Three-site Bose-Hubbard model subject to atom losses: Boson-pair dissipation channel and failure of the mean-field approach

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We employ the perturbation series expansion for derivation of the reduced master equations for the three-site Bose-Hubbard model subject to strong atom losses from the central site. The model describes a condensate trapped in a triple-well potential subject to externally controlled removal of atoms. We find that the π-phase state of the coherent superposition between the side wells decays via two dissipation channels, the single-boson channel (similar to the externally applied dissipation) and the boson-pair channel. The quantum derivation is compared to the classical adiabatic elimination within the mean-field approximation. We find that the boson-pair dissipation channel is not captured by the mean-field model, whereas the single-boson channel is described by it. Moreover, there is a matching condition between the zero-point energy bias of the side wells and the nonlinear interaction parameter which separates the regions where either the single-boson or the boson-pair dissipation channel dominate. Our results indicate that the M-site Bose-Hubbard models, for $M > 2$, subject to atom losses may require an analysis which goes beyond the usual mean-field approximation for correct description of their dissipative features. This is an important result in view of the recent experimental works on the single-site addressability of condensates trapped in optical lattices.

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I. INTRODUCTION

The mean-field approximation, formally obtained by replacing the boson creation and annihilation operators by complex scalars, is usually employed for description of bosonic many-body systems when the number of bosons is large (for instance, see Refs. [1–3]). Such a replacement can be justified by the $\sqrt{N}$ scaling of the boson operators, that is, when the populations are large, the commutators—the source of quantum corrections—are negligible [1]. The relation between the quantum and mean-field descriptions is a subject of intensive studies. The quantum description is necessary for the bifurcations, which modify significantly the quantum spectrum [4], the quantum collapses and revivals [5], and the many-body quantum corrections to the mean-field theory [6,7]. Making explicit the N scaling of the operators and identifying the N scaling of the parameters for a fixed particle density, reveals the link of the mean-field approximation to the Wentzel-Kramers-Brillouin semiclassical approach to the discrete Schrödinger equation [8], now employed in the Fock space with the inverse number of bosons $1/N$ playing the role of an effective Planck constant (see, for example, Ref. [9]). Therefore, the mean-field limit, as the semiclassical limit of a discrete Schrödinger equation, is also *singular*. Hence, besides the pronounced quantum corrections and fluctuations at the bifurcations and instabilities, one must be prepared to find even a *qualitative* disagreement between the mean-field description and the full quantum consideration even when the populations are large, as it is the case, for instance, in the nonlinear boson model which describes tunneling of boson pairs between two modes, see Refs. [10,11].

The main purpose of the present paper is to study the dissipation dynamics of the atom-filled lattice sites coupled to a common dissipated site. Our motivation is the recent advancement in the quantum microscopy techniques and the current experiments on the single-site addressability in the optical lattices [12,13], where controlled atom losses are induced in selected sites of a two-dimensional optical lattice. We develop a direct method based on the perturbation expansion for derivation of the reduced quantum master equations for the Bose-Hubbard models with dissipation (we consider the atom losses) and compare the quantum derivation with the mean-field description. We concentrate on the three-site Bose-Hubbard model, which is the simplest model describing atom-filled sites coupled to a common dissipated one and describes, for instance, cold atoms or Bose-Einstein condensate trapped in a triple-well potential subject to removal of atoms from the central well, see Fig. 1. The three-site Bose-Hubbard model was also noted to possess many features of complexity of a general quantum dynamics, as the Wigner-Dyson spectral statistics and quantum chaos [14,15]. It is also the simplest model where the boson-pair tunneling, originating from the nonlinearity of the model, is possible.

We derive the reduced quantum master equations for the coherent modes describing the condensate in the side wells, then consider the mean-field approach and compare the results of the two approaches. We note here that we consider an open quantum system and, as such, described by the quantum master equation [16,17]. However, in the case of the atom losses, the mean-field formulation is straightforward (this also applies to the case of the multiple-site atom losses of the experimental works of Refs. [12,13]). Contrary to the fact that the mean-field approximation applies with a good accuracy to the *two-site* Bose-Hubbard model with atoms losses and a noise [18,19], it is shown here that the correct and complete description of the three-site model (in general, the M-site Bose-Hubbard models, with $M \geqslant 3$) requires going beyond the usual mean-field approach. This disagreement stems from the quantum

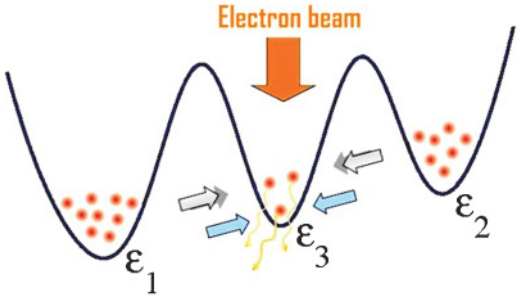

FIG. 1. (Color online) The three-site Bose-Hubbard model setup. A laser or electron beam removes atoms from the central well with a rate Γ. The ε_j represents the zero-point energy of the jth well. The two quantum channels of dissipation of the π-phase coherent mode, described in the text, are shown schematically by arrows.

boson-pair dissipation channel, due to the nonlinear interaction (this is similar to the boson-pair tunneling resulting in the qualitative failure of the mean-field approach in Ref. [11]). Moreover, there is a matching condition between the zero-point energy bias of the side wells and the nonlinear interaction parameter which separates the regions where either the single-boson or the boson-pair dissipation channel dominate. Hence, one has to use the full quantum consideration, i.e., the quantum master equation reduction methods, to describe the decay of the subsystem (in our case, the quantum modes describing the side wells), which then may be treated with further approximations, even *resembling* the mean-field approach. However, the point is that without invoking the quantum consideration at some stage, i.e., working just within the mean-field approach, one will be unable to describe the dissipative behavior of the filled sites coupled to a common dissipated site, which conclusion is also relevant to the recent experiments of Ref. [12].

The paper is organized as follows. In Sec. II we formulate the quantum master equation. The derivation of the reduced master equation for the side modes (i.e., the coherent superposition modes over the left and right wells in Fig. 1) is given in Sec. III. In Sec. IV a similar reduction is applied to the reduced master equation of Sec. III producing the master equation for the π-phase coherent mode. In Sec. V the adiabatic elimination within the mean-field approximation is studied. The concluding remarks and discussion is contained in Sec. VI.

II. THE PROBLEM FORMULATION IN TERMS OF THE MASTER EQUATION

A quantum channel representation for a local atom losses (e.g., from a single lattice site), see Fig. 1, can be given in the Fock space as follows [19]:

$$|k_j\rangle|0\rangle_R \to \sqrt{p}|k_j-1\rangle|1\rangle_R + \sqrt{1-p}|k_j\rangle|0\rangle_R,$$

$|k_j\rangle$ is the ket vector of the Fock space of a dissipating boson mode, $|X\rangle_R$ describes the atom counter, and $p = p(k_j,t)$ is the probability. Note that $p(k_j,\delta t)$, for small δt, depends linearly on the number of atoms in the dissipating mode. As the result, a Lindblad term with the generator $L = \sqrt{\Gamma}a_j$ appears in the master equation for the density matrix. Here Γ is the dissipation rate parameter. We consider the dissipation acting on the central well (denoted with $j = 3$ in Fig. 1) of the triple-well trap, thus the master equation for the density matrix reads

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \Gamma\left(a_3\rho a_3^\dagger - \frac{1}{2}n_3\rho - \frac{1}{2}\rho n_3\right). \tag{1}$$

By H we denote the three-site Bose-Hubbard Hamiltonian,

$$H = -J([a_1^\dagger + a_2^\dagger]a_3 + a_3^\dagger[a_1 + a_2]) + \sum_{j=1}^{3}\varepsilon_j a_j^\dagger a_j + U\sum_{j=1}^{3}(a_j^\dagger)^2 a_j^2, \tag{2}$$

where a_j and $a_j^\dagger$ $(j = 1,2,3)$ are the local boson modes describing occupation of the respective well, J is the linear tunneling rate, ε_j is the zero-point energy of the respective well, and U is the atomic interaction parameter proportional to the s-wave scattering length.

Due to the linear coupling of the central well to the side wells in the Bose-Hubbard model, it is convenient to use the new canonical basis $b_1 = (a_1 + a_2)/\sqrt{2}$, $b_2 = (a_1 - a_2)/\sqrt{2}$, and $b_3 = a_3$. Here the modes $b_{1,2}$ describe, respectively, the zero-phase and π-phase coherent superpositions between the side wells. The Bose-Hubbard Hamiltonian becomes

$$H = -J_{13}(b_1^\dagger b_3 + b_3^\dagger b_1) - J_{12}(b_1^\dagger b_2 + b_2^\dagger b_1) + \varepsilon n_3 + Un_3^2 + \frac{U}{2}([n_1+n_2]^2 + [b_1^\dagger b_2 + b_2^\dagger b_1]^2), \tag{3}$$

where $n_j = b_j^\dagger b_j$ and we have introduced the parameters $J_{13} = \sqrt{2}J$, $J_{12} = (\varepsilon_2 - \varepsilon_1)/2$, $\varepsilon = \varepsilon_3 - (\varepsilon_1 + \varepsilon_2)/2$ and dropped an inessential term proportional to $N = n_1 + n_2 + n_3$ (see the Appendix for more details).

III. THE REDUCED MASTER EQUATION FOR THE SIDE WELLS

In the strongly dissipative case, $\Gamma \gg J_{13}/\hbar$, the central well is emptied on the time scale $t \sim 1/\Gamma$, while the coherent modes b_1 and b_2 almost retain their populations (see also Ref. [19]). It turns out that in this case one can derive an approximate master equation for the side wells alone valid for the times $t \gg 1/\Gamma$, when the atom removal from the central-well occurs on the fastest time-scale in the system. Introducing the small parameter $\epsilon = J_{13}/\hbar\Gamma$ we require that $J_{12}/\hbar\Gamma = O(\epsilon)$ and $U\langle n_{1,2}\rangle/\hbar\Gamma = O(\epsilon)$ [the symbol $O(\epsilon)$ means the first-order in ϵ]. This approximation is valid for an arbitrary bias ε between the central and side wells.

Before we embark on the derivation of reduced master equations for the coherent modes (in this section for b_1 and b_2 and in Sec. IV for b_2 alone) let us make two remarks.

First, we will work with the quantum master equation for the density matrix of the system, whereas in the related mean-field approximation, Sec. V, the adiabatic elimination will be performed for the classical (i.e., mean-field) amplitudes. One could, in fact use the adjoint quantum master equation for the observables of the system (see, for instance, Ref. [17]) to come to the same results on the decay rates and Lamb shifts, and which derivation would directly correspond to working with the classical amplitudes (in this case, the

differences between the classical mean-field and the full quantum considerations are due to the contribution of the quantum commutators). However, we chose to work with the density matrix directly in the quantum case, since this approach is much more informative on the nature of the quantum state of the system. For instance, we will see that, already in the first order in ϵ, the density matrix for the whole system is not factorized and the central-well population strongly depends on that of the side wells. Nevertheless, in the higher orders of the perturbation theory (namely, in the second and third orders in ϵ), the central well acts as an effective reservoir for the side wells leading to the master equations for the coherent modes b_1 and b_2 in the standard Lindblad form.

Second, by analyzing the nature of the coupling between the modes b_1 and b_2 and their coupling to the dissipated mode $b_3 = a_3$ one can already make some conclusions about the expected results. Indeed, the Hamiltonian of the system in the form of Eq. (3) shows two channels of coupling between the coherent modes b_1 (zero-phase mode) and b_2 (π-phase mode): the linear single-boson coupling, given by $-J_{12}[b_1^\dagger b_2 + b_2^\dagger b_1]$, and the boson-pair coupling, given by $U[b_1^\dagger b_2 + b_2^\dagger b_1]^2/2$ (whereas the terms with $n_{1,2}$ do not induce any particle exchange between the modes). Hence, there must be two channels of dissipation from the side wells, correspondingly. This is also an essential feature of the Bose-Hubbard model formulated for the three or more lattice sites. Thus one can predict that atoms leave the side wells either one by one (the single-boson channel, the only possible for the coherent mode b_1) or by pairs (the boson-pair channel, which appears in the case of the mode b_2, since bosons must first go from mode b_2 to b_1 and only then can leave the side wells). Moreover, one could even guess the possible decay rates of the coherent modes $b_{1,2}$ by using the simpler and already studied two-site Bose-Hubbard model [18,19], i.e., at least that of the linear single-boson channel which is similar to the two-mode case. However, the derivation below produces also the form of the density matrix of the system which cannot be guessed beforehand. Moreover, while the general structure of the decay rates can be guessed, the actual coefficients must be derived [note for instance the $\sqrt{2}$ coefficient in the resonance condition $U = \sqrt{2}J_{12}$ of Sec. IV, discussed below Eq. (45)].

Let us now present the derivation of the reduced master equation for the mode b_1. First of all, let us pass to the interaction picture. Introduce

$$\begin{aligned} &\hat{H}_I(t) = U_0^\dagger(t) H_I U_0(t), \quad \hat{\rho} = U_0^\dagger(t)\rho(t)U_0(t), \\ &U_0(t) = U_{12}(t)\otimes U_3(t) = \exp\left\{-\frac{i}{\hbar}(H_{12}+H_3)\Delta t\right\}, \end{aligned} \tag{4}$$

where $\Delta t = t - t_0$, t_0 is some initial time, and H_{12}, H_3, and H_I are parts of the Hamiltonian (3) describing the side wells, the central well, and the interaction between them, respectively [we have also subtracted the nonessential term $U(n_1+n_2+n_3)$ from the system Hamiltonian to simplify the presentation]. The master equation in the interaction picture reads

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_I(t),\hat{\rho}] + \Gamma\left(\hat{b}_3(t)\hat{\rho}\hat{b}_3^\dagger(t) - \frac{1}{2}\hat{n}_3\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{n}_3\right), \tag{5}$$

where $\hat{b}_3(t) = U_0^\dagger(t) b_3 U_0(t)$. *Below we will work in the interaction picture and drop the hat, for simplicity.* The density matrix expansion reads $\rho = \rho_{12}^{(0)}\otimes\rho_3^{(0)} + \rho^{(1)} + \rho^{(2)} + O(\epsilon^3)$, where, for instance, $\rho_{12} = \mathrm{Tr}_3\{\rho\}$ (an arbitrary nonfactorized expansion, $\rho^{(0)} = \sum_{ij} c_{ij}\rho_{12i}^{(0)}\otimes\rho_{3j}^{(0)}$, leads to the same result). In the lowest orders in ϵ we have

$$\frac{d\rho_{12}^{(0)}}{dt} = 0, \quad \frac{d\rho_3^{(0)}}{dt} = \Gamma D(t)\{\rho_3^{(0)}\}, \tag{6}$$

$$\frac{d\rho^{(1)}}{dt} = -\frac{i}{\hbar}\left[H_I(t),\rho_{12}^{(0)}\otimes\rho_3^{(0)}\right] + \Gamma D(t)\{\rho^{(1)}\}, \tag{7}$$

$$\frac{d\rho_{12}^{(2)}}{dt} = -\frac{i}{\hbar}\mathrm{Tr}_3\{[H_I(t),\rho^{(1)}]\}. \tag{8}$$

The solution to the equation for $\rho_3^{(0)}$ is as follows:

$$\rho_3^{(0)}(t) = |0\rangle\langle 0| + e^{-\frac{\Gamma}{2}\Delta t}[c_1|1\rangle\langle 0| + \mathrm{H.c.}] + O(e^{-\Gamma\Delta t}), \tag{9}$$

where the last term represents all higher-order external products of the Fock basis states and a decaying renormalization correction to the first term. The coefficient c_1 has the following meaning: $\mathrm{Tr}_3\{b_3(t)\rho_3^{(0)}(t)\} = c_1 f_1(t) e^{-\frac{\Gamma}{2}\Delta t} + O(e^{-\Gamma\Delta t})$, with f_1 being the eigenvalue of the unitary transformation in Eq. (5): $U_3(t)|1\rangle = f_1(t)|1\rangle$, in our case $f_1(t) = e^{-i\varepsilon\Delta t/\hbar}$ [note also that $b_3(t) = f_1(t)b_3$].

Inserting Eq. (9) into Eq. (7) we get the general solution in the following form (suggested by the form of $\rho_3^{(0)}$ itself):

$$\rho^{(1)}(t) = V(t)\left[\rho_{12}^{(0+1)}\otimes|0\rangle\langle 0|\right]V^\dagger(t) + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right), \tag{10}$$

where $\rho_{12}^{(0+1)} = \rho_{12}^{(0)} + \rho_{12}^{(1)}$ and $V(t)$ reads

$$V(t) = \exp\{\alpha_1(t)b_1(t)b_3^\dagger(t)\} = I + \alpha_1(t)b_1(t)b_3^\dagger(t) + O(\epsilon^2), \tag{11}$$

with a scalar function $\alpha_1(t) = O(\epsilon)$. Indeed, taking derivative of the solution (11) and using Eqs. (7) and (9), we obtain the equation to be satisfied

$$\begin{aligned} \frac{d\rho^{(1)}}{dt} &= \frac{dV}{dt}\rho_{12}^{(0)}\otimes|0\rangle\langle 0|V^\dagger + V\rho_{12}^{(0)}\otimes|0\rangle\langle 0|\frac{dV^\dagger}{dt} \\ &+ V\frac{d\rho_{12}^{(1)}}{dt}\otimes|0\rangle\langle 0|V^\dagger = -\frac{i}{\hbar}\left[H_I(t),\rho_{12}^{(0)}\otimes|0\rangle\langle 0|\right] \\ &+ \Gamma D\left\{V\rho_{12}^{(0+1)}\otimes|0\rangle\langle 0|V^\dagger\right\} + O(\epsilon^2). \end{aligned} \tag{12}$$

We evaluate

$$\begin{aligned} &-\frac{i}{\hbar}\left[H_I(t),\rho_{12}^{(0)}\otimes|0\rangle\langle 0|\right] \\ &= \frac{iJ_{13}}{\hbar}\left[f_1^*(t)b_1(t)\rho_{12}^{(0)}\otimes|1\rangle\langle 0| - \mathrm{H.c.}\right] \end{aligned} \tag{13}$$

and, using Eq. (11),

$$\begin{aligned} &\Gamma D(t)\left\{V\rho_{12}^{(0+1)}\otimes|0\rangle\langle 0|V^\dagger\right\} \\ &= -\frac{\Gamma}{2}\left[\alpha_1(t)f_1^*(t)b_1(t)\rho_{12}^{(0)}\otimes|1\rangle\langle 0| + \mathrm{H.c.}\right] + O(\epsilon^2). \end{aligned} \tag{14}$$

Equations (7) and (13) give immediately

$$\frac{d\rho_{12}^{(1)}}{dt} = O\left(e^{-\frac{\Gamma}{2}\Delta t}\right). \tag{15}$$

Since $J_{12}/\hbar\Gamma = O(\epsilon)$ and $U\langle n_{1,2}\rangle/\hbar\Gamma = O(\epsilon)$, the derivative of $b_1(t)$ is of order ϵ,

$$\frac{db_1(t)}{dt} = -\frac{i}{\hbar}[H_{12}, b_1(t)] = O(\epsilon). \tag{16}$$

Now, by taking into account Eqs. (11), (13)–(16), one sees that Eq. (12) is satisfied by setting

$$\frac{d\alpha_1}{dt} = -\left[\frac{\Gamma}{2} + i\frac{\varepsilon}{\hbar}\right]\alpha_1 + \frac{iJ_{13}}{\hbar}. \tag{17}$$

Equation (17) gives

$$\alpha_1 = \frac{2iJ_{13}}{\hbar\Gamma}\left[1 + \frac{2i\varepsilon}{\hbar\Gamma}\right]^{-1} + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right). \tag{18}$$

The final step is to insert Eq. (10) into Eq. (8), use Eqs. (6) and (15) and take the trace over the central-well subspace, keeping only the terms up to $O(\epsilon^2)$. Returning to the Schrödinger picture, we get [up to a correction of the order $O(\epsilon^3)$]

$$\frac{d\rho_{12}}{dt} = -\frac{i}{\hbar}[H_{12}, \rho_{12}] - \frac{i}{\hbar}\left[\varepsilon_R b_1^\dagger b_1, \rho_{12}^{(0)}\right] + \Gamma_R D_1\left\{\rho_{12}^{(0)}\right\} + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right), \tag{19}$$

with Γ_R given by

$$\Gamma_R = |\alpha_1|^2\Gamma = \frac{4J_{13}^2}{\hbar^2\Gamma}\left[1 + \left(\frac{2\varepsilon}{\hbar\Gamma}\right)^2\right]^{-1}, \tag{20}$$

and $D_1\{\rho\} = b_1\rho b_1^\dagger - \frac{1}{2}\{b_1^\dagger b_1, \rho\}$. The interaction-induced Lamb shift ε_R of the zero-point energy of the coherent zero-phase mode b_1 reads

$$\varepsilon_R = -J_{13}\mathrm{Re}(\alpha_1) = -\varepsilon\frac{\Gamma_R}{\Gamma}. \tag{21}$$

Our aim is to further reduce Eq. (19) to a master equation for the coherent π-phase mode b_2, which has unusual dissipation features (see below). To this end, however, one has to consider the contribution to Eq. (19) coming from the next order in ϵ, i.e., $O(\epsilon^3)$. Indeed, similar to the derivation of this section, the reduced master equation for mode b_2 is obtained under the conditions that $J_{12}, U\langle n_{1,2}\rangle \ll \hbar\Gamma_R$, what makes the Hamiltonian part in the master equation (19) smaller than the Lindblad part, hence the former should be discarded in the present order $O(\epsilon^2)$. Thus, in the second-order approximation, the coherent mode b_2 has no dissipation dynamics at all (only the Hamiltonian evolution described by $H_2 = \frac{U}{2}n_2^2$). It only appears in the higher-order version of the master equation (19).

To derive the third-order correction to Eq. (19), we need to find the form of $\rho^{(2)}(t)$, which satisfies equation similar to Eq. (7), but now with the inhomogeneous term

$$\begin{aligned}
&-\frac{i}{\hbar}\left[H_I(t), V\rho_{12}^{(0+1)} \otimes |0\rangle\langle 0|V^\dagger\right] \\
&= \frac{iJ_{13}}{\hbar}\left\{\left[\sqrt{2}\alpha_1(t) f_2^*(t) b_1^2(t)\rho_{12}^{(0)} \otimes |2\rangle\langle 0| - \mathrm{H.c.}\right]\right. \\
&\quad - 2i\,\mathrm{Im}[\alpha_1(t)] b_1(t)\rho_{12}^{(0)} b_1^\dagger(t) \otimes |1\rangle\langle 1| \\
&\quad + \left[\alpha_1(t) b_1^\dagger(t) b_1(t)\rho_{12}^{(0)} - \mathrm{H.c.}\right] \otimes |0\rangle\langle 0| \\
&\quad \left. + \left[f_1^*(t) b_1(t)\rho_{12}^{(1)} \otimes |1\rangle\langle 0| - \mathrm{H.c.}\right]\right\} + O(\epsilon^2),
\end{aligned} \tag{22}$$

where $U_3(t)|2\rangle = f_2(t)|2\rangle$, i.e., $f_2 = e^{-2i\frac{(\varepsilon+U)}{\hbar}\Delta t}$. The first three lines of Eq. (22) give the terms additional to those in Eq. (13). Expression (22) also defines the general form of $\rho^{(2)}(t)$:

$$\begin{aligned}
\rho^{(2)} &= B_{00}(t) \otimes |0\rangle\langle 0| + B_{11}(t) \otimes |1\rangle\langle 1| \\
&\quad + \left(B_{10}(t) \otimes |1\rangle\langle 0| + B_{20}(t) \otimes |2\rangle\langle 0| + \mathrm{H.c.}\right),
\end{aligned} \tag{23}$$

where the operators $B_{ij}(t) = O(\epsilon^2)$ act on the subspace of the side wells. They satisfy, in view of Eqs. (20), (13), (15), (21) and (22), the following equations:

$$\frac{dB_{11}}{dt} = -\Gamma B_{11} + \Gamma_R b_1(t)\rho_{12}^{(0)} b_1^\dagger(t), \tag{24a}$$

$$\frac{dB_{00}}{dt} = \Gamma B_{11} - \frac{i}{\hbar}\left[\varepsilon_R n_1(t), \rho_{12}^{(0)}\right] - \frac{\Gamma_R}{2}\left(n_1(t)\rho_{12}^{(0)} + \rho_{12}^{(0)} n_1(t)\right), \tag{24b}$$

$$B_{10} = \alpha_1(t) f_1^*(t) b_1(t)\rho_{12}^{(1)}, \tag{24c}$$

$$B_{20} = \alpha_2(t) f_2^*(t) b_1^2(t)\rho_{12}^{(0)}, \tag{24d}$$

where $\alpha_1(t)$ is given by Eq. (18) and

$$\frac{d\alpha_2}{dt} = -\left[\frac{\Gamma}{2} + i\frac{\varepsilon + U}{\hbar}\right]\alpha_2 + i\frac{\sqrt{2}J_{13}}{\hbar}\alpha_1(t). \tag{25}$$

One can easily solve Eq. (24a) and by using integration by parts represent the result as

$$B_{11}(t) = \frac{\Gamma_R}{\Gamma} b_1\rho_{12}^{(0)} b_1^\dagger + O(\epsilon^3) + O(e^{-\Gamma\Delta t}). \tag{26}$$

Substituting Eq. (26) into Eq. (24b) and using the master equation (19) one obtains

$$B_{00}(t) = \rho_{12}^{(2)} + O(e^{-\Gamma\Delta t}). \tag{27}$$

Finally, the solution to Eq. (25) reads

$$\begin{aligned}
\alpha_2(t) &= -\frac{2\sqrt{2}J_{13}^2}{\hbar^2\Gamma^2}\left[1 + i\frac{2\varepsilon}{\hbar\Gamma}\right]^{-1}\left[1 + i\frac{2(\varepsilon+U)}{\hbar\Gamma}\right]^{-1} \\
&\quad + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right) = \frac{\alpha_1^2(t)}{\sqrt{2}} + O(\epsilon^3) + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right).
\end{aligned} \tag{28}$$

The fact that the operators B_{11} and B_{20} and B_{02} do not contribute to the equation for ρ_{12} and the explicit expressions for B_{00} and B_{10}, Eqs. (26) and (27), lead to *the same* Lindblad form of the reduced master equation, i.e., in the Schrödinger picture we get

$$\frac{d\rho_{12}}{dt} = -\frac{i}{\hbar}[H_{12} + \varepsilon_R n_1, \rho_{12}] + \Gamma_R D_1\{\rho_{12}\} + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right), \tag{29}$$

which is now valid up to correction of order $O(\epsilon^4)$. Equations (23), (26)–(28) also show that the density matrix of the full system ρ can be rewritten as (now in the Schrödinger picture)

$$\rho(t) = V[\rho_{12}(t) \otimes |0\rangle\langle 0|]V^\dagger + O(\epsilon^3) + O\left(e^{-\frac{\Gamma}{2}\Delta t}\right), \tag{30}$$

where $\rho_{12}(t)$ is taken up to the second order in ϵ, $V(t) = \exp\{\alpha_1 b_1 b_3^\dagger\} = I + \alpha_1 b_1 b_3^\dagger + \frac{1}{2}\alpha_1^2 b_1^2 (b_3^\dagger)^2$ with α_1 given by Eq. (18). Equations (29) and (30) are the main results of this section. Obviously, the full density matrix (30) is not

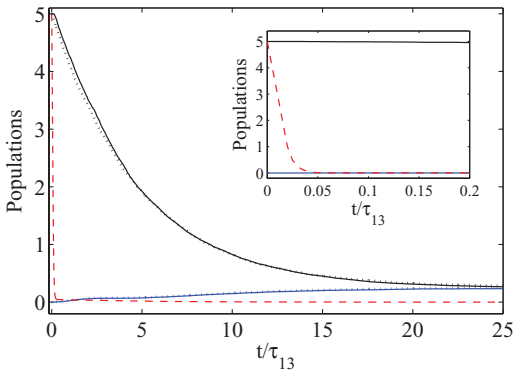

FIG. 2. (Color online) Comparison between the full three-mode Eq. (1) and the reduced two-mode Eq. (29). The population of a_3 is given by the dashed lines. The populations of modes a_1 and a_2 (solid lines) of the three-mode model are compared to that of the two-mode model (dotted lines). Here $\Gamma = 5J/\hbar$, $U = 0.2J$, $\varepsilon_j = 0$. We have used 5000 quantum trajectories. The inset shows the short-time dynamics of populations for larger $\Gamma = 50J/\hbar$. The initial state is the Fock state with the total of ten atoms (five occupying the mode a_1 and five occupying the mode a_3).

factorized, nevertheless, the reduced density matrix of a subsystem satisfies the Markovian master equation (29) in the Lindblad form. Note that the difference between the density matrix ρ_{12} and the one obtained by tracing the full density matrix of Eq. (30) is a constant term of the second order given by B_{11} in Eq. (26), which does not contribute to Eq. (29). Hence, Eq. (30) is consistent with the approximation made.

In Fig. 2, we use the Monte Carlo wave-function method [20] and find that an excellent agreement of the reduced master equation (29) with the full Eq. (1) extends also to the intermediate values of Γ (we have there $\epsilon = 0.2$). We have also verified that, for large Γ, the modes $a_{1,2}$ (and, hence, $b_{1,2}$) stay practically unchanged while the dissipating mode $b_3 = a_3$ is quickly emptied (the inset of Fig. 2).

Let us note the specific features of Eq. (29). We see that in the strongly dissipative case, quite similar to Eq. (1), mode b_2 can retain a significant part of its population, while b_1 loses almost all atoms, on the time scale $t \sim 1/\Gamma_R$, and after that stays practically empty. In the long run, on the time scale much longer that $1/\Gamma_R$, the population of mode b_2 drops to the single-atom level (this is seen also in Fig. 2, where it happens on the short time-scale due to a small Γ).

In the linear case Eq. (29) acts like a dispersive beam-splitter (see, for example, Ref. [21]). Thus, a strong loss in the central well induces quantum correlations between the side wells, i.e., for times $t \gg 1/\Gamma_R$ the cold atoms occupy the state

$$|\Psi_n\rangle \sim (b_2^\dagger)^n|0\rangle \sim (a_1^\dagger - a_2^\dagger)^n|0\rangle \tag{31}$$

which is unaffected by the dissipation described by Eq. (29).

In the nonlinear case, the dissipation of the π-phase mode b_2 is surprisingly nontrivial. This can be clearly demonstrated in the case when the decay of the zero-phase mode b_1 occurs on the faster scale than the intermode dynamics. To uncover the details, let us use the higher-order validity of Eq. (29) and reduce it to a single-mode master equation for b_2, by assuming that $U\langle n_2\rangle/\hbar \ll \Gamma_R$ and $J_{12}/\hbar \ll \Gamma_R$.

IV. THE REDUCED MASTER EQUATION FOR π-PHASE COHERENT MODE

We now reduce the master equation (29) to that for the mode b_2 alone, in the case of a strong dissipation of mode b_1 as compared to the Hamiltonian dynamics of the coherent modes b_1 and b_2. The derivation is similar to that of the previous section. First, we pass to the interaction representation:

$$\begin{aligned} &\hat{H}_{II}(t) = U^\dagger(t)H_{II}U(t), \quad \hat{\rho}_{12} = U^\dagger(t)\rho_{12}(t)U(t), \\ &U(t) = U_1(t)\otimes U_2(t) = \exp\left\{-\frac{i}{\hbar}(H_1+H_2)\Delta t\right\}, \end{aligned} \tag{32}$$

where the respective Hamiltonian terms, derived from Hamiltonian (3) with account of the Lamb shift (21), read

$$H_1 = \varepsilon_R n_1 + \frac{U}{2}n_1^2, \quad H_2 = \frac{U}{2}n_2^2,$$

$$H_{II} = -J_{12}(b_1^\dagger b_2 + b_2^\dagger b_1) + 2Un_1n_2 + \frac{U}{2}[b_1^\dagger b_2]^2 + \frac{U}{2}[b_2^\dagger b_1]^2. \tag{33}$$

Introducing the small parameter $\epsilon' = O(J_{12}/\hbar\Gamma_R)$, and assuming that $U\langle n_2\rangle/(\hbar\Gamma_R) = O(\epsilon')$, one can derive the equations for the density matrix in the interaction representation, which turn out to be similar to those of the previous section [see Eqs. (6)–(9)] with the obvious changes. The form of the density matrix $\rho_{12}^{(1)}$ is also similar to that of Eqs. (7) and (11):

$$\rho_{12}^{(1)}(t) = W(t)\left[\rho_2^{(0+1)}\otimes|0\rangle\langle 0|\right]W^\dagger(t) + O\left(e^{-\frac{\Gamma_1}{2}\Delta t}\right), \tag{34}$$

where $W(t)$ reads

$$W(t) = I + s_1(t)b_2(t)b_1^\dagger(t) + s_2(t)b_2^2(t)[b_1^\dagger(t)]^2, \tag{35}$$

with some scalar functions $s_j(t) = O(\epsilon')$. Using the same routine as in the previous section, one obtains the equations for the parameters $s_j(t)$:

$$\frac{ds_1}{dt} = -\left[\frac{\Gamma_R}{2} + i\frac{\varepsilon_R}{\hbar}\right]s_1 + i\frac{J_{12}}{\hbar}, \tag{36a}$$

$$\frac{ds_2}{dt} = -\left[\Gamma_R + i\frac{2\varepsilon_R}{\hbar}\right]s_2 - i\frac{U}{2\hbar}. \tag{36b}$$

These can be easily solved to give

$$\begin{aligned} s_1 &= \frac{2iJ_{12}}{\hbar\Gamma_R}\left[1 + i\frac{2\varepsilon_R}{\hbar\Gamma_R}\right]^{-1} + O\left(e^{-\frac{\Gamma_R}{2}\Delta t}\right), \\ s_2 &= -\frac{iU}{2\hbar\Gamma_R}\left[1 + i\frac{2\varepsilon_R}{\hbar\Gamma_R}\right]^{-1} + O\left(e^{-\frac{\Gamma_R}{2}\Delta t}\right). \end{aligned}$$

Define the following parameters:

$$\Gamma_j = \frac{K_j^2}{\hbar^2\Gamma_R}\left[1 + \left(\frac{2\varepsilon_R}{\hbar\Gamma_R}\right)^2\right]^{-1}, \quad \varkappa_j = -\varepsilon_R\frac{\Gamma_j}{\Gamma_R}, \tag{37}$$

where $K_1 = 2J_{12}$ and $K_2 = U$. Then, in a similar way as in the previous section, by taking the partial trace over the subspace of mode b_1, one derives a closed master equation for mode b_2.

In the Schrödinger picture we have

$$\frac{d\rho_2}{dt} = -\frac{i}{\hbar}[\tilde{H}_2,\rho_2] + \Gamma_1 D_1\{\rho_2\} + \Gamma_2 D_2\{\rho_2\} + O\left(e^{-\frac{\Gamma_R}{2}\Delta t}\right), \tag{38}$$

where the Hamiltonian, augmented by the Lamb shifts, and the two dissipation channels read

$$\tilde{H}_2 = \frac{U}{2}n_2^2 + \varkappa_1 n_2 + \varkappa_2 n_2(n_2-1), \tag{39}$$

$$D_1\{\rho\} = b_2\rho b_2^\dagger - \tfrac{1}{2}n_2\rho - \tfrac{1}{2}\rho n_2, \tag{40}$$

$$D_2\{\rho\} = b_2^2\rho(b_2^\dagger)^2 - \tfrac{1}{2}(b_2^\dagger)^2 b_2^2\rho - \tfrac{1}{2}\rho(b_2^\dagger)^2 b_2^2. \tag{41}$$

Finally, let us gather together the conditions used in derivation of the reduced equation (38). We have

$$\frac{UN}{\hbar}, \quad \frac{J_{12}}{\hbar} \ll \Gamma_R \sim \frac{J_{13}^2}{\hbar^2\Gamma}, \quad \frac{J_{13}}{\hbar\Gamma} \ll 1. \tag{42}$$

The validity conditions (42) can be recast in terms of the characteristic tunneling times and the nonlinear time. Defining $\tau_{13} = \hbar/J$, $\tau_{12} = \hbar/J_{12}$ and $\tau_{nl} = \hbar/U$, we have: $\frac{\tau_{13}}{\tau_{12}}$, $\frac{N\tau_{13}}{\tau_{nl}} \ll (\Gamma\tau_{13})^{-1} \ll 1$. The rates of the two dissipation channels of mode b_2 have the following orders: $\Gamma_1 \sim (\tau_{13}/\tau_{12})^2/\Gamma$ and $\Gamma_2 \sim (\tau_{13}/\tau_{nl})^2/\Gamma$.

We note that a master equation similar to Eq. (38) has already appeared before in connection with one- and two-photon absorption in quantum optics, where its special cases were studied [22]. It was shown that two-particle absorption has properties drastically different from the single-particle one. In particular, the decay is nonexponential and, irrespectively of the number of particles in the initial state of the mode b_2, number of particles in this mode drops to the single-particle level during the same time-interval [22].

Equation (38) has a number of specific features. First, we see that in the symmetric potential (when $\Gamma_1 = 0$) the decay occurs due to loss of two particles at once and the quantum parity, being average of the quantum parity operator

$$P = (-1)^{b_2^\dagger b_2} \tag{43}$$

remains constant. For example, for the state with the $\langle P(0)\rangle = -1$ (odd parity), one will have $\langle P(t)\rangle = -1$; the superposition state with only the odd (even) number of atoms will remain the state with the odd (even) number of atoms during all the evolution time. Second, for a biased potential ($\Gamma_1 \neq 0$) there is a resonance between the two different dissipation channels, under the condition $\Gamma_1 = 2\Gamma_2$, resulting in a polynomial decay of population. To see this, consider evolution of the average population

$$\frac{d\langle n_2\rangle}{dt} = -(\Gamma_1 - 2\Gamma_2)\langle n_2\rangle - 2\Gamma_2\langle n_2\rangle^2 - 2\Gamma_2(\Delta n_2)^2, \tag{44}$$

where $(\Delta n_2)^2 = \langle n_2^2\rangle - \langle n_2\rangle^2$. The initial state of mode b_2, to be used in Eq. (38), is a Fock state with a good approximation (mode b_1 is emptied on a much faster time scale). Hence, discarding Δn_2 (which is justified by numerical simulations, see Fig. 3), we get an approximation

$$\langle n_2(t)\rangle \approx \frac{\langle n_2(t_0)\rangle e^{-\gamma(t-t_0)}}{1 + \frac{2\Gamma_2}{\Gamma_1 - 2\Gamma_2}\langle n_2(t_0)\rangle[1 - e^{-(\Gamma_1-2\Gamma_2)(t-t_0)}]} \tag{45}$$

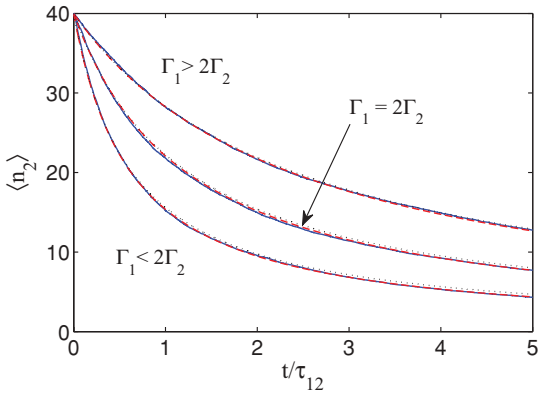

FIG. 3. (Color online) Comparison of the two-mode master equation (29) with the single-mode master equation (38) and the analytical approximation of Eq. (45). The solid lines give the two-mode equation, the dashed gives the single-mode, and the dotted give the analytical approximation. Here $N = 40$, $\Gamma_R\tau_{12} = 200$, and, from the top to bottom, $\Lambda \equiv UN/2J_{12} = 20, N/\sqrt{2}, 40$. To compare the results in the domain of their validity, i.e., after the atoms are removed from mode b_1, we have used the Fock initial state with all atoms occupying mode b_2 (a Gaussian with a small dispersion leads to the same results).

giving, for $\Gamma_1 = 2\Gamma_2$, the t^{-1} decay: $\langle n_2(t)\rangle^{-1} \approx \langle n_2(t_0)\rangle^{-1} + \Gamma_1(t - t_0)$. Thus, we have a quantum resonance between two different (linear and nonlinear) dissipation channels of a subsystem (mode b_2). The matching between them is expressed in terms of matching between the linear bias and nonlinear interaction coefficient: $U = \sqrt{2}J_{12} = (\varepsilon_2 - \varepsilon_1)/\sqrt{2}$. In Fig. 3 we show an excellent agreement of the analytical approximation, Eq. (45), with the numerical simulations, i.e., the Monte Carlo wave-function method [20], of the single-mode (38) and the two-mode (29) master equations.

V. THE MEAN-FIELD APPROXIMATION

The quantum derivations of the reduced master equations from the first principles, presented above, are involved. One then may inquire if the mean-field approximation, commonly applied to the many-boson models with large number of bosons and which is much simpler to analyze, can substitute the quantum derivation. Here we remind that the two-site model with a local dissipation is perfectly described by the mean-field approximation [18,19]. This, however, turns out to be not the case for the three-site Bose-Hubbard model, as we will show below.

The mean-field Hamiltonian can be obtained from Hamiltonian (3) by replacing the boson operators with c-numbers ($b_j \to \beta_j$) [1], we get

$$\mathcal{H} = -J_{13}(\beta_1^*\beta_3 + \beta_3^*\beta_1) - J_{12}(\beta_1^*\beta_2 + \beta_2^*\beta_1) + \varepsilon|\beta_3|^2 + U|\beta_3|^4 + \frac{U}{2}([|\beta_1|^2 + |\beta_2|^2]^2 + [\beta_1^*\beta_2 + \beta_2^*\beta_1]^2). \tag{46}$$

Note that the total number of atoms is given as $\sum_{j=1}^3 |\beta_j|^2 = N$. The local atom loss (dissipation) part of Eq. (1) can be simply added to the mean-field Hamiltonian equations as an atom loss of mode β_3 (as we will see shortly, it is describable

classically; see also Refs. [18,19]), that is

$$\frac{d\beta_j}{dt} = -\frac{i}{\hbar}\frac{\partial \mathcal{H}}{\partial \beta_j^*} - \delta_{j,3}\frac{\Gamma}{2}\beta_3. \tag{47}$$

Thus, the mean-field equations read

$$\frac{d\beta_1}{dt} = \frac{iJ_{13}}{\hbar}\beta_3 + \frac{iJ_{12}}{\hbar}\beta_2 - \frac{iU}{\hbar}(|\beta_1|^2 + 2|\beta_2|^2)\beta_1 - \frac{iU}{\hbar}\beta_2^2\beta_1^*, \tag{48a}$$

$$\frac{d\beta_2}{dt} = \frac{iJ_{12}}{\hbar}\beta_1 - \frac{iU}{\hbar}(|\beta_2|^2 + 2|\beta_1|^2)\beta_2 - \frac{iU}{\hbar}\beta_1^2\beta_2^*, \tag{48b}$$

$$\frac{d\beta_3}{dt} = \frac{iJ_{13}}{\hbar}\beta_1 - \left(\frac{\Gamma}{2} + \frac{i\varepsilon}{\hbar}\right)\beta_3 - \frac{2iU}{\hbar}|\beta_3|^2\beta_3. \tag{48c}$$

A. The first reduction: Equations for β_1 and β_2

Consider the strongly dissipated case of Sec. III. The small parameter is $\epsilon = \frac{J_{13}}{\hbar\Gamma}$ with the conditions $\frac{J_{12}}{\hbar\Gamma} = O(\epsilon)$ and $\frac{UN}{\hbar\Gamma} = O(\epsilon)$. For $t \gg 1/\Gamma$, $\beta_3 = O(\epsilon)$ and one can integrate Eq. (48c) by rewriting it in the integral form and neglecting the higher-order nonlinear term. Using the integration by parts, we get

$$\beta_3(t) = \frac{2iJ_{13}}{\hbar\Gamma}\left(1 + \frac{2i\varepsilon}{\hbar\Gamma}\right)^{-1}\beta_1(t) + O(\epsilon^2) + O\left(e^{-\frac{\Gamma\Delta t}{2}}\right). \tag{49}$$

This result corresponds to the expression for the full density matrix (10) with Eqs. (11) and (18), where the amplitude β_3 is locked to that of β_1 and β_2 with the same coefficient as in the full quantum case [α_1 of Eq. (18)]. Equation (49) can be now inserted into Eqs. (48a) and (48b). We obtain a reduced system describing the coherent modes:

$$\frac{d\beta_1}{dt} = -\left(\frac{\Gamma_R}{2} + \frac{i\varepsilon_R}{\hbar}\right)\beta_1 + \frac{iJ_{12}}{\hbar}\beta_2 - \frac{iU}{\hbar}\beta_2^2\beta_1^* - \frac{iU}{\hbar}(|\beta_1|^2 + 2|\beta_2|^2)\beta_1 + O(\epsilon^3) + O\left(e^{-\frac{\Gamma\Delta t}{2}}\right), \tag{50a}$$

$$\frac{d\beta_2}{dt} = \frac{iJ_{12}}{\hbar}\beta_1 - \frac{iU}{\hbar}(|\beta_2|^2 + 2|\beta_1|^2)\beta_2 - \frac{iU}{\hbar}\beta_1^2\beta_2^*, \tag{50b}$$

where Γ_R is given by Eq. (20) and the Lamb shift ε_R by Eq. (21) (i.e., by the corresponding quantum results).

B. The second reduction: Equation for β_2

Now, let us perform the second reduction to an equation for the amplitude β_2, similar as in the quantum case of Sec. IV. In Sec. IV we have assumed that the new small parameter is $\epsilon' = \frac{J_{12}}{\hbar\Gamma_R}$ with the additional condition on the nonlinearity $\frac{U\langle n_2\rangle}{\hbar\Gamma_R} = O(\epsilon')$. However, let us for a while broaden the derivation and discard the condition on the nonlinearity. For times $t \gg 1/\Gamma_R$ boson mode β_1 is practically empty, i.e., $\beta_1 = O(\epsilon')$. This allows us to simplify Eqs. (50a) and (50b) as follows:

$$\frac{d\beta_1}{dt} = -\left[\frac{\Gamma_R}{2} + \frac{i\varepsilon_R}{\hbar}\right]\beta_1 + \frac{iJ_{12}}{\hbar}\beta_2 - \frac{iU}{\hbar}\left[\beta_2^2\beta_1^* + 2|\beta_2|^2\beta_1\right] + O(\epsilon'^3) + O\left(e^{-\frac{\Gamma_R\Delta t}{2}}\right), \tag{51a}$$

$$\frac{d\beta_2}{dt} = \frac{iJ_{12}}{\hbar}\beta_1 - \frac{iU}{\hbar}|\beta_2|^2\beta_2 - \frac{iU}{\hbar}\beta_1^2\beta_2^* + O(\epsilon'^3) + O\left(e^{-\frac{\Gamma_R\Delta t}{2}}\right), \tag{51b}$$

where, for simplicity, we have dropped the term $O(\epsilon^3)$. Already from this system it is clear that the mean-field approach will not account for the boson-pair dissipation channel (41) of mode b_2, present in the full quantum Eq. (38). Indeed, Eq. (51a) must be somehow integrated with the result to be inserted into Eq. (51b). However, one can notice that the parameter U enters Eq. (51a) only in the conjunction with a factor $\sim|\beta_2|^2$, hence all the terms in Eqs. (51a) and (51b) scale as $\sqrt{N}$ (the scale of β) if we fix the nonlinearity parameter $\Lambda = UN$, whereas in the quantum case the N scaling of the boson-pair channel is $O(1)$.

Now, under the condition on the nonlinearity as in the full quantum case, i.e., $\frac{U\langle n_2\rangle}{\hbar\Gamma_R} = O(\epsilon')$, one can proceed to derive the reduced equation for the amplitude β_2. To this end, the system of equations for β_1 and β_1^* in the required order can be written in the following form:

$$\frac{d}{dt}\begin{pmatrix}\beta_1\\ \beta_1^*\end{pmatrix} = \frac{\Gamma_R}{2}M\begin{pmatrix}\beta_1\\ \beta_1^*\end{pmatrix} + \frac{iJ_{12}}{\hbar}\begin{pmatrix}\beta_2\\ -\beta_2^*\end{pmatrix}, \tag{52}$$

where

$$M = \begin{pmatrix}-\gamma - \frac{4iU}{\hbar\Gamma_R}|\beta_2|^2 & -\frac{2iU}{\hbar\Gamma_R}(\beta_2)^2\\ \frac{2iU}{\hbar\Gamma_R}(\beta_2^*)^2 & -\gamma^* + \frac{4iU}{\hbar\Gamma_R}|\beta_2|^2\end{pmatrix}, \tag{53}$$

with $\gamma = (1 + \frac{2i\varepsilon_R}{\hbar\Gamma_R})$. Equation (53) can be put into the integral form and integrated for times $t \gg 1/\Gamma_R$, in this case the matrix $\frac{\Gamma_R}{2}M(t)$ enters the exponent under the integral, with the result

$$\begin{pmatrix}\beta_1\\ \beta_1^*\end{pmatrix} = -\frac{2iJ_{12}}{\hbar\Gamma_R}M^{-1}\begin{pmatrix}\beta_2\\ -\beta_2^*\end{pmatrix} + O\left(e^{-\frac{\Gamma_R\Delta t}{2}}\right) + O(\epsilon'^2). \tag{54}$$

We need only the first row of the matrix M^{-1}:

$$(M^{-1})_{11} = -\gamma^{-1} + \gamma^{-2}\frac{4iU}{\hbar\Gamma_R}|\beta_2|^2 + O(\epsilon'^2), \tag{55}$$

$$(M^{-1})_{12} = -|\gamma|^{-2}\frac{2iU}{\hbar\Gamma_R}(\beta_2)^2 + O(\epsilon'^2). \tag{56}$$

From Eqs. (54)–(56) we obtain

$$\beta_1 = \gamma^{-1}\frac{2iJ_{12}}{\hbar\Gamma_R}\beta_2 + O(\epsilon'^2) + O\left(e^{-\frac{\Gamma_R\Delta t}{2}}\right). \tag{57}$$

Inserting this expression into Eq. (51b) we arrive at the reduced equation for the amplitude β_2, the mean-field analog of the coherent π phase mode b_2:

$$\frac{d\beta_2}{dt} = -\left(\frac{\Gamma_1}{2} + \frac{i\varkappa_1}{\hbar}\right)\beta_2 - \frac{iU}{\hbar}|\beta_2|^2\beta_2 + O(\epsilon'^3) + O\left(e^{-\frac{\Gamma_R\Delta t}{2}}\right), \tag{58}$$

with Γ_1 and $\varkappa_1$ given in Eq. (37). Observe that, while the single-boson dissipation channel, Eq. (40), is accounted by the mean-field Eq. (58) (the first term on the right hand side), the boson-pair channel, Eq. (41), is not.

In conclusion of this section, we have shown that, while the single-boson dissipation channel of the coherent mode b_2 can be described by the mean-field approach, the boson-pair dissipation channel cannot be captured by the mean-field approximation and, thus, it has *quantum nature*.

VI. DISCUSSION OF THE RESULTS

We have considered the derivation of the reduced master equations in the limit of strong dissipation on the example of the Bose-Hubbard model with a local external dissipation (i.e., the atom loss from the central site). The method we have used is not based on the assumption of the factorization of the full density matrix, instead we demonstrate that one can effectively solve the master equation directly in the lowest orders of a small parameter (inversely proportional to the local dissipation rate). On this way, one is able to obtain the reduced master equations for the subsystems (the coherent modes) of the higher-order in the small parameter (e.g., we have derived the equation up to the third order).

The derivation reveals the following features. First of all, the full density matrix is not factorized (which is the usual assumption, see for instance Ref. [17]) but is expressed in the form of a "dressed" factorized density matrix, where the population of the strongly dissipated mode depends on that of the other modes. Nevertheless, the reduced density matrix of a subsystem is shown to satisfy a master equation in the Lindblad form. This feature appears in the two reduced master equations derived in the present paper, thus suggesting an universality. Moreover, the Lamb shifts and the dissipation rates of the subsystem turn out to be given by the similar expressions in the two cases, suggesting even the universality of the form of expressions for these quantities.

We have analyzed the relation between the full quantum derivation of the reduced master equation for the density matrix of a subsystem and the mean-field adiabatic elimination procedure. We have found that the mean-field approximation applied to the Bose-Hubbard model cannot capture all dissipation channels of a subsystem, even if the external dissipation applied to the system is describable classically (in our case, the single-boson dissipation channel describing the removal of atoms). Namely, in the three-site Bose-Hubbard model the π-phase coherent mode has the boson-pair dissipation channel, which is not captured by the mean-field approximation, and the single-boson dissipation, captured by it. This is a quite distinct situation from the two-site model, where the dissipation dynamics is described by the mean-field approximation with a good accuracy [18,19]. Here we note, however, that in the two-site model with dissipation the boson-pair tunneling is not possible. Though we have considered the three-site Bose-Hubbard model, the boson-pair dissipation channel is a *general feature* of the multisite models, since it is the result of a nonlinearity of the model and the fact that more than one filled site is coupled to the dissipated site, which serves as a common reservoir.

The failure of the mean-field approximation to the quantum master equation means that the dissipation channels not accounted by the mean-field have a *genuine quantum nature*. This imposes severe limitations on the applicability of the semiclassical mean-field approach to the Bose-Hubbard models considered as open quantum systems, even when the external dissipation is describable classically. Invoking the link to the discrete WKB approach, mentioned in the Introduction, one has to develop a more general approach by going to the first-order approximation in the effective Planck constant $1/N$, i.e., a $1/N$-correction to the mean-filed equations, to capture the boson-pair dissipation channel. This is an important conclusion, in view of the current experiments on the single-site addressability and controlled measurement via the local atom losses of Bose-Einstein condensate loaded in the optical lattices [12,13], describable by the open multisite Bose-Hubbard models.

Finally, we would like to make a connection to the recent studies of the non-Hermitian quantum Hamiltonians [23], where a dissipation is introduced by hand. Such an approach may prove to be useful for capturing some quantum features, on this way discarding the need to work with the density matrix evolution which is also numerically expensive to simulate. However, so far only the two-mode Bose-Hubbard model (a dissipated dimer) was studied and it remains to see whether this approach is capable of capturing, for instance, the boson-pair dissipation channel in the three-site case considered here.

APPENDIX: THE EFFECT OF THE TERM PROPORTIONAL TO THE TOTAL NUMBER OF BOSONS

Let us show that the form of the master equation does not depend on the discarded term proportional to the total number of bosons $N = n_1 + n_2 + n_3$. The consideration below applies to any situation where a term linear in the number of bosons N is discarded from a N-conserving Hamiltonian H. To be specific, consider the master equation (1) with the Hamiltonian (2). Set $H = H_R + cN$, where the effect of the last term is to be considered [in Eq. (3) $c = (\varepsilon_1 + \varepsilon_2)/2$]. Let us introduce a transformation of the density matrix as follows:

$$\rho(t) = \exp\left\{-i\frac{c}{\hbar}tN\right\}\rho_R(t)\exp\left\{i\frac{c}{\hbar}tN\right\} \equiv Z(t)\rho_R(t)Z^\dagger(t), \tag{A1}$$

where $Z(t)$ is a unitary operator. It does not change the form of the master equation (except the cN term is dropped). Indeed, since $[H,N] = 0$ and $Z^\dagger(t)a_3Z(t) = e^{-ict/\hbar}a_3$, we have

$$\frac{d\rho_R}{dt} = -\frac{i}{\hbar}[H_R,\rho_R] + \Gamma\left(a_3\rho_R a_3^\dagger - \frac{1}{2}n_3\rho_R - \frac{1}{2}\rho_R n_3\right). \tag{A2}$$

The only effect of the N-dependent term on the original master equation is in the N-dependent phases in the density matrix, according to Eq. (A1), which, in our case, have no effect whatsoever. Note that the phase differences appear between the quantum states with the *different* total number of bosons, thus it is questionable if they are observable at all.

It is clear that the above argument is valid for any master equation where the Lindblad term is expressed through the boson creation and annihilation operators.

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