{(1)*} \mathcal{N}_{im}^{(2)} - \mathcal{N}_{jl}^{(1)*} \mathcal{M}_{im}^{(2)}] W_{lm}^{em} \}^*. \quad (30)$$

Equation (28) is one of the main results of this paper. It demonstrates that the total stress tensor due to electric and magnetic sources can be expressed as the sum of the individual stress tensors plus an interference term.

A situation of interest is the determination of the flow of momentum in the far zone of the electromagnetic source (i.e., at distances such that  $|\mathbf{r}| \gg |\mathbf{r}'|$ ). At such distances, we write  $\mathbf{r} = r\mathbf{u}$ , and it is to be noted that

$$\partial_j \frac{e^{ikR}}{R} \sim \partial_j \frac{e^{ikr}}{r} e^{-ik\mathbf{u}\mathbf{r}'}. \quad (31)$$

The derivatives in the operators  $\mathcal{M}_{ij}$  and  $\mathcal{N}_{ij}$  can be evaluated directly, and the operators take the form

$$\mathcal{M}_{ij}, \mathcal{M}_{ij}^* \rightarrow k^2 \delta_{ij} - k^2 u_i u_j, \quad (32)$$

$$\mathcal{N}_{ij} \rightarrow -k^2 \epsilon_{inj} u_n, \quad \mathcal{N}_{ij}^* \rightarrow k^2 \epsilon_{inj} u_n, \quad (33)$$

where  $u_i$  is the  $i$ th component of the unit vector  $\mathbf{u}$  in the direction of observation. With these definitions, we note that

$C_{ij}$  may be written as

$$C_{ij} = -k^4 [(\delta_{il} - u_i u_l) \epsilon_{jnm} u_n + \epsilon_{inl} u_n (\delta_{jm} - u_j u_m)] W_{lm}^{em} - k^4 [(\delta_{jl} - u_j u_l) \epsilon_{imn} u_n + \epsilon_{jln} u_n (\delta_{im} - u_i u_m)] W_{lm}^{em*}. \quad (34)$$

With the stress tensor, the momentum flow  $P_j$  in the direction of unit vector  $\mathbf{u}$  is written as

$$P_j = u_i T_{ij}. \quad (35)$$

However, we can readily show that  $u_i C_{ij} = 0$  because, for instance,

$$u_i [\delta_{il} - u_i u_l] = u_l - u_l = 0, \quad (36)$$

$$\epsilon_{inl} u_i u_n = \mathbf{u} \times [\mathbf{u} \times] = 0. \quad (37)$$

The momentum flow in the direction  $\mathbf{u}$  also has a term of the form  $u_i C_{ll}$ ; this term can also be shown to vanish using Eq. (37). In the far zone of a mixed electric and magnetic source, we therefore find that

$$P_j \sim P_j^e + P_j^m. \quad (38)$$

In other words, *the total momentum flow in the far zone is simply the combined momentum of the electric and magnetic source terms, without any interference effects playing a role.*

#### IV. CONCLUSION

We have derived a more general expression for the Maxwell stress tensor and the conservation of momentum for partially coherent electromagnetic wave fields that includes both the electric (polarization) and magnetic (magnetization) properties of a primary radiation source. The stress tensor may, in general, be divided into a purely electric part, a purely magnetic part, and an interference term between them. In the far zone, however, the interference part disappears completely and the total momentum flow from the source is the sum of the individual contributions from the electric and magnetic parts separately.

Though magnetization is typically negligible for ordinary materials at optical frequencies, the increasing emphasis on metamaterials with nontrivial magnetic properties (for instance, [10–12]) suggests that our results will be useful as the field develops.

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