Finite-temperature Cherenkov radiation in the presence of a magnetodielectric medium

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A canonical approach to Cherenkov radiation in the presence of a magnetodielectric medium is presented in classical, nonrelativistic, and relativistic quantum regimes. The equations of motion for the canonical variables are solved explicitly for both positive and negative times. Maxwell and related constitute equations are obtained. In the large-time limit, the vector potential operator is found and expressed in terms of the medium operators. The energy loss of a charged particle, emitted in the form of radiation, in finite temperature is calculated. A Dirac equation concerning the relativistic motion of the particle in presence of the magnetodielectric medium is derived and the relativistic Cherenkov radiation at zero and finite temperature is investigated. Finally, it is shown that the Cherenkov radiation in nonrelativistic and relativistic quantum regimes, unlike its classical counterpart, introduces automatically a cutoff for higher frequencies beyond which the power of radiation emission is zero.

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I. INTRODUCTION

Cherenkov radiation is the radiation with continuous spectrum and specific angular distribution emitted by the medium due to the motion of charged particles moving in the medium with a velocity exceeding the phase velocity of light in the transparent medium. The Cherenkov radiation in transparent media was experimentally observed by Cherenkov [1]. Theoretical explanation on this phenomenon was first developed by Frank and Tamm [2]. They showed that the particle should radiate when its velocity exceeds the velocity of light in the medium as the emitted rays make an angle θ with the charge velocity given by $\cos\theta = c/vn$, where v is the speed of the particle, n is the index of refraction of the medium. Also they are polarized with the electric field in the plane of this angle. The radiation shock-front, called the Cherenkov cone, is analogous to the Mach cone formed as objects move with supersonic speeds through air. The Cherenkov theory has drawn a great deal of attention all around the world. This theory has been widely used in high-energy particle physics, optics, cosmic-ray physics, high-power radiation sources, and so on [3–6]. Typical examples are the discoveries of the antiproton [7] and the J particle [8].

The classical theory of Cherenkov effect is sufficiently accurate in the optical part of spectrum. For methodological reasons, it is equally important to consider the quantum theory of this effects. The phenomenological quantum theory of the Cherenkov radiation developed by Ginzburg but the dissipative character of the medium was neglected [9,10]. The source-theory explanation of this effect was given by Schwinger *et al.* [11]. Unfortunately, due to the method of combining the denominators of the propagators in parametric form, the resulting integrals are exceedingly complicated and approximations were necessarily made.

The first formulation of the finite-temperature quantum field theory was presented by Dolan and Jackiw [12], Weinberg [13], and Bernard [14] and the first application of it concerned the effective potentials in Higgs theories. Also, the inclusion of temperature has been carried out in QED of Cherenkov radiation only in a nondispersive dielectric medium [15]. The main purpose of the present work is to develop a canonical theory of the finite temperature Cherenkov radiation to evaluate the electromagnetic field arising from the uniform motion of a charged particle in the presence of a magnetodielectric medium in different regimes, i.e, classical, nonrelativistic, and relativistic quantum regimes. To achieve this goal we first generalize a Lagrangian introduced in [16] to include the external charges. This prepares not only the grounds to extend the Ginzburg theory to include the dissipative character and the permeable character of a medium but extracts a Dirac equation for a relativistic particle in the presence of the magnetodielectric medium to survey the relativistic effects of the Cherenkov radiation to finite temperature regime.

The layout of the paper is as follows. In Sec. II, a Lagrangian for the total system is proposed and a classical treatment of the Cherenkov radiation to finite temperature regime is investigated. In Sec. III, we use the Lagrangian introduced in Sec. II to canonically quantize the electromagnetic field rising from the moving external charges embedded in the magnetodielectric medium. Maxwell and constitute equations are obtained. We find that for sufficiently large times the vector potential operator can be expressed in terms of the initial medium operators. The consistency of these solutions for the vector potential operator depend on the validity of certain velocity sum rules. We also show how to relate the results to the damping polarization model and phenomenological quantization theories. By considering finite temperature effects, energy loss of a charged particle emitted in the form of radiation is calculated. Subsequently, this formalism is generalized somehow to describe the relativistic moving particles. Finally, we discuss the main results and conclude in Sec. III.

II. CLASSICAL THEORY

Cherenkov radiation has the property that it occurs with uniform motion of a charged particle in a spatially homogeneous medium [17]. At first we attempt to treat the theory of Cherenkov radiation in the presence of a linear homogeneous magnetodielectric medium on basis of the classical theory. In the first part of this section we generalize the approach

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presented in [16] to the case where there are some external charges in the medium and in the second part we examine the emission of electromagnetic wave, if it occurs, by the moving particle. In the third section we will concentrate our attention on the Cherenkov radiation in the finite temperature situation.

A. Classical dynamics

Classical and quantum electrodynamics in a linear magnetodielectric can be accomplished by modeling the medium with two independent reservoirs that interact with electromagnetic field. Each reservoir contains a continuum of three dimensional harmonic oscillators describing the polarizability and magnetizability properties of the medium [18]. Therefore, in order to have a classical treatment of electrodynamics in a homogenous magnetodielectric medium, we begin with the following classical Lagrangian for the total system (medium + electromagnetic field + external charges):

$$L(t) = L_{res} + L_{em} + L_q + L_{int}. \tag{1}$$

The first term L_{res} is the reservoir part

$$\begin{aligned} L_{res} = &\int_0^\infty d\omega \int d^3\mathbf{r}\, \frac{1}{2}[\dot{\mathbf{X}}_\omega(\mathbf{r},t)\cdot\dot{\mathbf{X}}_\omega(\mathbf{r},t) \\ &- \omega^2 \mathbf{X}_\omega(\mathbf{r},t)\cdot\mathbf{X}_\omega(\mathbf{r},t)] \\ &+ \int_0^\infty d\omega \int d^3\mathbf{r}\, \frac{1}{2}[\dot{\mathbf{Y}}_\omega(\mathbf{r},t)\cdot\dot{\mathbf{Y}}_\omega(\mathbf{r},t) \\ &- \omega^2 \mathbf{Y}_\omega(\mathbf{r},t)\cdot\mathbf{Y}_\omega(\mathbf{r},t)], \end{aligned} \tag{2}$$

where the dynamical variables $\mathbf{X}_\omega(\mathbf{r},\mathbf{t})$ and $\mathbf{Y}_\omega(\mathbf{r},\mathbf{t})$ correspond to the electric and magnetic characters of the medium, respectively. The second term L_{em} is the electromagnetic field

$$L_{em} = \int d^3\mathbf{r} \left[\frac{1}{2}\epsilon_0 \mathbf{E}^2(\mathbf{r},t) - \frac{\mathbf{B}^2(\mathbf{r},t)}{2\mu_0}\right]. \tag{3}$$

The third term L_q is the Lagrangian of the external charges with mass m_α and position $\mathbf{r}_\alpha$

$$L_q = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2(t) + \sum_\alpha [q_\alpha \dot{\mathbf{r}}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha,t) - q_\alpha \phi(\mathbf{r}_\alpha,t)], \tag{4}$$

and finally L_{int} is the interaction term, which includes the linear interaction between the medium and electromagnetic field trough coupling functions $f(\omega)$ and $g(\omega)$ and also the interaction between the external charges and electromagnetic field

$$\begin{aligned} L_{int} = &\int_0^\infty d\omega \int d^3\mathbf{r}\; f(\omega)\mathbf{X}_\omega(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) \\ &+ \int_0^\infty d\omega \int d^3\mathbf{r}\; g(\omega)\mathbf{Y}_\omega(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t). \end{aligned} \tag{5}$$

In Eqs. (3) and (5), $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$ and $\mathbf{B} = \nabla\times\mathbf{A}$ are the total electric and magnetic fields and $\mathbf{A}$ and ϕ are the vector and the scaler potentials. For simplicity we work in the reciprocal space and write the fields in terms of their spatial Fourier transforms. The range of the variable $\mathbf{k}$ in the reciprocal space is restricted to the half-space denoted by a prime over the integral, i.e., $\int' d^3\mathbf{k}$ [19], thus in the reciprocal half-space the Lagrangian (1) can be written as

$$\underline{L}(t) = \underline{L}_q(t) + \underline{L}_{res}(t) + \underline{L}_{em}(t) + \underline{L}_{int}(t), \tag{6}$$

$$\begin{aligned} \underline{L}_q = &\frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2(t) + \frac{1}{(2\pi)^{3/2}} \\ &\times \sum_\alpha \int' d^3\mathbf{k}\{q_\alpha[\dot{\mathbf{r}}_\alpha\cdot\underline{\mathbf{A}}(\mathbf{k},t) - \underline{\phi}(\mathbf{k},t)]e^{\imath\mathbf{k}\cdot\mathbf{r}_\alpha} + \text{c.c.}\}, \end{aligned} \tag{7}$$

$$\begin{aligned} \underline{L}_{res}(t) = &\int_0^\infty d\omega \int' d^3\mathbf{k}(|\dot{\underline{\mathbf{X}}}_\omega|^2 - \omega^2|\underline{\mathbf{X}}_\omega|^2) \\ &+ \int_0^\infty d\omega \int' d^3\mathbf{k}(|\dot{\underline{\mathbf{Y}}}_\omega|^2 - \omega^2|\underline{\mathbf{Y}}_\omega|^2), \end{aligned} \tag{8}$$

$$\begin{aligned} \underline{L}_{em}(t) = &\int' d^3\mathbf{k}\left(\epsilon_0|\dot{\underline{\mathbf{A}}}|^2 + \epsilon_0|\mathbf{k}\underline{\phi}|^2 - \frac{|\mathbf{k}\times\underline{\mathbf{A}}|^2}{\mu_0}\right) \\ &+ \epsilon_0\int' d^3\mathbf{k}(-\imath\mathbf{k}\cdot\dot{\underline{\mathbf{A}}}\underline{\phi}^* + \text{c.c.}), \end{aligned} \tag{9}$$

$$\begin{aligned} \underline{L}_{int} = &-\int_0^\infty d\omega \int' d^3\mathbf{k}\{f(\omega)\underline{\mathbf{X}}_\omega^*(\mathbf{k},t)[\imath\mathbf{k}\underline{\varphi}(\mathbf{k},t) \\ &+ \dot{\underline{\mathbf{A}}}(\mathbf{k},t)] + \text{c.c.}\} + \int_0^\infty d\omega \int' d^3\mathbf{k}\{g(\omega)\underline{\mathbf{Y}}_\omega^*(\mathbf{k},t) \\ &\times [\imath\mathbf{k}\times\underline{\mathbf{A}}(\mathbf{k},t)] + \text{c.c.}\}, \end{aligned} \tag{10}$$

where we have applied $\underline{\mathbf{X}}^*(\mathbf{k},t) = \underline{\mathbf{X}}(-\mathbf{k},t)$ and the similar relations for the other dynamical fields [19]. we can obtain the classical equations of the motion simply from Euler-Lagrange equations. For the vector and scalar potentials $\mathbf{A}(\mathbf{k},t)$, $\phi(\mathbf{k},t)$ we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\delta\underline{L}}{\delta[\dot{\underline{\mathbf{A}}}_i^*(\mathbf{k},t)]}\right) - \frac{\delta\underline{L}}{\delta[\underline{\mathbf{A}}_i^*(\mathbf{k},t)]} = 0, \quad i = 1,2,3 \\ &\Rightarrow \mu_0\epsilon_0\ddot{\underline{\mathbf{A}}}(\mathbf{k},t) + \mu_0\epsilon_0\imath\mathbf{k}\dot{\underline{\phi}}(\mathbf{k},t) - \mathbf{k}\times[\mathbf{k}\times\underline{\mathbf{A}}(\mathbf{k},t)] \\ &= \mu_0\dot{\underline{\mathbf{P}}}(\mathbf{k},t) + \imath\mu_0\mathbf{k}\times\underline{\mathbf{M}}(\mathbf{k},t) + \mu_0\underline{\mathbf{J}}(\mathbf{k},t) \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\delta\underline{L}}{\delta[\dot{\underline{\phi}}^*(\mathbf{k},t)]}\right) - \frac{\delta\underline{L}}{\delta[\underline{\phi}^*(\mathbf{k},t)]} = 0 \\ &\Rightarrow -\epsilon_0\imath\mathbf{k}\cdot\dot{\underline{\mathbf{A}}}(\mathbf{k},t) + \epsilon_0\mathbf{k}^2\underline{\phi}(\mathbf{k},t) = -\imath\mathbf{k}\cdot\underline{\mathbf{P}}(\mathbf{k},t) + \underline{\rho}(\mathbf{k},t), \end{aligned} \tag{12}$$

where

$$\underline{\mathbf{P}}(\mathbf{k},t) = \int_0^\infty d\omega\, f(\omega)\,\underline{\mathbf{X}}_\omega(\mathbf{k},t), \tag{13}$$

$$\underline{\mathbf{M}}(\mathbf{k},t) = \int_0^\infty d\omega\, g(\omega)\,\underline{\mathbf{Y}}_\omega(\mathbf{k},t), \tag{14}$$

$$\underline{\mathbf{J}}(\mathbf{k},t) = \frac{1}{(2\pi)^{3/2}}\sum_\alpha q_\alpha\dot{\mathbf{r}}_\alpha e^{-\imath\mathbf{k}\cdot\mathbf{r}_\alpha}, \tag{15}$$

$$\underline{\rho}(\mathbf{k},t) = \frac{1}{(2\pi)^{3/2}}\sum_\alpha q_\alpha e^{-\imath\mathbf{k}\cdot\mathbf{r}_\alpha}, \tag{16}$$

are, respectively, the Fourier transforms of the electric and magnetic polarization densities of the medium. The Fourier transforms of the external current and charge densities satisfy the charge conservation in the reciprocal space $\dot{\underline{\rho}} = -\imath\mathbf{k}\cdot\underline{\mathbf{J}}$. Similarly from the Euler-Lagrange equations for the fields $\mathbf{r}_\alpha$ and $\underline{\mathbf{X}}_\omega$, $\underline{\mathbf{Y}}_\omega$ we find

$$\frac{d}{dt}\left(\frac{\delta\underline{L}}{\delta[\dot{r}_{\alpha,i}(t)]}\right) - \frac{\delta\underline{L}}{\delta[r_{\alpha,i}(t)]} = 0, \quad i = 1,2,3$$
$$\Rightarrow m_\alpha\ddot{\mathbf{r}}_\alpha(t) = q_\alpha\mathbf{E}(\mathbf{r}_\alpha,t) + q_\alpha\dot{\mathbf{r}_\alpha}\times\mathbf{B}(\mathbf{r}_\alpha,t) \tag{17}$$

and

$$\frac{d}{dt}\left(\frac{\delta\underline{L}}{\delta[\dot{\underline{\mathbf{X}}}^*_{\omega i}(\mathbf{k},t)]}\right) - \frac{\delta\underline{L}}{\delta[\underline{\mathbf{X}}^*_{\omega i}(\mathbf{k},t)]} = 0, \quad i = 1,2,3$$
$$\Rightarrow \ddot{\underline{\mathbf{X}}}_\omega(\mathbf{k},t) + \omega^2\underline{\mathbf{X}}_\omega(\mathbf{k},t) = -f(\omega)\,[\dot{\underline{\mathbf{A}}}(\mathbf{k},t) + \imath k\underline{\varphi}(\mathbf{k},t)], \tag{18}$$

$$\frac{d}{dt}\left(\frac{\delta\underline{L}}{\delta[\dot{\underline{\mathbf{Y}}}^*_{\omega i}(\mathbf{k},t)]}\right) - \frac{\delta\underline{L}}{\delta[\underline{\mathbf{Y}}^*_{\omega i}(\mathbf{k},t)]} = 0, \quad i = 1,2,3$$
$$\Rightarrow \ddot{\underline{\mathbf{Y}}}_\omega(\mathbf{k},t) + \omega^2\underline{\mathbf{Y}}_\omega(\mathbf{k},t) = g(\omega)\,\imath\mathbf{k}\times\dot{\underline{\mathbf{A}}}(\mathbf{k},t). \tag{19}$$

The formal solution of the field equation (18) is

$$\underline{\mathbf{X}}_\omega(\mathbf{k},t) = \dot{\underline{\mathbf{X}}}_\omega(\mathbf{k},0)\frac{\sin\omega t}{\omega} + \underline{\mathbf{X}}_\omega(\mathbf{k},0)\cos\omega t + f(\omega)\int_0^t dt'\,\frac{\sin\omega(t-t')}{\omega}\underline{\mathbf{E}}(\mathbf{k},t'), \tag{20}$$

where the first term is the inhomogeneous solution of Eq. (18) and the second term is the homogeneous one. We will show that after quantization the homogeneous solution becomes a noise operator. However, since we are only interested in the induced polarization, we keep the inhomogeneous solution and using Eq. (13), we find the electric polarization density of the medium in reciprocal space

$$\underline{\mathbf{P}}(\mathbf{k},t) = \epsilon_0\int_0^\infty dt\;\chi_e(t-t')\underline{\mathbf{E}}(\mathbf{k},t'), \tag{21}$$

where χ_e is the electric causal susceptibility of the medium and in terms of the coupling function f can be written as

$$\chi_e(t-t') = \begin{cases} \frac{1}{\epsilon_0}\int_0^\infty d\omega\frac{\sin\omega(t-t')}{\omega}f^2(\omega) & t > t' \\ 0 & t < t' \end{cases} \tag{22}$$

which is the origin of the significant Kramers-Kronig relations [17]. In a similar fashion the magnetic polarization density of the medium in reciprocal space can be obtained straightforwardly using the formal solution of Eq. (19) as

$$\underline{\mathbf{M}}(\mathbf{k},t) = \frac{1}{\mu_0}\int_0^\infty dt\;\chi_m(t-t')\underline{\mathbf{B}}(\mathbf{k},t'), \tag{23}$$

where χ_m is the magnetic causal susceptibility of the medium which in terms of the coupling function g can be written as

$$\chi_m(t-t') = \begin{cases} \mu_0\int_0^\infty d\omega\frac{\sin\omega(t-t')}{\omega}g^2(\omega) & t > t' \\ 0 & t < t' \end{cases}. \tag{24}$$

The electric permittivity and the inverse magnetic permeability of the magnetodielectric medium are defined in terms of χ_e and χ_m as

$$\epsilon(\omega) = 1 + \chi_e(\omega) \tag{25}$$

and

$$\kappa(\omega) = 1 - \chi_m(\omega), \tag{26}$$

where

$$\chi_{e,m}(\omega) = \int_0^\infty dt\,\chi_{e,m}(t)e^{\imath\omega t}.$$

By using Eqs. (22) and (24), we can obtain the following important relations in the frequency domain:

$$\chi_e(\omega) = \frac{1}{\epsilon_0}\int_0^\infty d\omega'\frac{f^2(\omega')}{\omega'^2-\omega^2-\imath 0^+}, \tag{27}$$

$$\chi_m(\omega) = \mu_0\int_0^\infty d\omega'\frac{g^2(\omega')}{\omega'^2-\omega^2-\imath 0^+}. \tag{28}$$

These are complex functions of frequency which satisfy Kramers-Kronig relations and have the properties of response functions, i.e., $\epsilon(-w^*) = \epsilon^*(\omega)$ and $\kappa(-w^*) = \kappa^*(\omega)$ and $\mathrm{Im}\epsilon(\omega) > 0$, $\mathrm{Im}\mu(\omega) > 0$ provided that $f^2(-\omega^*) = f^2(\omega)$ and $g^2(-\omega^*) = g^2(\omega)$. It can be shown that these functions have no poles in the upper half-plane and tend to zero as $\omega \longrightarrow \infty$. If we are given definite electric permittivity and inverse magnetic permeability of the medium then we can inverse the relations (22) and (24) and find the corresponding coupling functions $f(\omega)$ and $g(\omega)$ as

$$f(\omega) = \sqrt{\frac{2\omega\epsilon_0}{\pi}\mathrm{Im}\epsilon(\omega)}, \tag{29}$$

$$g(\omega) = \sqrt{-\frac{2\omega}{\pi\mu_0}\mathrm{Im}\kappa(\omega)}, \tag{30}$$

where the minus sign in the second expression is used since for a magnetodielectric medium $\mathrm{Im}\mu(\omega) > 0$, therefore $\mathrm{Im}\kappa(\omega) < 0$. In order to illustrate the relations between the coupling functions and the electric permittivity and the inverse magnetic permeability of a magnetodielectric medium, let us restrict our attention to a single resonance electric permittivity which can be obtained from the Lorentz oscillator model

$$\epsilon(\omega) = 1 + \frac{\omega_{pe}^2}{\omega_{0e}^2-\omega^2-\imath\gamma_e\omega} \tag{31}$$

and a single resonance magnetic permeability [20]

$$\mu(\omega) = 1 + \frac{\omega_{pm}^2}{\omega_{0m}^2-\omega^2-\imath\gamma_m\omega}, \tag{32}$$

where ω_{pe} and ω_{pm} are the coupling strengths, ω_{0m}, ω_{0m} are the transverse resonance frequencies, and γ_e, γ_m are the absorption parameters. Now by using Eqs. (29) and (30) the coupling functions $f(\omega)$ and $g(\omega)$ can be obtained as

$$f^2(\omega) = \frac{2\gamma_e\epsilon_0\omega_{pe}^2\omega^2/\pi}{\left(\omega_{0e}^2-\omega^2\right)^2+\gamma_e^2\omega^2}, \tag{33}$$

$$g^2(\omega) = \frac{2\gamma_m\omega_{pm}^2\omega^2/\pi\mu_0}{\left(\omega_{0m}^2+\omega_{pm}^2-\omega^2\right)^2+\gamma_m^2\omega^2}. \tag{34}$$

Following the standard approach to classical electrodynamics, we choose the Coulomb gauge $\mathbf{k}\cdot\mathbf{A} = 0$, so that the vector potential $\mathbf{A}$ is a purely transverse field. By using the Euler-Lagrange equation for $\dot{\varphi}^*$, we eliminate φ from Eq. (11)

and substituting Eqs. (21) and (23) into Eq. (11), obtain the following inhomogeneous wave equation:

$$\mu_0\epsilon_0\ddot{\underline{\mathbf{A}}}(\mathbf{k},t)+k^2\underline{\mathbf{A}}(\mathbf{k},t)-\mu_0k^2\int_0^t dt'\chi_m(t-t')\,\underline{\mathbf{A}}(\mathbf{k},t')$$
$$+\mu_0\frac{\partial}{\partial t}\int_0^t dt'\chi_e(t-t')\,\dot{\underline{\mathbf{A}}}(\mathbf{k},t')=\mu_0\underline{\mathbf{J}}^{\perp}(\mathbf{k},t), \qquad (35)$$

where the transverse current is defined as $\underline{\mathbf{J}}^{\perp}=\sum_{\lambda=1}^{2}\underline{\mathbf{J}}\cdot e_\lambda(\mathbf{k})$ with unit polarization vectors $e_\lambda(\mathbf{k}),\lambda=1,2$, which are orthogonal to $e_3(\mathbf{k})=\frac{\mathbf{k}}{k}=\hat{k}$ and to one another. This equation can be solved in terms of initial conditions using the Laplace transforms. For any time-dependent operator $\Omega(t)$ the forward Laplace transform is defined as

$$\tilde{\Omega}^f(s)=\int_0^\infty dte^{-st}\Omega(t), \qquad (36)$$

obviously, the Laplace transform contains all information about the time evolution of Ω for positive t. In the following we wish to determine the time evolution of the relevant operators of our model for any time, either positive or negative. Hence, we also introduce the backward Laplace transform

$$\tilde{\Omega}^b(s)=\int_0^\infty dte^{-st}\Omega(-t). \qquad (37)$$

Let $\tilde{\epsilon}(s)$ and $\tilde{\kappa}(s)$ be the Laplace transformations of $\epsilon(t)$ and $\kappa(t)$, respectively. Then $\tilde{\underline{A}}^f(\mathbf{k},s)$ and $\tilde{\underline{A}}^b(\mathbf{k},s)$, i.e., the forward and backward Laplace transformation of $\underline{A}(\mathbf{k},t)$, can be obtained as follows:

$$\tilde{\underline{\mathbf{A}}}^{f,b}(\mathbf{k},s)$$
$$=\frac{s\tilde{\epsilon}(s)}{s^2\tilde{\epsilon}(s)+k^2c^2\tilde{\kappa}(s)}\underline{\mathbf{A}}(\mathbf{k},0)\pm\frac{1}{s^2\tilde{\epsilon}(s)+k^2c^2\tilde{\kappa}(s)}\dot{\underline{\mathbf{A}}}(\mathbf{k},0)$$
$$+\frac{1}{\epsilon_0}\sum_\lambda\frac{(\tilde{\underline{\mathbf{J}}}^{f,b}(\mathbf{k},s)\cdot e_\lambda(\mathbf{k}))e_\lambda(\mathbf{k})}{s^2\tilde{\epsilon}(s)+k^2c^2\tilde{\kappa}(s)}. \qquad (38)$$

The time-dependent vector potential is obtained from a contour integration over the Bromwich contour by an inverse Laplace transformation. From Eq. (38) we find the vector potential for $t>0$ [21],

$$\underline{\mathbf{A}}(\mathbf{k},t)=\xi(t)\underline{A}(\mathbf{k},0)+\zeta(t)\dot{\underline{A}}(\mathbf{k},0)-\frac{1}{2\pi\epsilon_0}\sum_\lambda\int_{-\infty}^{+\infty}d\omega e^{-\imath\omega t}$$
$$\times\frac{[\tilde{\underline{\mathbf{J}}}^f(\mathbf{k},-\imath\omega+0)\cdot e_\lambda(\mathbf{k})]}{\omega^2\epsilon(\omega)-k^2c^2\kappa(\omega)}e_\lambda(\mathbf{k}), \qquad (39)$$

where

$$\xi(t)=\frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty}dse^{st}\frac{s\tilde{\epsilon}(s)}{s^2\tilde{\epsilon}(s)+k^2c^2\tilde{\kappa}(s)}$$
$$=\sum_j\mathrm{Re}\left(e^{-\imath\Omega_j t}\frac{v_g^j}{v_p^j}\right) \qquad (40)$$

and

$$\zeta(t)=\frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty}ds\frac{e^{st}}{s^2\tilde{\epsilon}(s)+k^2c^2\tilde{\kappa}(s)}$$
$$=\frac{1}{kc}\sum_j\mathrm{Im}\left(e^{-\imath\Omega_j t}\frac{v_g^j}{c\kappa(\Omega_j)}\right). \qquad (41)$$

In these expressions, we changed the integration variable from s to $-\imath\omega+\eta$, with a small but positive η. Therefore, we introduce the electric permittivity and the inverse magnetic permeability of the magnetodielectric medium in the frequency domain as $\epsilon(\omega)=\tilde{\epsilon}(-\imath\omega+0)$ and $\kappa(\omega)=\tilde{\kappa}(-\imath\omega+0)$ for real ω, and, correspondingly, $\tilde{\epsilon}(\imath\omega+0)$ and $\tilde{\kappa}(\imath\omega+0)$ as their complex conjugations $\epsilon^*(\omega)$ and $\kappa^*(\omega)$, respectively. We define for each allowed frequency Ω_j, the group velocity $v_g^j=\frac{\partial\omega}{\partial k}$ and the phase velocity $v_p^j=\frac{\omega}{k}$ where the frequencies $\Omega_j(\mathbf{k})$ and $\Omega_j^*(\mathbf{k})$ are the complex-frequency solutions of the dispersion relation $\omega^2\epsilon(\omega)-\mathbf{k}^2c^2\kappa(\omega)$ which has no zeros in the upper half-plane. It is worth emphasizing that for a lossy medium, we lose the usual dispersion relation in which a limited number of discrete frequencies ω are associated with each wave vector k. Thus k and ω must be considered as independent real variables [22].

Now the vector potential for $t<0$ is obtained from the inverse Laplace transform of Eq. (38) as

$$\underline{\mathbf{A}}(\mathbf{k},t)=\xi(t)\underline{\mathbf{A}}(\mathbf{k},0)-\zeta(t)\dot{\underline{\mathbf{A}}}(\mathbf{k},0)-\frac{1}{2\pi\epsilon_0}\sum_\lambda\int_{-\infty}^{+\infty}d\omega e^{-\imath\omega t}$$
$$\times\frac{[\tilde{\underline{\mathbf{J}}}^b(\mathbf{k},+\imath\omega+0)\cdot e_\lambda(\mathbf{k})]}{\omega^2\epsilon^*(\omega)-k^2c^2\kappa^*(\omega)}e_\lambda(\mathbf{k}), \qquad (42)$$

The coefficients $\eta(t)$ and $\zeta(t)$ in Eqs. (39) and (42) damp out exponentially in time since all Ω_j in the exponentials have negative imaginary parts. Therefore, for large times, the medium and electromagnetic field tend to an equilibrium which is determined by the characteristic damping time $\tau_j=1/\mathrm{Im}\Omega_j$. After a few times the maximum characteristic damping time, only the third term survives in these equations since they have poles on the imaginary axis in the complex s plane. Also, these terms in Eqs. (39) and (42) are zero for negative and positive t, respectively [23]. Thus, we may combine the two expressions into a single one and use the Fourier transform to obtain the vector potential in real space for all t

$$\mathbf{A}(\mathbf{r},t)=\int_0^{+\infty}d\omega\int d^3\mathbf{k}\,e^{\imath\mathbf{k}\cdot\mathbf{r}-\imath\omega t}\underline{\mathbf{A}}^+(\mathbf{k},\omega)+\text{c.c.}, \qquad (43)$$

with the positive-frequency Fourier component

$$\underline{\mathbf{A}}^+(\mathbf{k},\omega)=\frac{-1}{(2\pi)^{5/2}\epsilon_0}\sum_\lambda\left[\frac{\tilde{\underline{\mathbf{J}}}^f(\mathbf{k},-\imath\omega+0)\cdot e_\lambda(\mathbf{k})}{\omega^2\epsilon(\omega)-k^2c^2\kappa(\omega)}\right.$$
$$\left.+\frac{\tilde{\underline{\mathbf{J}}}^b(\mathbf{k},\imath\omega+0)\cdot e_\lambda(\mathbf{k})}{\omega^2\epsilon^*(\omega)-k^2c^2\kappa^*(\omega)}\right]e_\lambda(\mathbf{k}). \qquad (44)$$

It is not difficult to show that the transverse electric field can be written as

$$\mathbf{E}^{\perp}(\mathbf{r},t)=-\imath\int_0^{+\infty}d\omega\,\omega\int d^3\mathbf{k}\,[e^{\imath\mathbf{k}\cdot\mathbf{r}-\imath\omega t}\underline{\mathbf{A}}^+(\mathbf{k},\omega)-\text{c.c.}]. \qquad (45)$$

From Eq. (39) we can see that the coefficient $\xi(t)$ in Eq. (40) takes the value 1 at time $t=0$ and therefore the coefficient $\zeta(t)$ takes the value 0. These constraints are satisfied, if certain velocity sum rules are adopted. In this way, we find the following modified velocity sum rules for all wave vectors $\mathbf{k}$:

$$\sum_j \mathrm{Re}\left(\frac{v_g^j}{v_p^j}\right) = 1 \tag{46}$$

and

$$\sum_j \mathrm{Im}\left(\frac{v_g^j}{\kappa(\Omega_j)}\right) = 0, \tag{47}$$

which are resembling the quantum relations obtained in [21] and [24], now generalized to a magnetodielectric medium.

B. Classical theory of Cherenkov radiation ($T = 0$)

Theoretically, when considering the Cherenkov radiation, one usually treats the charge motion with a constant velocity which corresponds to the so-called Tamm Frank problem [2]. Consider a point charge e uniformly moving in a magnetodielectric medium with a velocity $\mathbf{v}$. Therefore, according to Eq. (15)

$$\underline{\mathbf{J}}(\mathbf{k},t) = \frac{e\mathbf{v}}{(2\pi)^{3/2}} e^{-\imath \mathbf{k}\cdot\mathbf{v}t}, \tag{48}$$

then

$$\tilde{\underline{\mathbf{J}}}^f(\mathbf{k}, -\imath\omega + 0) = \frac{e\mathbf{v}}{(2\pi)^{1/2}}\delta(\omega - \mathbf{k}\cdot\mathbf{v}). \tag{49}$$

Now substituting Eq. (49) into Eq. (45), we obtain

$$\mathbf{E}^{\perp}(\mathbf{r},t) = \frac{-\imath e}{8\pi^3\epsilon_0}\sum_\lambda \int_0^{+\infty} d\omega\,\omega \times \int d^3\mathbf{k}\left[\frac{e^{\imath\mathbf{k}\cdot\mathbf{r}}e^{-\imath\omega t}\mathbf{v}\cdot e_\lambda(\mathbf{k})}{\omega^2\epsilon(\omega) - k^2c^2\kappa(\omega)} - \text{c.c.}\right] \times \delta(\omega - \mathbf{k}\cdot\mathbf{v}) e_\lambda(\mathbf{k}). \tag{50}$$

In this case the energy loss of a point charged particle per unit length emitted in the form of radiation, is defined by the braking force acting on the charge at its location [17,25]

$$\begin{aligned}\frac{dW}{dt} &= e\mathbf{v}\cdot\mathbf{E}^{\perp}|_{\mathbf{r}=\mathbf{v}t}\\ &= \frac{\imath e^2}{8\pi^3\epsilon_0}\sum_\lambda\int_0^{+\infty} d\omega\,\omega\\ &\quad\times\int d^3\mathbf{k}\left[\frac{e^{\imath(\mathbf{k}\cdot\mathbf{v}-\omega)t}[\mathbf{v}\cdot e_\lambda(\mathbf{k})]^2}{k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)} - \text{c.c.}\right]\delta(\omega - \mathbf{k}\cdot\mathbf{v}).\end{aligned} \tag{51}$$

Letting θ be the angle between $\mathbf{v}$ and $\mathbf{k}$, then $\sum_\lambda[\mathbf{v}\cdot e_\lambda(\mathbf{k})]^2 = \mathbf{v}^2(1-\cos^2\theta)$. Therefore we find

$$\begin{aligned}\frac{dW}{dt} &= \frac{\imath e^2 v}{4\pi^2\epsilon_0}\int_0^{+\infty} d\omega\omega\int_0^{+\infty} dkk\int_{-1}^{+1} d(\cos\theta)\\ &\quad\times\left[\frac{e^{\imath(kv\cos\theta-\omega)t}(1-\cos^2\theta)}{k^2c^2\kappa(\omega)-\omega^2\epsilon(\omega)} - \text{c.c.}\right]\delta\left(\cos\theta - \frac{\omega}{kv}\right)\\ &= \frac{e^2 v}{2\pi^2\epsilon_0}\int_0^{+\infty} d\omega\omega\int_0^{+\infty} dkk\,\frac{\mathrm{Im}[k^2c^2\kappa(\omega)-\omega^2\epsilon(\omega)]}{|k^2c^2\kappa(\omega)-\omega^2\epsilon(\omega)|^2}\\ &\quad\times\left(1 - \frac{\omega^2}{k^2v^2}\right)\end{aligned} \tag{52}$$

that the electromagnetic waves are emitted at an angle to the path of the particle determined by

$$\cos\theta = \frac{\omega}{kv}. \tag{53}$$

The transparent magnetodielectric medium can be considered in principle as a limiting case of the lossy dispersive medium. Actually, there are some ranges of frequencies over which the imaginary parts of the permittivity and permeability of the magnetodielectric medium can be ignored. In a transparent magnetodielectric medium the energy loss of a charge a only a result of radiation. In this case the imaginary parts of the electric permittivity and the inverse magnetic permeability must tend to zero, thus

$$\begin{aligned}&\lim_{\mathrm{Im}\,\epsilon(\omega),\,\mathrm{Im}\,\mu(\omega)\to 0}\frac{\mathrm{Im}[k^2c^2\kappa(\omega)-\omega^2\epsilon(\omega)]}{|k^2c^2\kappa(\omega)-\omega^2\epsilon(\omega)|^2}\\ &= \pi\mu(\omega)\delta(n^2(\omega)\omega^2 - k^2c^2) = \sum_j\frac{\pi\mu(\Omega_j)v_j^g}{2\Omega_j n(\Omega_j)c}\delta(\omega-\Omega_j),\end{aligned} \tag{54}$$

where $n^2(\omega) = \epsilon(\omega)\mu(\omega)$ and the frequencies $\Omega_j(\mathbf{k})$ are the complex-frequency solutions of the dispersion relation $\omega^2\epsilon(\omega) - k^2c^2\kappa(\omega)$. Now substituting Eq. (54) into Eq. (52) and performing the integration over ω and converting the integration over k to an integration over Ω_j, we obtain

$$\frac{dW}{dt} = \frac{e^2 v}{4\pi\epsilon_0 c^2}\sum_j\int_0^{+\infty} d\Omega_j\,\Omega_j\,\mu(\Omega_j)\left(1 - \frac{c^2}{v^2 n^2(\Omega_j)}\right). \tag{55}$$

It is seen from the Dirac δ function in Eqs. (54) and (52) that radiation is possible only if the inequality $v > c/n(\Omega_j)$ is satisfied. We define the Cherenkov cone $\cos\theta = c/vn(\Omega_j)$ corresponding to any frequency Ω_j for which $v > c/n(\Omega_j)$, where θ is the angle between the wave vector $\mathbf{k}$ of the radiated electromagnetic wave and the velocity of the particle $\mathbf{v}$. It is easy to show that when there is only one frequency for each k, then Eq. (55) tends to the result of [9,17,26].

C. Finite temperature Cherenkov radiation in classical regime

Our considerations so far have been applied to zero temperature. The generalization of the formalism for this case is straightforward. It is known that the medium and electromagnetic field are in the thermal equilibrium in this regime. The inclusion of temperature may be done in the usual manner [27–29]. The finite temperature expression, as is well known, is found by replacing the frequency integral by a sum over Matsubara frequencies according to the transition

$$\hbar\int_0^\infty\frac{d\xi}{2\pi}f(\imath\xi_l)\to k_BT{\sum_{l=0}^{\infty}}' f(\imath\xi_l), \quad \xi_l = 2\pi k_BTl/\hbar, \tag{56}$$

where T and k_B are the temperature and Boltzmann constant and the prime on the summation mark denotes that the zeroth term is given half-weight as is conventional. The effect of finite temperature on the energy loss in the form of Cherenkov radiation can be easily taken into account by using Eqs. (56) and (52)

$$\frac{dW}{dt} = \frac{\imath e^2 v k_B T}{\pi \epsilon_0 \hbar} \sum_{l=0}^{\infty'} \xi_l \int_0^{+\infty} dkk \frac{\mathrm{Im}\left[k^2c^2\kappa(\imath\xi_l) + \xi_l^2\epsilon(\imath\xi_l)\right]}{\left|k^2c^2\kappa(\imath\xi_l) + \xi_l^2\epsilon(\imath\xi_l)\right|^2} \times \left(1 + \frac{\xi_l^2}{k^2v^2}\right). \tag{57}$$

We know that the function $\coth(\hbar\omega/2k_BT)$ has an infinite number of poles at $\omega_l = \imath\xi_l$ and elsewhere is analytic and bounded. This enables us to write Eq. (57) as

$$\frac{dW}{dt} = \frac{e^2 v}{2\pi^2\epsilon_0} \int_0^{+\infty} d\omega\, \omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \times \int_0^{+\infty} dkk \frac{\mathrm{Im}[k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)]}{|k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)|^2}\left(1 - \frac{\omega^2}{k^2v^2}\right). \tag{58}$$

Then with suitable rearrangement of the exponentials in the hyperbolic cotangent we obtain

$$\frac{dW}{dt} = \left(\frac{dW}{dt}\right)_{T=0} + \left(\frac{dW}{dt}\right)_{T\neq 0}, \tag{59}$$

where

$$\left(\frac{dW}{dt}\right)_{T=0} = \frac{e^2 v}{2\pi^2\epsilon_0} \int_0^{+\infty} d\omega\, \omega \int_0^{+\infty} dkk \times \frac{\mathrm{Im}[k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)]}{|k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)|^2}\left(1 - \frac{\omega^2}{k^2v^2}\right) \tag{60}$$

and

$$\left(\frac{dW}{dt}\right)_{T\neq 0} = \frac{e^2 v}{2\pi^2\epsilon_0} \int_0^{+\infty} d\omega \frac{2\omega}{e^{(\hbar\omega/2k_BT)} - 1} \int_0^{+\infty} dkk \times \frac{\mathrm{Im}[k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)]}{|k^2c^2\kappa(\omega) - \omega^2\epsilon(\omega)|^2}\left(1 - \frac{\omega^2}{k^2v^2}\right). \tag{61}$$

The last formula differs from the zero-temperature formula only by the multiplicative factor $2[e^{(\hbar\omega/2k_BT)} - 1]^{-1}$ which has the asymptotic behavior 0 and $4k_BT/\hbar\omega$ at low temperatures $k_BT \ll \hbar\omega$ and at high temperatures $k_BT \gg \hbar\omega$, respectively. In fact, the Matsubara frequency sum naturally separates into a term which is temperature independent and a term containing the Bose-Einstein distribution. In some sense the replacement of Matsubara frequency sum by an integral, as in (58), is equivalent to switching from imaginary time (discrete frequencies in Euclidean space) to real time (continuous energies in Minkowski space).

A transparent magnetodielectric medium can be considered in principle as a limiting case of a lossy dispersive medium. In this case using Eq. (54), we obtain

$$\frac{dW}{dt} = \frac{e^2 v}{4\pi\epsilon_0 c^2} \sum_j \int_0^{+\infty} d\Omega_j\, \Omega_j\, \mu(\Omega_j) \times \coth\left(\frac{\hbar\Omega_j}{2k_BT}\right)\left(1 - \frac{c^2}{v^2 n^2(\Omega_j)}\right) \tag{62}$$

which is the finite temperature generalization of Eq. (55). It is easy to show that for a nondispersive medium, Eq. (55) tends to the result of [30].

III. QUANTUM THEORY

The classical theory of Cherenkov radiation effects is sufficiently accurate in the optical part of the spectrum [10]. For methodological and physical reasons, it is equally important to consider the quantum theory of this effects. Quantum theory enables us to derive the classical equation with the appropriate corrections. We extract the Maxwell equations and constitute relations and the vector potential field operator in the first part of this section, and in the following we calculate the radiation intensity within a nonrelativistic and relativistic theory.

A. Canonical quantization

In the description of the canonical quantization of the electromagnetic field, we choose the Coulomb gauge $\mathbf{k}\cdot\underline{\mathbf{A}}(\mathbf{k},t) = 0$. In this gauge the vector potential $\underline{\mathbf{A}}$ is a purely transverse field and can be decomposed along the unit polarization vectors $e_\lambda(\mathbf{k})\lambda = 1,2$

$$\underline{\mathbf{A}}(\mathbf{k},t) = \sum_{\lambda=1}^{2} \underline{A}_\lambda(\mathbf{k},t)\,\mathbf{e}_\lambda(\mathbf{k}). \tag{63}$$

The dynamical fields $\underline{\mathbf{X}}_\omega$ and $\underline{\mathbf{Y}}_\omega$ have both transverse and longitudinal parts and can be expanded as

$$\underline{\mathbf{X}}_\omega(\mathbf{k},t) = \sum_{\lambda=1}^{3} \underline{X}_{\omega\lambda}(\mathbf{k},t)\,\mathbf{e}_\lambda(\mathbf{k}). \tag{64}$$

Furthermore, by using the Lagrange's equation for the scalar potential ϕ, we find a Lagrangian depending on a reduced number of dynamical variables in the reciprocal space

$$\begin{aligned}\underline{L} =& \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2(t) + \sum_{\lambda=1}^{3}\int_0^\infty d\omega \int' d^3\mathbf{k}(|\dot{\underline{X}}_{\omega\lambda}|^2 - \omega^2|\underline{X}_{\omega\lambda}|^2 \\ &+ |\dot{\underline{Y}}_{\omega\lambda}|^2 - \omega^2|\underline{Y}_{\omega\lambda}|^2) + \sum_{\lambda=1}^{2}\int' d^3\mathbf{k}\left(\epsilon_0|\dot{\underline{A}}_\lambda|^2 - \frac{|\mathbf{k}\underline{A}_\lambda|^2}{\mu_0}\right) \\ &+ \int d^3\mathbf{k}[\underline{A}_\lambda \underline{J}_\lambda^{*\perp}(\mathbf{k},t) + \mathrm{H.c.}] + \sum_{\lambda=1}^{2}\int' d^3\mathbf{k}\{-\dot{\underline{A}}_\lambda \underline{P}_\lambda^{*\perp} \\ &\times(\mathbf{k},t) + [\imath\mathbf{k}\times(\underline{A}_\lambda\mathbf{e}_\lambda(\mathbf{k})]\cdot\underline{\mathbf{M}}^*(\mathbf{k},t) + \mathrm{H.c.}\} \\ &+ \int d^3\mathbf{k}\left(\frac{(\imath\mathbf{k}\cdot\underline{\mathbf{P}})\underline{\rho}^*}{\epsilon_0|k|^2} + \mathrm{H.c.}\right) - \int d^3\mathbf{k}\frac{|\underline{\rho}|^2}{\epsilon_0|k|^2} \\ &- \int d^3\mathbf{k}\frac{|\imath\mathbf{k}\cdot\underline{\mathbf{P}}|^2}{\epsilon_0|k|^2}.\end{aligned} \tag{65}$$

The Lagrangian (66) can now be used to obtain the corresponding canonical conjugate variables of the fields within the half $\mathbf{k}$-space as

$$-\underline{D}_\lambda^\perp(\mathbf{k},t) = \frac{\delta\underline{L}}{\delta(\dot{\underline{A}}_\lambda^*)} = \epsilon_0\dot{\underline{A}}_\lambda(\mathbf{k},t) - P_\lambda^\perp(\mathbf{k},t), \tag{66}$$

$$\underline{Q}_{\omega\lambda}(\mathbf{k},t) = \frac{\delta\underline{L}}{\delta(\dot{\underline{X}}_{\omega\lambda}^*)} = \dot{\underline{X}}_{\omega\lambda}, \quad \underline{\Pi}_{\omega\lambda}(\mathbf{k},t) = \frac{\delta\underline{L}}{\delta(\dot{\underline{Y}}_{\omega\lambda}^*)} = \dot{\underline{Y}}_{\omega\lambda}, \tag{67}$$

$$\mathbf{p}_\alpha(t) = \frac{\delta L}{\delta(\dot{\mathbf{r}}_\alpha)} = m_\alpha \dot{\mathbf{r}}_\alpha + q_\alpha \mathbf{A}(\mathbf{r}_\alpha,t). \tag{68}$$

Now the fields can be quantized canonically in a standard fashion by demanding equal-time commutation relations among the variables and their conjugates. For electromagnetic field components, and the dynamical variables of the external charges, we find, respectively,

$$[\underline{A}^*_\lambda(\mathbf{k},t), -\underline{D}^\perp_{\lambda'}(\mathbf{k}',t)] = \imath\hbar\delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}'), \tag{69}$$

$$[r_{\alpha i}(t), p_{\alpha j}(t)] = \imath\hbar\delta_{ij} \tag{70}$$

and for the reservoir fields

$$[\underline{X}^*_{\omega\lambda}(\mathbf{k},t), \underline{Q}_{\omega'\lambda'}(\mathbf{k}',t)] = \imath\hbar\delta_{\lambda\lambda'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'), \tag{71}$$

$$[\underline{Y}^*_{\omega\lambda}(\mathbf{k},t), \underline{\Pi}_{\omega'\lambda'}(\mathbf{k}',t)] = \imath\hbar\delta_{\lambda\lambda'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}') \tag{72}$$

with all other equal-time commutators being zero. Using the Lagrangian (66) and the expression for the canonical conjugate variables in (69), we obtain the Hamiltonian of the total system

$$\begin{aligned} H = &\sum_{\lambda=1}^{2}\int' d^3k\left(\frac{|\underline{D}^\perp_\lambda - \underline{P}^\perp_\lambda|^2}{\epsilon_0} + \frac{|\mathbf{k}\underline{A}_\lambda|^2}{\mu_0}\right) \\ &+ \sum_\alpha \frac{(\mathbf{p}_\alpha - q_\alpha\mathbf{A}(\mathbf{r}_\alpha,t))^2}{2m_\alpha} + \sum_{\lambda=1}^{3}\int_0^\infty d\omega \\ &\times \int' d^3\mathbf{k}(|\dot{\underline{X}}_{\omega\lambda}|^2 + \omega^2|\underline{X}_{\omega\lambda}|^2 + |\dot{\underline{Y}}_{\omega\lambda}|^2 + \omega^2|\underline{Y}_{\omega\lambda}|^2) \\ &- \sum_{\lambda=1}^{2}\int' d^3\mathbf{k}[(\imath\mathbf{k}\times(\underline{A}_\lambda\mathbf{e}_\lambda(\mathbf{k}))\cdot\underline{\mathbf{M}}^*(\mathbf{k},t) + \text{H.c.}] \\ &+ \int' d^3\mathbf{k}\frac{|\imath\mathbf{k}\cdot\underline{\mathbf{P}}|^2}{\varepsilon_0|k|^2} + \int' d^3\mathbf{k}(\frac{(-\imath\mathbf{k}\cdot\underline{\mathbf{P}})\underline{\rho}^*}{\varepsilon_0|k|^2} + \text{H.c.}) \\ &+ \int' d^3\mathbf{k}\frac{|\underline{\rho}|^2}{\varepsilon_0|k|^2}. \end{aligned} \tag{73}$$

If we apply Heisenberg equation to the operators $\underline{D}_\lambda$ and $\underline{A}_\lambda$, and use the commutation relation (69), Maxwell equations in the reciprocal space can be obtained as

$$\dot{\underline{A}}_\lambda(\mathbf{k},t) = \frac{\imath}{\hbar}[H,\underline{A}_\lambda(\mathbf{k},t)] = -\frac{\underline{D}^\perp_\lambda(\mathbf{k},t) - \underline{P}^\perp_\lambda(\mathbf{k},t)}{\epsilon_0}, \tag{74}$$

$$\begin{aligned}\dot{\underline{D}}^\perp_\lambda(\mathbf{k},t) = \frac{\imath}{\hbar}[H,\underline{D}^\perp_\lambda(\mathbf{k},t)] = \frac{|k|^2}{\mu_0}\underline{A}_\lambda(\mathbf{k},t) \\ - \mathbf{e}_\lambda(\mathbf{k})\cdot[\imath\mathbf{k}\times\underline{\mathbf{M}}(\mathbf{k},t)] - \underline{J}^\perp_\lambda(\mathbf{k},t). \end{aligned} \tag{75}$$

Multiplying both sides of these equations by the polarization unit vectors and summing over the polarization indices we find

$$\underline{\mathbf{D}}^\perp(\mathbf{k},t) = \epsilon_0\underline{\mathbf{E}}^\perp(\mathbf{k},t) + \underline{\mathbf{P}}^\perp(\mathbf{k},t), \tag{76}$$

$$\dot{\underline{\mathbf{D}}}^\perp(\mathbf{k},t) = \imath\mathbf{k}\times\underline{\mathbf{H}}(\mathbf{k},t) - \underline{\mathbf{J}}^\perp(\mathbf{k},t), \tag{77}$$

where $\underline{\mathbf{D}}^\perp$ is the transverse displacement field, $\underline{\mathbf{E}}^\perp = -\dot{\underline{\mathbf{A}}}$ is the transverse electric field and $\mu_0\underline{\mathbf{H}}(\mathbf{k},t) = \imath\mathbf{k}\times\underline{\mathbf{A}}(\mathbf{k},t) - \mu_0\mathbf{M}$ is the magnetic induction field and

$$\underline{\mathbf{J}}^\perp(\mathbf{k},t) = \frac{1}{2(2\pi)^{3/2}}\sum_\alpha\sum_{\lambda=1}^{2} q_\alpha(\dot{\mathbf{r}}_\alpha e^{-\imath\mathbf{k}\cdot\mathbf{r}_\alpha} + e^{-\imath\mathbf{k}\cdot\mathbf{r}_\alpha}\dot{\mathbf{r}}_\alpha)\cdot\mathbf{e}_\lambda(\mathbf{k}). \tag{78}$$

In the presence of external charges, the longitudinal components of the electric and displacement fields can be written respectively as

$$\underline{\mathbf{E}}^\parallel(\mathbf{k},t) = -\frac{\hat{k}(\hat{k}\cdot\underline{\mathbf{P}})}{\epsilon_0} - \frac{\imath\mathbf{k}\underline{\rho}(\mathbf{k},t)}{\epsilon_0|k|^2}, \tag{79}$$

$$\underline{\mathbf{D}}^\parallel(\mathbf{k},t) = \epsilon_0\underline{\mathbf{E}}^\parallel(\mathbf{k},t) + \underline{\mathbf{P}}^\parallel(\mathbf{k},t) = -\frac{\imath\mathbf{k}\underline{\rho}(\mathbf{k},t)}{|k|^2}. \tag{80}$$

If we differentiate Eq. (76) with respect to the time variable t and use Eq. (77), we find the quantum counterpart of Eq. (11) for the vector potential in the reciprocal space as

$$\begin{aligned}\mu_0\epsilon_0\ddot{\underline{\mathbf{A}}}(\mathbf{k},t) + |k|^2\underline{\mathbf{A}}(\mathbf{k},t) - \mu_0\dot{\underline{\mathbf{P}}}^\perp(\mathbf{k},t) \\ - \mu_0\imath\mathbf{k}\times\underline{\mathbf{M}}(\mathbf{k},t) = \mu_0\underline{\mathbf{J}}^\perp(\mathbf{k},t).\end{aligned} \tag{81}$$

Equation (81) is the Langevin equation for the vector potential $\underline{\mathbf{A}}(\mathbf{k},\mathbf{t})$, wherein, the explicit form of the electric and magnetic polarization densities of the medium is known. The quantum Langevin equation can be considered as the basis of the macroscopic description of a quantum particle coupled to an environment or a heat bath [31]. Similarly, it is easy to show that the Heisenberg equation of motion for the external charged particles is

$$\begin{aligned} m_\alpha\ddot{\mathbf{r}}_\alpha &= \frac{\imath}{\hbar}[H,\mathbf{p}_\alpha - q_\alpha\mathbf{A}(\mathbf{r}_\alpha,t)] \\ &= q_\alpha\mathbf{E}(\mathbf{r}_\alpha,t) + \frac{1}{2}q_\alpha[\dot{\mathbf{r}}_\alpha\times\mathbf{B}(\mathbf{r}_\alpha,t) - \mathbf{B}(\mathbf{r}_\alpha,t)\times\dot{\mathbf{r}}_\alpha]. \end{aligned} \tag{82}$$

Using the commutation relations (71), (72) and applying the total Hamiltonian (73), it can be shown that the combination of the Heisenberg equations of the canonical variables $\mathbf{X}(\omega,t)$ and $\mathbf{Y}(\omega,t)$ lead to the same equations (18) and (19) with the solutions (20) and (23), respectively. Now by substituting Eqs. (20) and (23) in the integrands of Eqs. (13), we find the electric and magnetic polarization densities of the medium in reciprocal space

$$\underline{\mathbf{P}}(\mathbf{k},t) = \epsilon_0\int_0^\infty dt\,\chi_e(t-t')\underline{\mathbf{E}}(\mathbf{k},t') + \underline{\mathbf{P}}^N(\mathbf{k},t), \tag{83}$$

$$\underline{\mathbf{M}}(\mathbf{k},t) = \frac{1}{\mu_0}\int_0^\infty dt\,\chi_m(t-t')\underline{\mathbf{B}}(\mathbf{k},t') + \underline{\mathbf{M}}^N(\mathbf{k},t), \tag{84}$$

where χ_e and χ_m are the same electric and magnetic susceptibilities of the medium defined in Eqs. (22) and (24), the noises

$$\underline{\mathbf{P}}^N(\mathbf{k},t) = \int_0^\infty d\omega f(\omega)\left(\dot{\underline{\mathbf{X}}}_\omega(\mathbf{k},0)\frac{\sin\omega t}{\omega} + \underline{\mathbf{X}}_\omega(\mathbf{k},0)\cos\omega t\right), \tag{85}$$

$$\underline{\mathbf{M}}^N(\mathbf{k},t) = \int_0^\infty d\omega g(\omega)\left(\dot{\underline{\mathbf{Y}}}_\omega(\mathbf{k},0)\frac{\sin\omega t}{\omega} + \underline{\mathbf{Y}}_\omega(\mathbf{k},0)\cos\omega t\right) \tag{86}$$

are the electric and magnetic polarization noise densities associated with absorption, with the causal behavior of the medium, respectively. To facilitate the calculations, let us introduce the following annihilation operators:

$$a_\lambda(\mathbf{k},t) = \sqrt{\frac{1}{2\hbar\epsilon_0 c|k|}}[\epsilon_0 c|k|\underline{A}_\lambda(\mathbf{k},t) - \imath\underline{D}^\perp_\lambda(\mathbf{k},t)], \tag{87}$$

$$d_\lambda(\mathbf{k},\omega,t) = \sqrt{\frac{1}{2\hbar\omega}}[\omega \underline{X}_{\omega\lambda}(\mathbf{k},t) + \imath \underline{Q}_{\omega\lambda}(\mathbf{k},t)], \tag{88}$$

$$b_\lambda(\mathbf{k},\omega,t) = \sqrt{\frac{1}{2\hbar\omega}}[\omega \underline{Y}_{\omega\lambda}(\mathbf{k},t) + \imath \underline{\Pi}_{\omega\lambda}(\mathbf{k},t)]. \tag{89}$$

From equal-time commutation relations for the fields (69)–(72), we obtain the following equal-time commutation relations for the creation and annihilation operators:

$$[a_\lambda(\mathbf{k},t), a^\dagger_{\lambda'}(\mathbf{k}',t)] = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}'), \tag{90}$$

$$[d_\lambda(\mathbf{k},\omega,t), d^\dagger_{\lambda'}(\mathbf{k}',\omega',t)] = \delta_{\lambda\lambda'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'), \tag{91}$$

$$[b_\lambda(\mathbf{k},\omega,t), b^\dagger_{\lambda'}(\mathbf{k}',\omega',t)] = \delta_{\lambda\lambda'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'). \tag{92}$$

The commutation relations (90)–(92) in contrast to the previous relations (69)–(72), which were correct only in the half $\mathbf{k}$-space, are now valid in the whole reciprocal space. Inverting Eqs. (87) and (88), we can write the canonical variables $\underline{\mathbf{A}}$, $\underline{\mathbf{X}}_\omega$, and $\underline{\mathbf{Y}}_\omega$ in terms of the creation and annihilation operators as

$$\underline{\mathbf{A}}(\mathbf{k},t) = \sqrt{\frac{\hbar}{2\epsilon_0 c|k|}}\sum_{\lambda=1}^{2}[a_\lambda(\mathbf{k},t) + a^\dagger_\lambda(-\mathbf{k},t)]e_\lambda(\mathbf{k}), \tag{93}$$

$$\underline{\mathbf{X}}_\omega(\mathbf{k},t) = \sqrt{\frac{\hbar}{2\omega}}\sum_{\lambda=1}^{3}[d_\lambda(\mathbf{k},\omega,t) + d^\dagger_\lambda(-\mathbf{k},\omega,t)]e_\lambda(\mathbf{k}), \tag{94}$$

$$\underline{\mathbf{Y}}_\omega(\mathbf{k},t) = \sqrt{\frac{\hbar}{2\omega}}\sum_{\lambda=1}^{3}[b_\lambda(\mathbf{k},\omega,t) + b^\dagger_\lambda(-\mathbf{k},\omega,t)]e_\lambda(\mathbf{k}). \tag{95}$$

Now by employing the Fourier transforms of these recent relations, the Hamiltonian of the total system (73), in the real space, can be recast into the final form

$$\begin{aligned} H =& \int d^3\mathbf{r}\bigg[-\frac{\mathbf{D}^\perp(\mathbf{r},t)\cdot\mathbf{P}(\mathbf{r},t)}{\epsilon_0} + \frac{\mathbf{P}^2(\mathbf{r},t)}{2\epsilon_0} \\ &- \nabla\times\mathbf{A}(\mathbf{r},t)\cdot\mathbf{M}(\mathbf{r},t)\bigg] + \sum_\alpha \frac{[\mathbf{p}_\alpha - q_\alpha\mathbf{A}(\mathbf{r}_\alpha,t)]^2}{2m_\alpha} \\ &+ \frac{1}{8\pi\epsilon_0}\int d^3\mathbf{r}\int d^3\mathbf{r}'\frac{[\nabla\cdot\mathbf{P}(\mathbf{r},t)][\nabla'\cdot\mathbf{P}(\mathbf{r}',t)]}{|\mathbf{r}-\mathbf{r}'|} \\ &- \frac{1}{4\pi\epsilon_0}\sum_\alpha q_\alpha\int d^3\mathbf{r}\frac{[\nabla\cdot\mathbf{P}(\mathbf{r},t)]}{|\mathbf{r}-\mathbf{r}_\alpha|} \\ &+ \frac{1}{8\pi\epsilon_0}\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\mathbf{r}-\mathbf{r}_\alpha|} + H_F + H_e + H_m, \end{aligned} \tag{96}$$

where

$$\mathbf{P}(\mathbf{r},t) = \sum_{\lambda=1}^{3}\int_0^\infty d\omega\int d^3\mathbf{k}\sqrt{\frac{\hbar}{2\omega}}f(\omega) \times [d_\lambda(\mathbf{k},\omega,t)e^{\imath\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}]\mathbf{e}_\lambda(\mathbf{k}), \tag{97}$$

$$\mathbf{M}(\mathbf{r},t) = \sum_{\lambda=1}^{3}\int_0^\infty d\omega\int d^3\mathbf{k}\sqrt{\frac{\hbar}{2\omega}}g(\omega) \times [b_\lambda(\mathbf{k},\omega,t)e^{\imath\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}]\mathbf{e}_\lambda(\mathbf{k}), \tag{98}$$

and

$$H_F = \sum_{\lambda=1}^{2}\int d^3\mathbf{k}\,\hbar c|k|\,a^\dagger_\lambda(\mathbf{k},t)a_\lambda(\mathbf{k},t), \tag{99}$$

$$H_e = \sum_{\lambda=1}^{3}\int d\omega\int d^3\mathbf{k}\,\hbar\omega\,d^\dagger_\lambda(\mathbf{k},\omega,t)d_\lambda(\mathbf{k},\omega,t), \tag{100}$$

$$H_m = \sum_{\lambda=1}^{3}\int d\omega\int d^3\mathbf{k}\,\hbar\omega\,b^\dagger_\lambda(\mathbf{k},\omega,t)b_\lambda(\mathbf{k},\omega,t) \tag{101}$$

are the Hamiltonian of the electromagnetic field and the medium in the normal ordering form.

We now proceed to solve the wave equation (81) along the lines of the classical wave equation (35) using the Laplace transform technique. After some lengthy and elaborated calculations (see Appendix), the vector potential in the large-time limit, i.e., when the medium and electromagnetic field tend to an equilibrium state, can be obtained as

$$\begin{aligned} \mathbf{A}(\mathbf{r},t) =& \frac{-\imath}{\epsilon_0}\sum_{\lambda=1}^{2}\int d^3\mathbf{k}\int d\omega\omega\sqrt{\frac{\hbar}{2(2\pi)^3\omega}}f(\omega) \\ &\times\left(\frac{d_\lambda(\mathbf{k},\omega,0)e^{-\imath\omega t}e^{\imath\mathbf{k}\cdot\mathbf{r}}}{-\omega^2\epsilon(\omega)+c^2k^2\kappa(\omega)} - \text{H.c.}\right)\mathbf{e}_\lambda(\mathbf{k}) \\ &+ \frac{\imath}{\epsilon_0}\sum_{\lambda=1}^{2}\int d^3\mathbf{k}\int d\omega\sqrt{\frac{\hbar|k|^2}{2(2\pi)^3\omega}}g(\omega) \\ &\times\left(\frac{b_\lambda(\mathbf{k},\omega,0)e^{-\imath\omega t}e^{\imath\mathbf{k}\cdot\mathbf{r}}}{-\omega^2\epsilon(\omega)+c^2k^2\kappa(\omega)} - \text{H.c.}\right)\mathbf{s}_\lambda(\mathbf{k}), \end{aligned} \tag{102}$$

where $\mathbf{s}_\lambda(\mathbf{k}) = \hat{k}\times\mathbf{e}_\lambda(\mathbf{k})$. In the large-time limit, the vector potential operator will be a function of the medium operators only, i.e., the radiation is due to the medium, which still satisfies Maxwells equations, as expected. Also the canonical commutation relation (69) is preserved in this limit if in addition to the velocity sum rules (46) and (47), which still legitimate in the quantum domain, the following velocity sum rule for a magnetodielectric medium is also satisfied (see Appendix):

$$\begin{aligned} &[A^*(k,t), -D^\perp(k',t)] \\ &= \int_0^{+\infty} d\omega\frac{\omega^3\text{Im}\epsilon(\omega) - k^2c^2\omega\text{Im}\kappa(\omega)}{|-\omega^2\epsilon(\omega)+c^2k^2\kappa(\omega)t|^2} = \frac{\pi}{2} \\ &\Rightarrow \sum_j\text{Re}\left[\frac{v_g^j v_p^j}{c^2}\right] = 1. \end{aligned} \tag{103}$$

The form of vector potential operator given in (102) agrees with previous works, if one replaces the medium annihilation operators $d_\lambda(\mathbf{k},\omega,0)$ and $b_\lambda(\mathbf{k},\omega,0)$ in the large-time (96) with the diagonalizing annihilation operators $K_{e,\lambda}(\mathbf{k},\omega)$ and $K_{m,\lambda}(\mathbf{k},\omega)$, derived by the damped polarization formalism [24,32] and also if one makes similar replacements for the creation operators $f^e_\lambda(\mathbf{k},\omega)$ and $f^m_\lambda(\mathbf{k},\omega)$ derived by the phenomenological formalism [33,34], where again the same expressions for the field and medium operators are recovered.

B. Nonrelativistic quantum theory of Cherenkov radiation

We consider a charge particle with mass m and electric charge e uniformly moving in a linear homogeneous magnetodielectric medium described by the Hamiltonian (96). In fact, we consider a total system of two noninteracting parts, that is, the free electron and a system, which consist of the electromagnetic field and the magnetodielectric medium in interaction. Therefore the Hamiltonian operator of the total system (96), i.e., electromagnetic field, the medium, and the particle, in the large-time limit, can be rewritten as

$$
\begin{aligned}
H &= H_0 + H_{int},\\
H_0 &= H_{\mathrm{ele}} + H_F,\\
H_{\mathrm{ele}} &= \frac{\mathbf{p}^2}{2m},\\
H_F &= :\sum_{\lambda=1}^{3}\int d\omega \int d^3\mathbf{k}\,\hbar\omega\,\{d_\lambda^\dagger(\mathbf{k},\omega,t)d_\lambda(\mathbf{k},\omega,t)\\
&\quad + b_\lambda^\dagger(\mathbf{k},\omega,t)b_\lambda(\mathbf{k},\omega,t)\},\\
H_{int} &= -\frac{e\mathbf{p}\cdot\mathbf{A}(\mathbf{x},t)}{m},
\end{aligned}
\tag{104}
$$

where $\mathbf{x}$ is the position operator of the particle and we have ignored the term $\mathbf{A}^2$ since the Cherenkov radiation can be considered as a first order process in which the number of photons changes by ± 1. Also the direct Coulomb interaction between the electron and the medium has been omitted from the Hamiltonian (96) since it can give rise to radiative transitions only in third or higher orders.

The moving charged particle with the momentum $\hbar\mathbf{q}$ has the quantum state $|\mathbf{q}\rangle$

$$
\mathbf{p}\,|\mathbf{q}\rangle = \hbar\mathbf{q}\,|\mathbf{q}\rangle\,, \tag{105}
$$

where $|\mathbf{q}\rangle$ is the momentum eigenvector of the particle which in a coordinate representation can be written as

$$
\langle\mathbf{x}|\mathbf{q}\rangle = \psi_{\mathbf{q}}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}e^{\imath\,\mathbf{q}\cdot\mathbf{x}}, \tag{106}
$$

therefore the Hamiltonian $H_{\mathrm{ele}} = \mathbf{p}^2/2m$ for a free particle has the eigenvector $|\mathbf{q}\rangle$ with the energy eigenvalues $E_{\mathbf{q}} = \hbar^2\mathbf{q}^2/2m$. The unperturbed Hamiltonian $H_0 = H_{\mathrm{ele}} + H_F$ has the eigenstate

$$
|\mathrm{ele}+\mathrm{rad}\rangle = |\mathrm{ele}\rangle|\mathrm{rad}\rangle
$$

which are the direct product of the eigenstates of H_{ele} and H_F. In order to separate out the emission of Cherenkov radiation from various other processes which might occur, such as, for example, ionization, emission of bremsstrahlung, etc., we restrict our attention to first order transitions. Therefore we apply quantum mechanical perturbation theory up to the first order approximation to treat the transition probability per unit time for a free particle of momentum $\hbar\mathbf{q}$ to emit a photon of momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$ thereby changing its momentum to $\hbar(\mathbf{q}-\mathbf{k})$ as following:

$$
\begin{aligned}
\Gamma_{\mathbf{q}\to\mathbf{q}-\mathbf{k}} &= \frac{2\pi}{\hbar}\,|\langle 1_{\mathbf{k}}|\langle\mathbf{q}-\mathbf{k}|H_{int}|\mathbf{q}\rangle|0\rangle|^2\,\delta\\
&\quad\times\left(\frac{\hbar^2\mathbf{q}^2}{2m} - \frac{\hbar^2}{2m}\,|\mathbf{q}-\mathbf{k}|^2 - \hbar\omega\right),
\end{aligned}
\tag{107}
$$

where the states $|0\rangle$ and $|1_{\mathbf{k}}\rangle$ present the vacuum state of the electromagnetic field and the excited state of the electromagnetic field with a single photon with wave vector $\mathbf{k}$ and frequency ω, respectively. The argument of the Dirac δ function displays the conservation of energy and the square of the matrix element of Eq. (107) is obtained by using Eqs. (102) and (104) as

$$
\begin{aligned}
&|\langle 1_{\mathbf{k}}|\langle\mathbf{q}-\mathbf{k}|H_{int}|\mathbf{q}\rangle|0\rangle|^2\\
&\quad= \frac{\hbar e^2}{16\pi^3\epsilon_0^2 m^2\omega}\{|\beta(\omega,|k|)|^2|\langle\mathbf{q}-\mathbf{k}|e^{-\imath\,\mathbf{k}\cdot\mathbf{x}}\mathbf{p}\cdot\mathbf{e}_\lambda(\mathbf{k})|\mathbf{q}\rangle|^2\\
&\qquad + k^2|\gamma(\omega,|k|)|^2|\langle\mathbf{q}-\mathbf{k}|e^{-\imath\,\mathbf{k}\cdot\mathbf{x}}\mathbf{p}\cdot\mathbf{s}_\lambda(\mathbf{k})|\mathbf{q}\rangle|^2\}
\end{aligned}
\tag{108}
$$

by using Eqs. (106) and (105) the matrix elements in above equation are just $\hbar\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k})$ and $\hbar\mathbf{k}\cdot\mathbf{s}_\lambda(\mathbf{k})$, respectively. Let θ be the angle between $\mathbf{q}$ and $\mathbf{k}$ and let $\mathbf{v} = \hbar\mathbf{q}/m$ be the particle velocity, we find

$$
\begin{aligned}
\Gamma_{\mathbf{q}\to\mathbf{q}-\mathbf{k}} &= \frac{e^2 v(1-\cos^2\theta)}{4\pi^3\epsilon_0\hbar k}\left(\frac{\omega^2\mathrm{Im}\epsilon(\omega) - k^2c^2\mathrm{Im}\kappa(\omega)}{|-\omega^2\epsilon(\omega)+k^2c^2\kappa(\omega)|^2}\right)\\
&\quad\times\delta\left[\cos\theta - \frac{\omega}{kv}\left(1+\frac{\hbar k^2}{2m\omega}\right)\right],
\end{aligned}
\tag{109}
$$

therefore, photon is emitted at an angle to the path of the particle given by

$$
\cos\theta = \frac{\omega}{kv}\left(1+\frac{\hbar k^2}{2m\omega}\right). \tag{110}
$$

If the energy of the photon $\hbar\omega$ is much less than the rest mass of the particle mc^2 then this is approximately Eq. (53) which gives the classical Cherenkov angle. The total energy radiated per unit time is found to be

$$
\begin{aligned}
\frac{dW}{dt} &= \frac{e^2 v}{2\pi^2\epsilon_0}\sum_{\lambda=1}^{2}\int d^3\mathbf{k}\int_0^{+\infty} d\omega\hbar\omega\Gamma_{\mathbf{q}\to\mathbf{q}-\mathbf{k}}\\
&= \frac{e^2 v}{2\pi^2\epsilon_0}\int_0^{+\infty} kdk\int_0^{+\infty}\omega d\omega\\
&\quad\times\left[1-\frac{\omega^2}{k^2v^2}\left(1+\frac{\hbar k^2}{2m\omega}\right)^2\right]\\
&\quad\times\mathrm{Im}\left(\frac{1}{-\omega^2\epsilon(\omega)+k^2c^2\kappa(\omega)}\right).
\end{aligned}
\tag{111}
$$

We note that the integration on the azimuthal angle is trivial. The integration on polar angle is done with the help of the Dirac δ function in Eq. (111) and

$$
\left(\frac{\omega^2\mathrm{Im}\epsilon(\omega) - k^2c^2\mathrm{Im}\kappa(\omega)}{|-\omega^2\epsilon(\omega)+k^2c^2\kappa(\omega)|^2}\right) = \mathrm{Im}\left(\frac{1}{-\omega^2\epsilon(\omega)+k^2c^2\kappa(\omega)}\right). \tag{112}
$$

It is easily shown that Eq. (111) is consistent with the result of [35] and in the classical limit as $\hbar \longrightarrow 0$, reduces to the classical results (52).

C. Relativistic quantum theory of Cherenkov radiation

The description of particles used in the preceding section is valid only when the particles are moving at velocities small compared to the velocity of light. The preceding formalism must be generalized somehow to describe the relativistic moving particles, that is, we should drive the Dirac equation for external particles embedded in the magnetodielectric medium. For this purpose, we substitute the following Lagrangian for external particle instead of the Lagrangian (4) [36,37]

$$L_q = \frac{\imath\hbar c}{2}\int d^3\mathbf{x}\left[\sum_{\mu=0}^{3}\left(\bar\psi(\mathbf{x},t)\gamma^\mu\frac{\partial\psi(\mathbf{x},t)}{\partial x^\mu} - \frac{\partial\bar\psi(\mathbf{x},t)}{\partial x^\mu}\gamma^\mu\psi(\mathbf{x},t)\right)\right] - mc^2\bar\psi\psi + e\int d^3\mathbf{x}\left\{\sum_{j=1}^{3}[c\bar\psi(\mathbf{x},t)\gamma^j\psi(\mathbf{x},t)\mathbf{A}^j(\mathbf{x},t)] - \bar\psi(\mathbf{x},t)\gamma^0\psi(\mathbf{x},t)\varphi(\mathbf{x},t)\right\}, \tag{113}$$

where $\gamma^\mu, \mu = 0,\ldots,3$ are the Dirac matrices with $\gamma^0 = \beta$, $\gamma^j = \beta\alpha_j$, and $\bar\psi = \psi^\dagger\beta$. In a standard representation we have

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0\end{pmatrix}, \tag{114}$$

where $\sigma_j, j = 1,2,3$ are Pauli spin matrices and I is the unit matrix. It is worth mentioning that the Lagrangian of the magnetodielectric medium (8) need not be written in a covariant form since the medium is at rest and a nonrelativistic description is enough although it can be written in a covariant form straightforwardly. We proceed along the lines of the preceding section and instead of the canonical momentum of the particle $\mathbf{p}_\alpha$ we define the canonical conjugate variable of the Dirac particle $\imath\hbar\psi^\dagger$ as

$$\frac{\imath\hbar\psi^\dagger}{2} = \frac{\partial L}{\partial\dot\psi}. \tag{115}$$

The quantization procedure for the Dirac field can be achieved by imposing equal-time anticommutation relation among the field components

$$\{\psi_\alpha(\mathbf{x},t),\psi^\dagger_\beta(\mathbf{x}',t)\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') \tag{116}$$

together with $\{\psi_\alpha(\mathbf{x},t),\psi_\beta(\mathbf{x}',t)\} = 0$. Here we have chosen the anticommutation relation, since we are developing a theory of particles that obey Fermi-Dirac statistics. Using the Lagrangian (65) and (113) and the expression for the canonical conjugate variables in (69)–(72) and (116), we obtain the Hamiltonian of the total system

$$\begin{aligned} H =& \int d^3\mathbf{x}\,(-\imath\hbar c\psi^\dagger(\mathbf{x},t)\boldsymbol{\alpha}\cdot\nabla\psi(\mathbf{x},t) + mc^2\psi^\dagger(\mathbf{x},t)\beta\psi(\mathbf{x},t)) \\ &+ \sum_{\lambda=1}^{3}\int_0^\infty d\omega\int' d^3\mathbf{k}(|\underline{\dot X}_{\omega\lambda}|^2 + \omega^2|\underline{X}_{\omega\lambda}|^2 + |\underline{\dot Y}_{\omega\lambda}|^2 \\ &+ \omega^2|\underline{Y}_{\omega\lambda}|^2) - \sum_{\lambda=1}^{2}\int' d^3\mathbf{k}[(\imath\mathbf{k}\times(\underline{A}_\lambda\mathbf{e}_\lambda(\mathbf{k}))\cdot\underline{\mathbf{M}}^*(\mathbf{k},t) \\ &+ \mathrm{H.c.}] + \int' d^3\mathbf{k}\frac{|\imath\mathbf{k}\cdot\underline{\mathbf{P}}|^2}{\varepsilon_0\,|k|^2} + \int' d^3\mathbf{k}\left(\frac{(-\imath\mathbf{k}\cdot\underline{\mathbf{P}})\underline{\rho}^*}{\varepsilon_0\,|k|^2} + \mathrm{H.c.}\right) \\ &+ \int' d^3\mathbf{k}\frac{|\underline{\rho}|^2}{\varepsilon_0\,|k|^2} - \int' d^3\mathbf{k}(\underline{\mathbf{J}}^*(\mathbf{k},t)\cdot\underline{\mathbf{A}}(\mathbf{k},t) + \mathrm{H.c.}) \\ &+ \sum_{\lambda=1}^{2}\int' d^3k\left(\frac{|\underline{D}^\perp_\lambda - \underline{P}^\perp_\lambda|^2}{\epsilon_0}\right). \end{aligned} \tag{117}$$

It is easily shown that the Heisenberg equation for the dynamic variable of the electromagnetic and medium fields lead to the same constitute equations (83) and (84) and also Maxwell equations (76)–(79) in the reciprocal space. The Fourier transforms of the external current and charge densities are defined by

$$\underline{\mathbf{J}}(\mathbf{k},t) = \frac{ec}{(2\pi)^{3/2}}\int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x},t)\boldsymbol{\alpha}\,\psi(\mathbf{x},t)e^{-\imath\mathbf{k}\cdot\mathbf{x}}, \tag{118}$$

$$\underline{\rho}(\mathbf{k},t) = \frac{e}{(2\pi)^{3/2}}\int d^3\mathbf{x}\,\psi^\dagger(\mathbf{x},t)\,\psi(\mathbf{x},t)e^{-\imath\mathbf{k}\cdot\mathbf{x}}. \tag{119}$$

If we apply Heisenberg equation to the Dirac fields $\psi(\mathbf{x},t)$ and make use of the anticommutation relation (116), the Dirac equation in the presence of the electromagnetic field can be obtained as

$$\imath\hbar\dot\psi(\mathbf{x},t) = -c\boldsymbol{\alpha}\cdot[\imath\hbar\nabla + e\mathbf{A}(\mathbf{x},t)]\psi(\mathbf{x},t) + e\varphi(\mathbf{x},t) + mc^2\beta\psi(\mathbf{x},t), \tag{120}$$

where

$$\varphi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\left(\frac{-\imath\mathbf{k}\cdot\underline{\mathbf{P}}(\mathbf{k},t)}{\epsilon_0 k^2} + \frac{\underline{\rho}(\mathbf{k},t)}{\epsilon_0 k^2}\right)e^{\imath\mathbf{k}\cdot\mathbf{x}} \tag{121}$$

is the scalar potential of the electromagnetic field defined in Eq. (12). We now expand the Dirac field $\psi(\mathbf{x},t)$ in eigenfunctions of the Dirac equation in the absence of the electromagnetic field

$$\psi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\sum_{\mu=1}^{4}\int d^3\mathbf{q}\,c_\mu(\mathbf{q},t)\psi_\mu(\mathbf{q}), \tag{122}$$

where $\psi_\mu(\mathbf{q}) = u_\mu(\mathbf{q})e^{\imath\mathbf{q}\cdot\mathbf{x}}$ and $u_\mu(\mathbf{q})$ are four-component spinors of the Dirac equation with eigenvalues $E_\mathbf{q} = \pm\sqrt{\hbar^2c^2\mathbf{q}^2 + m^2c^4}$ with the normalization condition $u^\dagger_\mu(\mathbf{q})u_\nu(\mathbf{q}) = \delta_{\mu\nu}$ [28] and $c_\mu(\mathbf{q},t)$ are the annihilation operators of the particle with momentum $\hbar\mathbf{q}$. By substituting Eqs. (122) and (102) in (117), the Hamiltonian of the total system in the large-time limit can be recast in the following form:

$$\begin{aligned} H &= H_0 + H_{int}, \\ H_0 &= H_{\mathrm{ele}} + H_F, \\ H_{\mathrm{ele}} &= \sum_{\mu=1}^{4}\int d^3\mathbf{q}\,E_\mathbf{q}c^\dagger_\mu(\mathbf{q},t)c_\mu(\mathbf{q},t), \end{aligned} \tag{123}$$

$$\begin{aligned} H_F =& :\sum_{\lambda=1}^{3}\int d\omega\int d^3\mathbf{k}\hbar\omega d^\dagger_\lambda(\mathbf{k},\omega,t)d_\lambda(\mathbf{k},\omega,t) \\ &+ b^\dagger_\lambda(\mathbf{k},\omega,t)b_\lambda(\mathbf{k},\omega,t)\}: \end{aligned}$$

$$H_{int} = -\int d^3\mathbf{k}[\underline{\mathbf{J}}^*(\mathbf{k},t)\cdot\underline{\mathbf{A}}(\mathbf{k},t) + \mathrm{H.c.}]$$

$$= -ec\int d^3\mathbf{x}\int d^3\mathbf{k}\,\psi^*\boldsymbol{\alpha}\cdot\underline{\mathbf{A}}(\mathbf{k},t)\psi$$
$$= \frac{\imath ce}{\epsilon_0}\sum_{\lambda=1}^{2}\sum_{\mu,\mu'=1}^{4}\int d^3\mathbf{k}\int d^3\mathbf{q}\int_0^\infty d\omega\sqrt{\frac{\hbar\omega}{2(2\pi)^3}}f(\omega)$$
$$\times\left(\frac{u_\mu^\dagger(\mathbf{q})\boldsymbol{\alpha}\cdot\mathbf{e}_\lambda(\mathbf{k})u_{\mu'}(\mathbf{q}-\mathbf{k})}{-\omega^2\epsilon(\omega)+c^2k^2\kappa(\omega)}c_\mu^\dagger(\mathbf{q})c_{\mu'}(\mathbf{q}-\mathbf{k})\right.$$
$$\left.\times\, d_\lambda(\mathbf{k},\omega,0)e^{-\imath\omega t}-\text{H.c.}\right)-\frac{\imath ce}{\epsilon_0}\sum_{\lambda=1}^{2}\sum_{\mu,\mu'=1}^{4}\int d^3\mathbf{k}$$
$$\times\int d^3\mathbf{q}\int_0^\infty d\omega\sqrt{\frac{\hbar|k|^2}{2(2\pi)^3\omega}}g(\omega)$$
$$\times\left(\frac{u_\mu^\dagger(\mathbf{q})\boldsymbol{\alpha}\cdot\mathbf{s}_\lambda(\mathbf{k})u_{\mu'}(\mathbf{q}-\mathbf{k})}{-\omega^2\epsilon(\omega)+c^2k^2\kappa(\omega)}c_\mu^\dagger(\mathbf{q})c_{\mu'}(\mathbf{q}-\mathbf{k})\right.$$
$$\left.\times\, b_\lambda(\mathbf{k},\omega,0)e^{-\imath\omega t}-\text{H.c.}\right).\qquad(124)$$

The unperturbed Hamiltonian $H_0 = H_{\text{ele}} + H_F$ has the eigenstate

$$|\text{ele}+\text{rad}\rangle = |\text{ele}\rangle|\text{rad}\rangle$$

which are the direct product of the eigenstates of H_{ele} and H_F. we again apply first order perturbation theory to treat the transition probability per unit time for a free Dirac particle of momentum $\hbar\mathbf{q}$ to emit a photon of momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$ thereby changing its momentum to $\hbar(\mathbf{q}-\mathbf{k})$

$$\Gamma_{\mathbf{q}\to\mathbf{q}-\mathbf{k}} = \frac{2\pi}{\hbar}\,|\langle 1_\mathbf{k}|\langle\mathbf{q}-\mathbf{k}|H_{int}|\mathbf{q}\rangle|0\rangle|^2\,\delta\Big(\sqrt{\hbar^2c^2\mathbf{q}^2+m^2c^4}$$
$$-\sqrt{\hbar^2c^2|\mathbf{q}-\mathbf{k}|^2+m^2c^4}-\hbar\omega\Big),\qquad(125)$$

where the argument of the Dirac δ function displays the conservation of energy. We may proceed to calculate the energy loss per unit time as we did in the previous section but with a little modification here. The sum over final states must include a sum over the final spin states of the particle with positive energy $\mu = 1,2$, also we average over the initial spin states

$$\frac{dW}{dt} = \frac{1}{2}\sum_{\lambda=1}^{2}\sum_{\mu,\mu'=1}^{2}\int d^3\mathbf{k}\int_0^\infty d\omega\hbar\omega\Gamma_{\mathbf{q}\to\mathbf{q}-\mathbf{k}}.\qquad(126)$$

In order to calculate the above equation we must evaluate the following sums:

$$S = \frac{1}{2}\sum_{\lambda=1}^{2}\sum_{\mu,\mu'=1}^{2}|u_\mu^\dagger(\mathbf{q})\alpha\cdot\mathbf{e}_\lambda(\mathbf{k})u_{\mu'}(\mathbf{q}-\mathbf{k})|^2,\qquad(127)$$
$$S' = \frac{1}{2}\sum_{\lambda=1}^{2}\sum_{\mu,\mu'=1}^{2}|u_\mu^\dagger(\mathbf{q})\alpha\cdot\mathbf{s}_\lambda(\mathbf{k})u_{\mu'}(\mathbf{q}-\mathbf{k})|^2.\qquad(128)$$

For this purpose we introduce the annihilation operators [38]

$$\Lambda(\mathbf{q}) = \frac{c\alpha\cdot\mathbf{p}+\beta mc^2+|E_\mathbf{q}|}{2|E_\mathbf{q}|}\qquad(129)$$
$$\Lambda(\mathbf{q}-\mathbf{k}) = \frac{c\alpha\cdot(\mathbf{q}-\mathbf{k})+\beta mc^2+|E_{\mathbf{q}-\mathbf{k}}|}{2|E_{\mathbf{q}-\mathbf{k}}|},\qquad(130)$$

and we obtain

$$S = \frac{1}{8}\text{Tr}\{[\alpha\cdot\mathbf{e}_\lambda(\mathbf{k})]\Lambda(\mathbf{q}-\mathbf{k})[\alpha\cdot\mathbf{e}_\lambda(\mathbf{k})]\Lambda(\mathbf{q})\}$$
$$= \frac{1}{2}\left\{1-\frac{m^2c^4}{|E_\mathbf{q}||E_{\mathbf{q}-\mathbf{k}}|}+\frac{2[\mathbf{e}_\lambda(\mathbf{k})\cdot\mathbf{v}_1]^2}{c^2}-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right\},\qquad(131)$$
$$S' = \frac{1}{8}\text{Tr}\{(\alpha\cdot\mathbf{s}_\lambda(\mathbf{k})]\Lambda(\mathbf{q}-\mathbf{k})[\alpha\cdot\mathbf{s}_\lambda(\mathbf{k})]\Lambda(\mathbf{q})\}$$
$$= \frac{1}{2}\left\{1-\frac{m^2c^4}{|E_\mathbf{q}||E_{\mathbf{q}-\mathbf{k}}|}+\frac{2[\mathbf{s}_\lambda(\mathbf{k})\cdot\mathbf{v}_1]^2}{c^2}-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right\},\qquad(132)$$

where we have used $\mathbf{v} = \hbar c^2\mathbf{q}/E_\mathbf{q}$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities before an after the emission of the photon respectively. The sum over polarizations can be carried out as in Eq. (52). The result is

$$S = S' = \frac{\mathbf{v}_1^2}{c^2}(1-\cos^2\theta)$$
$$+\frac{1}{2}\left\{1-\sqrt{\left(1-\mathbf{v}_1^2/c^2\right)\left(1-\mathbf{v}_2^2/c^2\right)}-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right\},\qquad(133)$$

where again θ is the angle between $\mathbf{q}$ and $\mathbf{k}$ given by

$$\cos\theta = \frac{\omega}{vk}\left[1+\frac{\hbar\omega}{2mc^2}\left(\frac{k^2c^2}{\omega^2}-1\right)\sqrt{1-\frac{v^2}{c^2}}\right].\qquad(134)$$

We mention here that in the classical theory only the first term occurs on the right-hand side of Eq. (134) and the second and third terms are a consequence of nonrelativistic and relativistic quantum theory, respectively. These terms are small since the wave length of the electron is much smaller than the photon's wavelength [39,40]. Also the second term in Eq. (133) is a small correction to the result we found in the preceding section. We neglect this term and the rest of the calculation is similar to what presented in the preceding section. The only difference here is that Eq. (134) must be used instead of Eq. (110). The result is

$$\frac{dW}{dt} = \frac{e^2v}{2\pi^2\epsilon_0}\int_0^{+\infty}dkk\int_0^{+\infty}d\omega\omega$$
$$\times\left(1-\frac{\omega^2}{v^2k^2}\left[1+\frac{\hbar\omega}{2mc^2}\left(\frac{k^2c^2}{\omega^2}-1\right)\sqrt{1-\frac{v^2}{c^2}}\right]^2\right)$$
$$\times\,\text{Im}\left(\frac{1}{-\omega^2\epsilon(\omega)+k^2c^2\kappa(\omega)}\right).\qquad(135)$$

Comparing this expression and Eq. (111) with its classical counterpart, Eq. (52), it is seen that the only difference is in the argument of the δ function. In fact, we note that the integrations over k and ω in classical equation (52) diverge but its nonrelativistic and relativistic counterpart that is Eqs. (111) and (135) have no divergent behavior. To appreciate the physical significance of this subject, we consider a transparent and nondispersive magnetodielectric medium. We note that Eqs. (110) and (134) for $0 < v < c$ and $n > 1$ provide a cutoff in frequency $\omega < \omega_c^{nonrel,rel}$ which in nonrelativistic and relativistic regimes are respectively given by

$$\omega_c^{nonrel} = \frac{2mc^2(n\beta-1)}{\hbar n^2}\qquad(136)$$

and

$$\omega_c^{rel} = \frac{2mc^2(n\beta - 1)}{\hbar(n^2-1)\sqrt{1-\beta^2}}, \tag{137}$$

where $\beta = v/c$. These cutoffs are still bounded above by the electron energy mc^2 or $mc^2/\sqrt{1-\beta^2}$, respectively. Thus, we can use one of these cutoffs for the range of integrations to obtain a physically acceptable result for the classical radiation intensity.

It is easy to show that in a transparent magnetodielectric medium Eq. (135) becomes

$$\frac{dW}{dt} = \frac{e^2 v}{4\pi\epsilon_0 c^2}\sum_j \int_0^{+\infty} d\Omega_j \Omega_j \mu(\Omega_j) \times \left(1 - \frac{c^2}{n^2v^2}\left[1 + \frac{\hbar\Omega_j}{2mc^2}(n^2-1)\sqrt{1-\frac{v^2}{c^2}}\right]^2\right), \tag{138}$$

which tends to the correct relation in the classical and extreme relativistic limits.

D. Finite temperature Cherenkov radiation in quantum regime

Our considerations so far have been applied to zero temperature. The inclusion of temperature may be done in the usual manner [27]. In this case the transition probability (125) for a free Dirac particle of momentum $\hbar\mathbf{q}$ to emit a photon of momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$ thereby changing its momentum to $\hbar(\mathbf{q}-\mathbf{k})$ is obtained as

$$\Gamma_{\mathbf{q}\to\mathbf{q}-\mathbf{k}} = \frac{2\pi}{\hbar}|H_{int}|^2(N_{\mathbf{k}}+1)[1-n_F(\mathbf{q}-\mathbf{k})]\delta\left(\sqrt{\hbar^2c^2\mathbf{q}^2+m^2c^4} - \sqrt{\hbar^2c^2|\mathbf{q}-\mathbf{k}|^2+m^2c^4} - \hbar\omega\right), \tag{139}$$

where

$$N_{\mathbf{k}} = \frac{1}{e^{\hbar\omega/k_BT}-1} \tag{140}$$

and

$$n_F(\mathbf{q}-\mathbf{k}) = \frac{1}{e^{\sqrt{\hbar^2c^2|\mathbf{q}-\mathbf{k}|^2+m^2c^4}/k_BT}+1}. \tag{141}$$

Here the factor $N_{\mathbf{k}}+1$ comes from the phonon creation operator for transition from the initial state with $n_{\mathbf{k}}$ photons to the final state with $n_{\mathbf{k}}+1$ photons. In fact, we use the thermal average of $n_{\mathbf{k}}$ which is $N_{\mathbf{k}}$. This photon emission process take place as stimulated and spontaneous emission. Similarly, the factor $1-n_F(\mathbf{q}-\mathbf{k})$ is the probability that the electron state $\mathbf{q}-\mathbf{k}$ is empty, so that the operator $c_\mu^\dagger(\mathbf{q}-\mathbf{k})$ can create an electron in that state. There is also a factor $n_F(\mathbf{q})$, which is the probability that $\mathbf{q}$ is occupied with an electron that is unity. But in calculating the total energy radiated as Cherenkov radiation, we have to omit some important processes. They arise from other electrons in the system with the same spin state. These electrons can not be found in the state $\mathbf{q}$ since our electron is occupying it already. Thus the other electrons of the system will reduce the transition probability (139) and we must subtract the transition probability due to the presence of other electrons from Eq. (139) [27]. Therefore by using Eqs. (126)–(135), the total energy radiated in finite temperature is found to be

$$\frac{dW}{dt} = \frac{e^2v}{2\pi^2\epsilon_0}\int_0^{+\infty} dkk \int_0^{+\infty} d\omega\omega \times \left(1 - \frac{\omega^2}{v^2k^2}\left[1 + \frac{\hbar\omega}{2mc^2}\left(\frac{k^2c^2}{\omega^2}-1\right)\sqrt{1-\frac{v^2}{c^2}}\right]^2\right) \times \mathrm{Im}\left(\frac{1}{-\omega^2\epsilon(\omega)+k^2c^2\kappa(\omega)}\right)F_T(\omega), \tag{142}$$

where

$$F_T(\omega) = (N_{\mathbf{k}}+1)[1-n_F(\mathbf{q}-\mathbf{k})] - N_{\mathbf{k}}n_F(\mathbf{q}-\mathbf{k}) = \frac{e^{\hbar\omega/k_BT}}{e^{\hbar\omega/k_BT}-1}\left[\frac{e^{|E_{\mathbf{q}}-\hbar\omega|/k_BT} - e^{-\hbar\omega/k_BT}}{e^{|E_{\mathbf{q}}-\hbar\omega|/k_BT}+1}\right]. \tag{143}$$

The last equation differs from the zero temperature equation (135) only by the multiplicative factor $F_T(\omega)$ which has the following asymptotic behavior at low temperatures $k_BT \ll \hbar\omega$ and at high temperatures $k_BT \gg E_{\mathbf{q}}$, respectively:

$$F_T(\omega) \sim 1,$$

$$F_T(\omega) \sim \frac{E_{\mathbf{q}}}{2\hbar\omega} = \frac{mc^2}{2\hbar\omega\sqrt{1-(v/c)^2}}.$$

It is easy to show that in a transparent magnetodielectric medium, the finite temperature generalization of Eq. (135) becomes

$$\frac{dW}{dt} = \frac{e^2v}{4\pi\epsilon_0c^2}\sum_j\int_0^{+\infty} d\Omega_j\Omega_j\mu(\Omega_j) \times \left(1 - \frac{c^2}{n^2v^2}\left[1+\frac{\hbar\Omega_j}{2mc^2}(n^2-1)\sqrt{1-\frac{v^2}{c^2}}\right]^2\right)F_T(\Omega_j) \tag{144}$$

which in a nondispersive medium tends to the result of [15].

IV. CONCLUSION

In this paper, we have generalized a Lagrangian introduced in [11] to include the external charges. In this formalism the medium is modelled with two independent collections of vector fields. The classical electrodynamics in the presence of a polarizable and magnetizable medium is discussed and the susceptibility functions of the medium are calculated in terms of the coupling functions. The energy loss of a point charged particle per unit length emitted in the form of radiation, called Cherenkov radiation, is obtained in both zero and finite temperature in classical regime. A fully canonical quantization of both electromagnetic field and the dynamical variables, modeling the medium, is demonstrated. In the Heisenberg picture, the constitutive equations of the medium together with the Maxwell equations are obtained as the equations of motion of the total system. The wave equation for the vector potential is solved. It is shown how the vector potential operator in this theory can be expressed in terms of the medium operators at an initial time. The consistency of these solutions for the field operators are found to depend on the validity of certain velocity sum rules. It is also shown how this scheme is related to the damping polarization and phenomenological quantization theories. The large-time limit and quantum mechanical perturbation theory is applied to treat the finite temperature Cherenkov radiation in the domain of the nonrelativistic and relativistic quantum regimes. The total energy radiated per unit time is calculated which is consistent with both classical and extreme relativistic limits. The approach is based on a Lagrangian formalism and magnetic properties of the medium and the relativistic motion of the particle are included. This model can be applied to the case of Cherenkov radiation in a nonlinear medium which is under consideration.

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APPENDIX

In this appendix we give the analogous expression for the vector potential in the quantum domain. Furthermore, we evaluate the time-dependent coefficients for the vector potential operator. By using Eqs. (83) and (84) the wave equation (81), can be written as

$$\mu_0\epsilon_0\ddot{\underline{\mathbf{A}}} + |k|^2\,\underline{\mathbf{A}} + \mu_0\frac{\partial}{\partial t}\int_0^t dt'\chi_e(t-t')\dot{\underline{\mathbf{A}}}(\mathbf{k},t') - \mu_0\,|k|^2\int_0^t dt'\chi_m(t-t')\underline{\mathbf{A}}(\mathbf{k},t') = \mu_0\frac{\partial \underline{\mathbf{P}}^{N\perp}}{\partial t} + \mu_0\imath\mathbf{k}\times\underline{\mathbf{M}}^N + \mu_0\underline{\mathbf{J}}^{\perp}. \quad (A1)$$

We find the full time dependence of the vector potential by taking the inverse Laplace transform. The inverse Laplace transformation as defined before is a contour integration over the Bromwich contour. After transforming to frequency variables we obtain

$$\mathbf{A}(\mathbf{r},t) = \sum_{\lambda=1}^{2}\int d^3\mathbf{k}\sqrt{\frac{\hbar}{2(2\pi)^3\epsilon_0 ck}}[\eta(k,t)a_\lambda(\mathbf{k},0)e^{\imath\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}]\mathbf{e}_\lambda(\mathbf{k}) + \frac{1}{\epsilon_0}\sum_{\lambda=1}^{2}\int d^3\mathbf{k}\int d\omega\sqrt{\frac{\hbar}{2(2\pi)^3\omega}} \times[\beta(\omega,k,t)d_\lambda(\mathbf{k},0)e^{\imath\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}]\mathbf{e}_\lambda(\mathbf{k}) + \frac{\imath}{\epsilon_0}\sum_{\lambda=1}^{2}\int d^3\mathbf{k}\int d\omega\sqrt{\frac{\hbar k^2}{2(2\pi)^3\omega}}[\gamma(\omega,k,t) \times b_\lambda(\mathbf{k},0)e^{\imath\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}]\mathbf{s}_\lambda(\mathbf{k}) + \sum_\alpha\sum_{\lambda=1}^{2}\frac{q_\alpha}{2}\zeta(k,t) \times[\dot{\mathbf{r}}_\alpha\delta(\mathbf{r}-\mathbf{r}_\alpha) + \delta(\mathbf{r}-\mathbf{r}_\alpha)\dot{\mathbf{r}}_\alpha]\cdot\mathbf{e}_\lambda(\mathbf{k})\mathbf{e}_\lambda(\mathbf{k}), \quad (A2)$$

where

$$\eta(k,t) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} ds e^{st}\frac{s\tilde{\epsilon}(s) - \imath ck}{s^2\tilde{\epsilon}(s) + k^2c^2\tilde{\kappa}(s)} = \sum_j\left[\text{Re}\left(e^{-\imath\Omega_j t}\frac{v_g^j}{v_p^j}\right) - \imath\text{Im}\left(e^{-\imath\Omega_j t}\frac{v_g^j}{c\kappa(\Omega_j)}\right)\right], \quad (A3)$$

$$\beta(\omega,k,t) = \frac{f(\omega)}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} ds\frac{s e^{st}}{(s+\imath\omega)[s^2\tilde{\epsilon}(s) + k^2c^2\tilde{\kappa}(s)]} = \frac{-\imath\omega f(\omega)e^{-\imath\omega t}}{-\omega^2\tilde{\epsilon}(\omega) + k^2c^2\tilde{\kappa}(\omega)}, \quad (A4)$$

$$\gamma(\omega,k,t) = \frac{g(\omega)}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} ds\frac{e^{st}}{(s+\imath\omega)[s^2\tilde{\epsilon}(s) + k^2c^2\tilde{\kappa}(s)]} = \frac{g(\omega)e^{-\imath\omega t}}{-\omega^2\tilde{\epsilon}(\omega) + k^2c^2\tilde{\kappa}(\omega)}, \quad (A5)$$

$$\zeta(k,t) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} ds\frac{e^{st}}{s^2\tilde{\epsilon}(s) + k^2c^2\tilde{\kappa}(s)} = \frac{1}{kc}\sum_j\text{Im}\left(e^{-\imath\Omega_j t}\frac{v_g^j}{c\kappa(\Omega_j)}\right). \quad (A6)$$

The solution of the wave equation is the sum of a transient and a permanent part. The latter are expressed solely in terms of the initial medium operators. Long after the initial time, vector potential operator will be a function of the medium operators alone since it has poles on the imaginary axis in the complex s plane. With the velocity sum rules discussed in Sec. II, one can see that the coefficient $\eta(k,t)$ equals 1 at time $t = 0$ whereas the other coefficients have the initial value 0.

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